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EXTENDING THE STANDARD FORMAT OF ADAPTIVE LOGICS TO THE PRIORITIZED CASE*

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Abstract
This paper introduces a new format for reasoning with prioritized standards of normality. It is applicable in a broad variety of contexts, e.g. dealing with (possibly conflicting) prioritized belief bases or combining different reasoning methods in a prioritized way. The format is a generalization of the standard format of adaptive logics (see [4]). Every logic that is formulated within it has a straightforward semantics in the style of Shoham's selection semantics (see [22]) and a dynamic proof theory. Furthermore, it can count on a rich meta-theory that inherits the attractive features of the standard format, such as soundness and completeness, reflexivity, idempotence, cautious monotonicity, and many other properties.

1. *Introduction*

In this paper, we present a format for adaptive logics (henceforth ALs), that is a generalization of the standard format of ALs. The new format enables one to deal with prioritized defeasible reasoning. Let us in the following introduce the main ideas behind ALs and motivate the extension to the prioritized case.

1.1. *Adaptive Logics*

ALs are powerful formal systems that model and explicate several forms of human reasoning: reasoning with inconsistent premises [1], inductive generalization [6], abduction [18], reasoning on the basis of conflicting norms

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[19], argumentation [26], etc.[1] Many consequence relations from the literature have been reformulated as ALs, see e.g. [8, 12, 23, 30].

Generally speaking, ALs are developed to capture defeasible reasoning forms (DRFs), reasoning forms in which certain inferences may be retracted in view of later insights. A distinctive feature of ALs is their dynamic proof theory — most of the available systems that model DRFs lack a proof theory. Scholars often highlight the non-monotonic character of the consequence relation that is supposed to represent a DRF, but neglect the internal dynamics that is characteristic of the way we reason towards consequences. The growing insight into the given information (resp. premises) may cause the withdrawal of previously drawn inferences, even if no new information is available.[2] The proof theory of ALs nicely captures this internal dynamics.

One of the most important developments within the AL program is the definition of a canonical format, the so-called *standard format* for ALs. This format encompasses a generic proof theory and semantics. A rich and attractive meta-theory has been shown to hold generically for all ALs formulated in the standard format (see [4]): they are sound and complete, their consequence relation is idempotent, cautiously monotonic, etc. Most ALs have been successfully expressed within this format, whence it provides a good basis for a unifying study of DRFs. Let us list some of the key features of the standard format — technical details and a discussion of its meta-theory will be given in Section 2.

Every AL in standard format is characterized by a triple: (i) a lower limit logic (henceforth LLL), (ii) a set of abnormalities Ω and (iii) a strategy. The LLL is a monotonic logic, the rules of which are unconditionally valid in the AL. The AL strengthens its LLL by considering a certain set of formulas (the elements of Ω) as abnormal, and by interpreting premises "as normally as possible". The precise interpretation of the latter phrase depends on the strategy of the AL — the details will be spelled out in Section 2. Hence we can say that the AL equips its LLL with a certain standard of normality.

For example, the inconsistency-adaptive logic $\mathsf{CLuN}^{\mathsf{m}}$ from [1] strengthens its LLL, the paraconsistent logic CLuN, by interpreting premises as consistently as possible. Hence its standard of normality reads: "contradictions are false". While $\mathsf{CLuN}^{\mathsf{m}}$ does not lead to triviality in the face of an inconsistent premise set, it retains a large number of inferences that are valid in classical logic.

In adaptive proofs, this is realized by deriving formulas on a condition. For instance, $\mathsf{CLuN}^{\mathsf{m}}$ allows for the application of Disjunctive Syllogism to $A \vee \neg B$ and B on the condition that B behaves consistently. Whether we

[1] Unpublished papers in the reference section (and many others) are available from the internet address `http://logica.UGent.be/centrum/writings/`.

[2] Pollock dubs this the diachronic defeasibility of defeasible inferences [21].

can depend on the consistent behavior of a formula, may change with the insights we gain and the new information we obtain. Hence some formula occurring at a line of a proof may count as derived at some point and as not derived at another point in the proof. This is determined by its condition, other formulas derived so far in the proof, and the adaptive strategy.

ALs employ a selection semantics in the vein of Shoham [22]. From the set of the LLL-models of the premises, ALs select a subset of models that verify "as few abnormalities as possible". Again, what is meant by "as few as possible" depends on the strategy.

1.2. *Prioritized Adaptive Logics*

This paper deals with prioritized ALs. Agents often make use of various reasoning methods, where some of these methods take precedence over others. For instance, a scientist may reason towards inductive generalizations, but only in as far as this does not run counter to his (defeasible) background knowledge. He may even infer some abductive consequences from his induced generalizations. Also, some standards of normality may themselves have a prioritized flavor. For example, where we start from a (possibly conflicting) set of obligations, each having a certain weight, we may want to deal with the conflicts in a way that is sensitive to this weight.

We will henceforth use the name "prioritized ALs" to refer to ALs that model such processes. The first prioritized ALs were developed to capture reasoning with prioritized belief bases (see Section 3.3 for an example). There are also examples in the literature of prioritized logics for inductive generalization [6] and prioritized inconsistency-adaptive logics [5, 3]. Examples of prioritized ALs that combine different reasoning methods can be found in [17, 16, 25, 26].

While ALs in the standard format are well-studied, prioritized ALs have been comparatively neglected.[3] The standard format of ALs does not incorporate prioritized ALs. The most straightforward way to achieve a prioritized system is to superimpose ALs in standard format. Roughly speaking, this is done as follows: where $\mathsf{AL}_1, \mathsf{AL}_2, \ldots$ are ALs in standard format — each of them taking care of one particular set of abnormalities —, and $Cn_{\mathsf{AL}_i}(\Gamma)$ denotes the AL_i-consequence set of Γ, define the prioritized logic PAL by

$$Cn_{\mathsf{PAL}}(\Gamma) = \text{“}\ldots Cn_{\mathsf{AL}_3}(Cn_{\mathsf{AL}_2}(Cn_{\mathsf{AL}_1}(\Gamma)))\ldots\text{”}$$

In other words, PAL boils down to the application of AL_1 to Γ, next of AL_2 to the consequences obtained so far, next AL_3, and so on. Promising as this approach may seem, some rather discouraging results are available for PAL,

[3] There is some work forthcoming though, namely [27, 7, 24]

such as the lack of soundness, completeness, and idempotence.[4] Moreover, there has been a substantial lack of meta-theory on these combinations of ALs.

Starting from Section 3, we will depart from the above approach, and develop a new format for prioritized ALs that cannot be reduced to (the combination of) ALs in standard format. This new format is very close to the standard format in numerous respects. It also makes use of the characterization by a triple, but now replacing the set of abnormalities Ω by a sequence of sets of abnormalities $\langle \Omega_1, \Omega_2, \ldots \rangle$, where the different subscripts of the sets refer to their priority ranking. Both proof theory and semantics of the new format have the same overall structure as the standard format. The difference is that the strategy is adjusted to the prioritized setting.

Since the new format cannot be reduced to the standard format, we have to re-establish a lot of meta-theoretic results. However, in view of the strong similarity with the standard format, much of the work can be easily achieved through an adaptation of the meta-proofs from [4, 7]. As a result, the new format inherits most if not all of the nice properties of the standard format. Last but not least, every AL in standard format can be characterized as a logic in the new format as well. In view of this, we can safely claim that the new format provides a generalization of the old one.

2. *Flat Adaptive Logics*

In this section, the standard format of ALs is spelled out. This standard format unifies a broad range of what we will henceforth call *flat* ALs. Flat ALs stand in contradistinction to the prioritized ALs we introduce in Section 3. We only explain the general characteristics of the standard format here, and refer to [4] for more details, examples and meta-theoretic proofs. Before we start, let us introduce some conventions.

Throughout this paper, all formulas are assumed to be finite strings. Where W is the set of closed formulas of a formal language L, we define a logic L as a function $f : \wp(W) \to \wp(W)$. L may be characterized by a proof theory, by a semantics or by both. Where $\Gamma \subseteq W$ and $A \in W$, we use $\Gamma \vdash_{\mathsf{L}} A$ to denote that A is L-derivable from Γ. Let $Cn_{\mathsf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathsf{L}} A\}$ be the L-consequence set of Γ. Where M is a L-model and $A \in W$, we write $M \Vdash A$ to denote that A is true in M. M is a L-model of $\Gamma \subseteq W$ iff it is a L-model and $M \Vdash A$ for all $A \in \Gamma$. The set of L-models of Γ is denoted by $\mathcal{M}_{\mathsf{L}}(\Gamma)$. We say that $A \in W$ is a semantic L-consequence of Γ, $\Gamma \models_{\mathsf{L}} A$ iff A is verified by all L-models of Γ.

[4] [24] introduces some exceptional cases in which soundness and completeness is guaranteed.

2.1. *The Standard Format*

General Characterization. Henceforth, we say that L is a Tarski-logic iff it is reflexive, transitive and monotonic. Every adaptive logic in standard format is characterized by a triple:

1. A *lower limit logic* LLL: a compact Tarski-logic that has a proof theory and a characteristic semantics
2. A *set of abnormalities* Ω: a set of formulas, characterized by a (possibly restricted) logical form F; or a union of such sets
3. An *adaptive strategy*: Reliability or Minimal Abnormality

The strategy is indicated by a superscript: AL^{r} for ALs that have Reliability as their strategy, AL^{m} for those that have Minimal Abnormality as strategy. Many definitions and theorems are applicable to both classes of logics. In that case, we use the generic name AL.

The logic LLL is a function $\wp(\mathcal{W}) \to \wp(\mathcal{W})$, where $\mathcal{W}$ is the set of closed formulas of a formal language, henceforth denoted by $\mathcal{L}$. As mentioned in the introduction, AL equips LLL with a certain standard of normality. To express statements concerning normality in the object language, a distinct set of classical connectives is used. How this is done precisely, requires some explanation.

The additional classical connectives are noted by a check: $\check{\neg}, \check{\vee}, \check{\wedge}, \check{\supset}$, and for the predicative case also $\check{\exists}, \check{\forall}$. The language $\mathcal{L}_+$ is obtained by extending $\mathcal{L}$ with the checked connectives, where it is assumed that these symbols are not in $\mathcal{L}$. The set of closed formulas of $\mathcal{L}_+$, $\mathcal{W}_+$ is the closure of $\mathcal{W}$ under the checked connectives. Unless specified differently, we henceforth use Γ as a metavariable for subsets of $\mathcal{W}_+$.

To model inferences on the basis of $\mathcal{L}_+$, LLL is upgraded to $\mathsf{LLL}^+ : \wp(\mathcal{W}_+) \to \wp(\mathcal{W}_+)$. To prepare for semantics of $\mathsf{LLL}^+ : \wp(\mathcal{W}_+) \to \wp(\mathcal{W}_+)$, we define a model validity relation $\Vdash^+$ that extends the validity relation $\Vdash$ of LLL, as follows. Let M be an LLL-model. Define (1) for all $A \in \mathcal{W}$: $M \Vdash^+ A$ iff $M \Vdash A$, (2) for all $A \in \mathcal{W}_+$: $M \nVdash^+ A$ iff $M \Vdash^+ \check{\neg} A$, (3) for all $A, B \in \mathcal{W}_+$: ($M \Vdash^+ A$ or $M \Vdash^+ B$) iff $M \Vdash^+ A \check{\vee} B$, and likewise for the other checked connectives. Henceforth, we say that M is an LLL^+-model of $\Gamma \subseteq \mathcal{W}_+$, $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ iff M is an LLL-model and $M \Vdash^+ A$ for every $A \in \Gamma$. We write $\Gamma \models_{\mathsf{LLL}^+} A$ iff for all LLL^+-models M of Γ: $M \Vdash^+ A$.

In the standard format, a sound and complete axiomatization for LLL^+ is assumed to be given.[5] Note that in view of its semantics, LLL^+ is a compact

[5] Where LLL is supraclassical, one can obtain the axiomatization for LLL^+ by a generic procedure. However, for the sake of generality, we include logics LLL that have rather weak

Tarski-logic and it is a $\mathcal{L}$-conservative extension of LLL: for every $\Gamma \subseteq \mathcal{W}$, $Cn_{\mathsf{LLL}}(\Gamma) \cap \mathcal{W} = Cn_{\mathsf{LLL}^+}(\Gamma) \cap \mathcal{W}$.

Every logic AL is a function $\wp(\mathcal{W}_+) \rightarrow \wp(\mathcal{W}_+)$. Since AL was intended to explicate defeasible reasoning processes on the basis of premises in $\mathcal{L}$, premises of AL logic are often assumed to be subsets of $\mathcal{W}$. One possible interpretation of the relation between AL, $\mathcal{L}$ and $\mathcal{L}_+$ is that AL provides an explication of a reasoning based on formulas in $\mathcal{L}$, but that for this explication, it uses formulas in $\mathcal{L}_+$ — this will become clear when we present the AL-proof theory.

The set of abnormalities $\Omega \subseteq \mathcal{W}_+$ represents those formulas that AL assumes to be false "as much as possible", in view of the premises.[6] The phrase "as much as possible" can have various interpretations — every such interpretation corresponds to an adaptive strategy.[7]

Every flat AL also has an *upper limit logic* ULL : $\wp(\mathcal{W}_+) \rightarrow \wp(\mathcal{W}_+)$, which boils down to enforcing the standard of normality axiomatically. In the remainder of this paper, let $\Theta^{\check{\neg}} = \{\check{\neg} A \mid A \in \Theta\}$ for any $\Theta \subseteq \mathcal{W}_+$. Syntactically, ULL is defined as follows: $\Gamma \vdash_{\mathsf{ULL}} A$ iff $\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathsf{LLL}^+} A$. Semantically, we speak of *normal models* as those LLL^+-models M for which $M \Vdash^+ \check{\neg} A$ for every $A \in \Omega$. Γ is a *normal premise set* iff it has normal models. Finally, $\Gamma \models_{\mathsf{ULL}} A$ iff for every normal model M of Γ, $M \Vdash_{\mathsf{LLL}^+} A$.

Semantics. Before we come to the AL-semantics, we first need a few extra definitions. A Dab-formula $Dab(\Delta)$ is the checked disjunction of the members of a finite $\Delta \subseteq \Omega$. Where $\Delta = \{A\}$, $Dab(\Delta)$ denotes A; where $\Delta = \emptyset$, $\check{\vee}\, Dab(\Delta)$ denotes the empty string. Where $\Delta \neq \emptyset$, $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathsf{LLL}^+} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathsf{LLL}^+} Dab(\Delta')$.

Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-consequences of Γ, let $\Sigma(\Gamma) = \{\Delta_1, \Delta_2, \ldots\}$. We say that $U(\Gamma) = \bigcup \Sigma(\Gamma)$ is the set of *unreliable* formulas with respect to Γ. Finally, where M is a LLL^+-model, its *abnormal part* $Ab(M)$ is the set $\{B \in \Omega \mid M \Vdash^+ B\}$.

As mentioned in the introduction, ALs have a semantics similar to Shoham's preferential semantics [22]: from the set of LLL^+-models of Γ, AL

and non-standard connectives, whence it becomes a lot tougher to find a generic procedure that gives a sound and complete axiomatization for LLL^+. Nevertheless, for concrete cases, the adaptive logician's job of devising a syntax for LLL^+ will usually be fairly easy.

[6] In some papers on ALs, it is required that for some or all $A \in \Omega, \nvdash_{\mathsf{LLL}^+} A$ and $\nvdash_{\mathsf{LLL}^+} \check{\neg} A$. This restriction is useful to rule out ALs that have no sensible applications; however, there is no technical problem with allowing for degenerate cases in which all abnormalities are LLL^+-theorems, or all abnormalities are trivialized by LLL^+.

[7] Other strategies then Reliability and Minimality are e.g. Counting, Normal Selections and the Flip-Flop-Strategy. These are strictly speaking not part of the Standard Format, but can be obtained from it under a translation — see [7, Chapter 6].

selects a subset of models in view of their abnormal part. The precise criterion for a model to be selected depends on the strategy:

Definition 1: $M \in \mathcal{M}_{\mathsf{AL}^\mathsf{r}}(\Gamma)$ *iff* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ *and* $Ab(M) \subseteq U(\Gamma)$.

Definition 2: $M \in \mathcal{M}_{\mathsf{AL}^\mathsf{m}}(\Gamma)$ *iff* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ *and there is no* $M' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ *such that* $Ab(M') \subset Ab(M)$.

$\mathcal{M}_{\mathsf{AL}^\mathsf{r}}(\Gamma)$ is called the set of *reliable* models, $\mathcal{M}_{\mathsf{AL}^\mathsf{m}}(\Gamma)$ the set of $\subset$-*minimally abnormal* models, or more briefly, *minimally abnormal* models.

Although the above definition of $\mathcal{M}_{\mathsf{AL}^\mathsf{m}}(\Gamma)$ is more direct, we can also define the semantics of Minimal Abnormality in terms of the minimal *Dab*-consequences of Γ. This requires some notational preparation. Let $I \subseteq \mathbb{N}$ be an index set, $\Sigma = \{\Delta_i \mid i \in I\}$ and for every $i \in I$, $\Delta_i \subseteq \Omega$. We say that $\varphi \subseteq \Omega$ is a *choice set* of Σ iff for every $i \in I$, $\varphi \cap \Delta_i \neq \emptyset$. For the border case where $\Sigma = \emptyset$, this means that every set $\varphi \subseteq \Omega$ is a choice set of Σ, including the empty set.

φ is a $\subset$-*minimal* choice set of Σ iff there is no choice set ψ of Σ such that $\psi \subset \varphi$. In the context of the standard format, we speak of “minimal choice sets” to refer to “$\subset$-minimal choice sets”. The following is proven in [7, Chapter 5]:

Fact 1: *If every* $\Delta \in \Sigma$ *is finite, then* Σ *has minimal choice sets.* [7, Fact 5.2.1]

$\Phi(\Gamma)$ is the set of minimal choice sets of $\Sigma(\Gamma)$. Note that when $\Sigma(\Gamma) = \emptyset$, $\Phi(\Gamma) = \{\emptyset\}$. It is easily provable that $U(\Gamma) = \bigcup \Phi(\Gamma)$. Also, remark that since all the members of $\Sigma(\Gamma)$ are finite, $\Phi(\Gamma) \neq \emptyset$ for every $\Gamma \subseteq \mathcal{W}$ by Fact 1. The following theorem was proven in [4]:

Theorem 1: $M \in \mathcal{M}_{\mathsf{AL}^\mathsf{m}}(\Gamma)$ *iff* $(M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ *and* $Ab(M) \in \Phi(\Gamma))$.

From this it follows immediately that every minimally abnormal model is a reliable model: $Ab(M) \in \Phi(\Gamma)$ implies that $Ab(M) \subseteq U(\Gamma)$.

Proof Theory. The proof theory of ALs mirrors the dynamic character of defeasible reasoning forms. Every AL-proof consists of lines that have four elements: a line number i, a formula A, a justification (consisting of a series of line numbers and a derivation rule) and a condition $\Delta \subseteq \Omega$. Where Γ is the set of premises, the inference rules are given by:

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \vdots & \vdots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathsf{LLL}^+} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathsf{LLL}^+} B \check{\vee} Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

A *stage* of a proof can be seen as a (countable) sequence of lines, obtained by the application of the above rules. A proof is a (countable) sequence of stages. Every proof starts off with stage 1. Adding a line to a proof by applying one of the rules of inference brings the proof to a successor stage, which is the sequence of all lines written so far. Notably, a new line can be added anywhere in the proof as long as the inference rules are used. An *extension* of a proof at stage s is simply the same proof at a later stage s'. In view of the inference rules, the condition of any line l is necessarily finite, and the following lemma holds:

Lemma 1: *There is an* AL*-proof from* Γ *that contains a line at which* A *is derived on a condition* $\Delta \subseteq \Omega$ *iff* $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. [4, Lemma 1]

A distinguishing feature of adaptive proofs is the marking definition. At every stage of a proof, a marking definition — see below — determines for each line in the proof whether it is marked or not. If a line that has as its second element A is marked at stage s, this indicates that according to our best insights at this stage, A cannot be considered derivable. If the line is unmarked at stage s, we say that A is derived at stage s of the proof. To prepare for the marking definitions, we need some more conventions.

Where $\emptyset \neq \Delta \subset \Omega$, $Dab(\Delta)$ is a Dab-formula at stage s of a proof iff it is the second element of a line at stage s with an empty condition. $Dab(\Delta)$ is a *minimal* Dab-formula at stage s iff there is no other Dab-formula $Dab(\Delta')$ at stage s for which $\Delta' \subset \Delta$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-formulas at stage s of a proof, let $\Sigma_s(\Gamma) = \{\Delta_1, \Delta_2, \ldots\}$. $U_s(\Gamma) = \bigcup \Sigma_s(\Gamma)$ and let $\Phi_s(\Gamma)$ be the set of minimal choice sets of $\Sigma_s(\Gamma)$. By Fact 1, $\Phi_s(\Gamma) \neq \emptyset$ at every stage s of a proof from Γ.

Definition 3: AL^{r}*-Marking: a line* l *is marked at stage* s *iff, where* Δ *is its condition,* $\Delta \cap U_s(\Gamma) \neq \emptyset$.

Definition 4: AL^{m}*-Marking: a line* l *with formula* A *is marked at stage* s *iff, where its condition is* Δ*: (i) there is no* $\varphi \in \Phi_s(\Gamma)$ *such that* $\varphi \cap \Delta = \emptyset$*, or (ii) for a* $\varphi \in \Phi_s(\Gamma)$*, there is no line on which* A *is derived on a condition* Θ *for which* $\Theta \cap \varphi = \emptyset$.

Put differently: where the strategy is Minimal Abnormality, a line with formula A is *un*marked at stage s iff its condition has an empty intersection with at least one $\varphi \in \Phi_s(\Gamma)$, and for every $\psi \in \Phi_s(\Gamma)$, there is a line on which A is derived on a condition Δ such that $\Delta \cap \psi = \emptyset$. As a line may be marked at stage s, unmarked at a later stage s' and marked again at a still later stage s'', we also define a stable notion of derivability.

Definition 5: A *is* finally derived *from* Γ *on line* l *of a finite stage* s *iff (i)* A *is the second element of line* l*, (ii) line* l *is unmarked at stage* s*, and (iii) every extension of the proof at stage* s*, in which line* l *is marked may be further extended in such a way that line* l *is unmarked again.*

Definition 6: $\Gamma \vdash_{\mathsf{AL}} A$ *iff* A *is finally derived on a line of an* AL*-proof from* Γ.

2.2. *An Example: the logic* $\mathsf{CLuN}^{\mathsf{m}}$

General Characterization of $\mathsf{CLuN}^{\mathsf{m}}$. In the introduction, we already mentioned the idea behind inconsistency-adaptive logics. The example we will use to illustrate the standard format is one such logic: $\mathsf{CLuN}^{\mathsf{m}}$. For reasons of simplicity, we only consider the propositional fragment of this system.

The lower limit of $\mathsf{CLuN}^{\mathsf{m}}$ is CLuN, which stands for “Classical Logic with gluts for the Negation”. CLuN is a monotonic paraconsistent logic, defined by full positive CL together with excluded middle (e.g. by the axiom $(A \wedge \neg A) \supset \neg A$). This means that CLuN invalidates disjunctive syllogism: $\{A, \neg A \vee B\} \nvdash_{\mathsf{CLuN}} A$. CLuN^+ is obtained by enriching CLuN with the checked connectives as described above.

The set of abnormalities of $\mathsf{CLuN}^{\mathsf{m}}$ is $\{A \wedge \neg A \mid A \in \mathcal{W}\}$. Hence contradictions (with respect to any formula $A \in \mathcal{W}$) are avoided as much as possible. As a result, we obtain a much richer consequence set than $Cn_{\mathsf{CLuN}}(\Gamma)$, without trivializing inconsistent premises sets $\Gamma \subseteq \mathcal{W}$.

Example of a $\mathsf{CLuN}^{\mathsf{m}}$*-proof.* Consider the premise set $\Gamma_1 = \{p, \neg p \vee q, \neg q, \neg p \vee r, q \vee r\}$. Note that the following Dab-formula is CLuN^+-derivable from Γ_1, which implies that we are dealing with an inconsistent premise set:

$$(p \wedge \neg p) \check{\vee} (q \wedge \neg q) \tag{1}$$

On the semantic level, every CLuN^+-model of Γ_1 verifies either $p \wedge \neg p$ or $q \wedge \neg q$, or both. For every minimally abnormal CLuN^+-model M of Γ_1, either $Ab(M) = \{p \wedge \neg p\}$ or $Ab(M) = \{q \wedge \neg q\}$. Suppose that for some such model M, $Ab(M) = \{p \wedge \neg p\}$. In view of the premise set, $M \Vdash^+ \neg q$ and $M \Vdash^+ q \vee r$. Since also $M \nVdash^+ q \wedge \neg q$, $M \nVdash^+ q$ and $M \Vdash^+ r$. We leave it to the reader to see that also the second class of minimally abnormal models verify r. As a result, r is a semantic $\mathsf{CLuN}^{\mathsf{m}}$-consequence of Γ_1.

Consider the following $\mathsf{CLuN}^{\mathsf{m}}$-proof from Γ_1:

1	p	PREM	$\emptyset$
2	$\neg p \vee q$	PREM	$\emptyset$
3	$\neg q$	PREM	$\emptyset$
4	$\neg p \vee r$	PREM	$\emptyset$
5	$q \vee r$	PREM	$\emptyset$

Note that the fourth element is $\emptyset$, indicating that premises are introduced on the empty condition. We may now derive r from lines 1 and 4:

6	$(p \wedge \neg p) \check{\vee} r$	1,4;RU	$\emptyset$
7	r	6;RC	$\{p \wedge \neg p\}$

In the remainder of this paper, let us denote the stage consisting of lines 1 — n by stage n. At stage 7 of the proof, r is derived. However, we can continue the proof as follows, showing that the condition on line 7 is problematic:

6	$(p \wedge \neg p) \vee r$	1,4;RU	$\emptyset$
7	r	6;RC	$\{p \wedge \neg p\}\checkmark^8$
8	$(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$	1,2,3;RU	$\emptyset$

Where $i \in \mathbb{N}$, we will henceforth use $\checkmark^i$ to denote the marking of a line at stage i. At stage 8, line 7 is marked. Recall that in order to find out which lines are marked at stage s, we had to look at the set $\Phi_s(\Gamma_1)$. Since $\Sigma_8(\Gamma_1) = \{\{p \wedge \neg p, q \wedge \neg q\}\}$, the minimal choice sets at stage 8 are $\varphi_1 = \{p \wedge \neg p\}$ and $\varphi_2 = \{q \wedge \neg q\}$.

Clearly, the condition of line 7 has an empty intersection with φ_2. But r, the formula on line 7, has not been derived on a condition that has an empty intersection with φ_1. Hence the marking definition for Minimal Abnormality stipulates that line 7 is marked.

So how can line 7 become unmarked again? This is done by showing that r can be derived in the proof on a yet different condition:

6	$(p \wedge \neg p) \check{\vee} r$	1,4;RU	$\emptyset$
7	r	6;RC	$\{p \wedge \neg p\}$
8	$(p \wedge \neg p) \check{\vee} (q \wedge \neg q)$	1,2,3;RU	$\emptyset$
9	r	3,5;RC	$\{q \wedge \neg q\}$

Note that throughout the stages $8 - 9$, the set of minimal choice sets remains the same, which means that lines 7 and 9 are unmarked.

The difference with the Reliability Strategy can also be clarified by the above example: in $\mathsf{CLuN}^{\mathsf{r}}$, r is not finally derivable from Γ_1. The reason is that from stage 8 on, the set of unreliable formulas is $\{p \wedge \neg p, q \wedge \neg q\}$. In view of Definition 3, both lines 7 and 9 are marked if Reliability is the strategy. This is in agreement with the $\mathsf{CLuN}^{\mathsf{r}}$-semantics: there is a $M \in \mathcal{M}_{\mathsf{CLuN}^{\mathsf{r}}}(\Gamma) - \mathcal{M}_{\mathsf{CLuN}^{\mathsf{m}}}(\Gamma)$ for which $Ab(M) = \{p \wedge \neg p, q \wedge \neg q\}$ and $M \not\Vdash^{+} r$.

2.3. *Meta-theory of the Standard Format*

In this section, we mention some of the most significant meta-theoretic properties of the standard format. A number of well-known properties are inherent to ALs in standard format, such as soundness, reflexivity and the fixed point property. Furthermore, for AL^{m}, the Deduction Theorem holds, which means that one can introduce hypotheses in a proof, as in classical logic. We assume the reader to be familiar with these properties and refer to Section 5 for their exact formulation. A number of significant properties are less well-known, whence we discuss them here. We mention the original theorems and corollaries in the literature between square brackets.

$\mathcal{L}$-Completeness. In [24] and [7, Chapter 4], an example is presented of a Γ, A for which $\Gamma \nvdash_{\mathsf{AL}^{\mathsf{m}}} A$, whereas $\Gamma \models_{\mathsf{AL}^{\mathsf{m}}} A$. A similar example can be constructed for the Reliability Strategy.[8] Hence completeness in general does not hold for AL. Nevertheless, for all $\Gamma \subseteq \mathcal{W}$, completeness is provable — we will use the term $\mathcal{L}$-completeness to refer to this restricted form

[8] Note that according to Definitions 5 and 6, in order to finally derive A, one has to be able to derive it in a finite proof on an unmarked line. The mentioned examples are constructed such that this first requirement cannot be fulfilled: in order to derive A on a line l, one has to derive Dab-formulas that render line l marked. See Lemmas 7 and 12 for how this problem is avoided whenever $\Gamma \subseteq \mathcal{W}$.

of completeness. From the same examples, we can infer that some other properties such as e.g. Fixed Point also have to be restricted to $\Gamma \subseteq \mathcal{W}$. This should not be seen as a severe problem for ALs, since as we explained before, they were developed to explicate a reasoning process on the basis of premises in $\mathcal{L}$. For the sake of generality we state the meta-theory about AL for any $\Gamma \subseteq \mathcal{W}_+$ whenever possible.

Strong Reassurance. In Section 2.1, we explained that every AL selects a subset of the LLL^+-models of Γ. Now suppose a LLL^+-model M of Γ is not selected. In that case, it seems desirable to have as a property of the logic that there is a LLL^+-model M' of Γ that *is* selected, and that is less abnormal then M. Only then can the logic justify that M is not selected. This property is called “Strong Reassurance” in the literature.

Theorem 2: *If* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}}(\Gamma)$*, then there is a* $M' \in \mathcal{M}_{\mathsf{AL}}(\Gamma)$ *such that* $Ab(M') \subset Ab(M)$. [4, Th. 4 & 5] *(Strong Reassurance)*

Note that the abnormal part-relation and $\subset$ impose a partial order on the LLL^+-models of Γ: $M \prec M'$ iff $Ab(M) \subset Ab(M')$. Strong Reassurance boils down to the claim that $\prec$ is smooth on the set of LLL^+-models of Γ.[9] It also entails that whenever Γ has LLL^+-models, Γ has AL-models — this property is referred to as *Reassurance*. In other words, unless Γ is LLL^+-trivial, AL will not trivialize this premise set.

Cautious Indifference. Suppose we have established for some Γ, A, that $\Gamma \vdash_{\mathsf{AL}} A$. In that case, it seems desirable that the AL-closure of $\Gamma \cup \{A\}$ is not different from the AL-closure of Γ itself. That is, adding A as a premise to Γ should not lead to a different consequence set. This is warranted by the Cautious Indifference principle:

Theorem 3: *For every* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma' \subseteq Cn_{\mathsf{AL}}(\Gamma)$, *then* $Cn_{\mathsf{AL}}(\Gamma) = Cn_{\mathsf{AL}}$ $(\Gamma \cup \Gamma')$. *(Cautious Indifference)* [4, Th. 11.10]

Note that the fixed point property, i.e. that $Cn_{\mathsf{AL}}(\Gamma) = Cn_{\mathsf{AL}}(Cn_{\mathsf{AL}}(\Gamma))$, is derivable from Theorem 3 together with the reflexivity of AL. Cautious Indifference is often divided in two parts: Cautious Monotonicity ($Cn_{\mathsf{AL}}(\Gamma) \subseteq Cn_{\mathsf{AL}}(\Gamma \cup \Gamma')$) and Cumulative Transitivity ($Cn_{\mathsf{AL}}(\Gamma) \supseteq Cn_{\mathsf{AL}}(\Gamma \cup \Gamma')$). Cautious Monotonicity can be proven for the more general case where $\Gamma \subseteq \mathcal{W}_+$, whereas Cumulative Transitivity only holds for $\Gamma \subseteq \mathcal{W}$.

[9] A partial order $\prec$ on X is smooth with respect to a set $Y \subseteq X$ iff for all $A \in Y$ either A is $\prec$-minimal or there is a $\prec$-minimal element $B \in Y$ for which $B \prec A$.

The Hierarchy within the Standard Format. We say that L is *weaker than* L' (L' is stronger than L) iff for every $\Gamma \subseteq W$, $Cn_{\mathsf{L}}(\Gamma) \subseteq Cn_{\mathsf{L}'}(\Gamma)$, while for some $\Gamma \subseteq W$, $Cn_{\mathsf{L}}(\Gamma) \subset Cn_{\mathsf{L}'}(\Gamma)$. The following theorem summarizes the difference in strength between the different logics LLL^+, AL^{r}, AL^{m} and ULL:

Theorem 4: $Cn_{\mathsf{LLL}^+}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}}(\Gamma) \subseteq Cn_{\mathsf{ULL}}(\Gamma)$. [4, Th. 11.1]

Obviously, AL is in most cases stronger than LLL^+. Also, AL^{r} is slightly weaker than AL^{m}, as the example in Section 2.2 illustrated. A related property is that if a premise set Γ is normal, then AL is equipowerful to ULL:

Theorem 5: *If Γ is normal, then $Cn_{\mathsf{AL}}(\Gamma) = Cn_{\mathsf{ULL}}(\Gamma)$.*

Hence if AL can avoid abnormalities altogether, it will do so. Nevertheless, if the premise set is not normal, it will still in most cases render more consequences than LLL^+, without yielding triviality as ULL would. In other words, AL strengthens LLL^+ and approximates ULL as much as possible, adapting itself to the premises.

Equivalent Premise Sets. In [11], it is argued that ALs have certain advantages over numerous other formal approaches to defeasible reasoning methods. The most important argument there is one concerning transparency: there are various criteria to decide when two premise sets are AL-equivalent — criteria that do not hold for those other formalisms. For a lengthy discussion, we refer to the original paper; here we simply mention the three criteria for equivalence (the original Theorems from [11] are given between square brackets).

Theorem 6: *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$, $Cn_{\mathsf{AL}}(\Gamma) = Cn_{\mathsf{AL}}(\Gamma')$ if one of the following holds:*

- *(C1)* $\Gamma' \subseteq Cn_{\mathsf{AL}}(\Gamma)$ *and* $\Gamma \subseteq Cn_{\mathsf{AL}}(\Gamma')$ [Th. 6]
- *(C2) Where L is a Tarski-logic weaker than or identical to AL:* $Cn_{\mathsf{L}}(\Gamma) = Cn_{\mathsf{L}}(\Gamma')$ [Th. 7]
- *(C3) Where L is a Tarski-logic and for every $\Theta \subseteq \mathcal{W}$, $Cn_{\mathsf{AL}}(\Theta) = Cn_{\mathsf{L}}(Cn_{\mathsf{AL}}(\Theta))$:* $Cn_{\mathsf{L}}(\Gamma) = Cn_{\mathsf{L}}(\Gamma')$ [Th. 7]

3. *A Prioritized Selection of Models:* $\mathsf{AL}_{\sqsubset}$

3.1. *General Characterization of* $\mathsf{AL}_{\sqsubset}$

Recall that the aim of this paper is to develop a generic format for prioritized ALs that remains as close as possible to the existing standard format, and inherits its meta-theoretic properties. We will use $\mathsf{AL}_{\sqsubset}$ as a placeholder for logics in this format, for reasons that will become clear in this section. Every logic $\mathsf{AL}_{\sqsubset} : \wp(\mathcal{W}_+) \to \wp(\mathcal{W}_+)$ is characterized by a triple:

1. A lower limit logic LLL
2. A sequence of sets of abnormalities: $\langle \Omega_i \rangle_{i \in I}$, where $I \subseteq \mathbb{N}$ is an index set
3. A strategy: $\sqsubset$-Minimal Abnormality or $\sqsubset$-Reliability

For the remainder of this paper, let $\Omega = \bigcup_{i \in I} \Omega_i$. Each $\Omega_i \subseteq \mathcal{W}_+$ is characterized by a (possibly restricted) logical form, whence Ω fits the format of a set of abnormalities of a flat AL — see page 605. Henceforth, we will use the name AL to refer to the flat AL defined by (i) LLL, (ii) Ω and (iii) a strategy (Reliability or Minimal Abnormality).

Let us briefly discuss the elements of the above triple. First of all, just like AL, every logic $\mathsf{AL}_{\sqsubset}$ is built on top of a logic LLL^+, which is obtained from LLL as described in Section 2. The upper limit logic of $\mathsf{AL}_{\sqsubset}$ is identical to the upper limit logic of AL, and will hence also be denoted by ULL.

The sets of abnormalities $\Omega_1, \Omega_2, \ldots$ correspond to the different standards of normality mentioned in the introduction. We say that A is an abnormality of *rank* i iff $A \in \Omega_i$ and there is no $j < i$ such that $A \in \Omega_j$. The lower the rank of an abnormality, the higher the priority of the corresponding standard of normality. The logic $\mathsf{AL}_{\sqsubset}$ avoids abnormalities "as much as possible, *in view of their rank*". The adaptive strategy specifies the latter phrase. As for AL, the two strategies give rise to two subclasses of prioritized ALs: $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$ and $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$.

Since the $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-semantics is technically less involving than the $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-semantics, we start with the former in Section 3.2. In Section 3.3, we present an example of a logic in the new format: $\mathsf{K}^{\mathsf{m}}_{\sqsubset}$. After that, we discuss an alternative way to characterize the $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-models of a premise set. Finally, in Section 3.5, we show how a Reliability-variant is obtained from this alternative characterization.

3.2. *The* $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$*-semantics*

In Section 2, we explained that flat ALs select a subset of the LLL^{+}-models of a premise set in view of their abnormal part. For AL^{m}, a model M is selected iff its abnormal part $Ab(M)$ is minimal with respect to set-inclusion. The prioritized logic $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$ also selects LLL^{+}-models in view of their abnormal part, but takes into account the rank of abnormalities. In view of the prioritization $Ab(M)$ is not flat but is structured and may be represented by the tuple $\langle Ab(M) \cap \Omega_1, Ab(M) \cap \Omega_2, \ldots\rangle$. Just like the flat abnormal parts were partially ordered in the standard format by $\subset$, the structured abnormal parts of prioritized ALs may be partially ordered by the lexicographic order $\sqsubset_{\mathsf{lex}}$:[10]

Definition 7: $\langle \Delta \cap \Omega_i\rangle_{i\in I} \sqsubset_{\mathsf{lex}} \langle \Delta' \cap \Omega_i\rangle_{i\in I}$ *iff (1) there is an* $i \in I$ *such that for all* $j < i$, $\Delta \cap \Omega_j = \Delta' \cap \Omega_j$, *and (2)* $\Delta \cap \Omega_i \subset \Delta' \cap \Omega_i$. *We write* $\Delta \sqsubset \Delta'$ *iff* $\langle \Delta \cap \Omega_i\rangle_{i\in I} \sqsubset_{\mathsf{lex}} \langle \Delta' \cap \Omega_i\rangle_{i\in I}$.

Just as for flat ALs, the LLL^{+}-models were selected whose abnormal part was $\subset$-minimal, we now select the LLL^{+}-models whose abnormal part is $\sqsubset$-minimal:

Definition 8: $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$ *iff* $M \in \mathcal{M}_{\mathsf{LLL}^{+}}(\Gamma)$ *and there is no* $M' \in \mathcal{M}_{\mathsf{LLL}^{+}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$.

As we did with $\subset$-minimally abnormal models, we can speak of $\sqsubset$-minimally abnormal models. Lemma 2 below states that the $\subset$-order on $\wp(\Omega)$ is included in the $\sqsubset$-order on $\wp(\Omega)$.

Lemma 2: *Where* $\Delta, \Delta' \subseteq \Omega$*: if* $\Delta \subset \Delta'$, *then* $\Delta \sqsubset \Delta'$.

Proof. Suppose $\Delta \subset \Delta'$. Then for all $i \in I$, $\Delta \cap \Omega_i \subseteq \Delta' \cap \Omega_i$ and there is an $i \in I$ such that $\Delta \cap \Omega_i \subset \Delta' \cap \Omega_i$. Take the smallest $i \in I$ for which $\Delta \cap \Omega_i \subset \Delta' \cap \Omega_i$, whence for all $j < i$, $\Delta \cap \Omega_i = \Delta' \cap \Omega_i$. By Definition 7, $\Delta \sqsubset \Delta'$. □

By Lemma 2, we immediately obtain:

[10] Lexicographic orders are a well-known ordering type and are mentioned in any representative mathematical dictionary or encyclopedia (see e.g. [14, p. 1170]). Lexicographic orders have already previously proven to be useful for the formal explication of reasoning on the basis of prioritized information. Lehmann employed them to deal with priorities among defaults [15], Nebel [20] in order to deal with prioritized theory bases and Hansen [13] applied Nebel's preference order to the context of prioritized imperatives.

Theorem 7: *Every* $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$*-model of* Γ *is a* AL^{m}*-model of* Γ.

3.3. *An Example:* $\mathsf{K}^{\mathsf{m}}_{\sqsubset}$

Several ALs have been developed to explicate reasoning with prioritized belief bases — see [9], [30] and [29]. These are sequences of the form $\Psi = \langle \Theta_0, \Theta_1, \Theta_2, \ldots \rangle$, where each Θ_i is a set of formulas, and the index of the sets denotes their plausibility degree: Θ_0 is the set of facts, Θ_1 the set of most plausible beliefs, and so on. The ALs that deal with such belief bases typically use a certain logical operator or a sequence of such operators to express that a belief has a certain degree of plausibility. We will discuss only one such system, in order to illustrate the $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-format.

As before, we restrict the logic to the propositional level. We use the standard modal language $\mathcal{L}^M$ of Kripke's minimal normal modal logic K, axiomatized by the propositional fragment of CL together with the following axioms:

K $\Box(A \supset B) \supset (\Box A \supset \Box B)$
RN if $\vdash A$ then $\vdash \Box A$

As usually, we define $\Diamond A = \neg\Box\neg A$. Let $\mathcal{W}^M$ denote the set of modal wffs, and $\mathcal{W}^l$ the set of literals (sentential letters and their negations). To express the plausibility degree of a piece of information, sequences of diamonds are used: $\Diamond\Diamond \ldots \Diamond A$ The longer the sequence, the less plausible the information. A sequence of i diamonds will be abbreviated by $\Diamond^i$ — $\Diamond^0$ denotes the empty string. Starting from a prioritized belief base $\Psi = \langle \Theta_0, \Theta_1, \Theta_2, \ldots \rangle$, we translate this into the premise set $\Psi^{\Diamond} = \bigcup_{i \in \mathbb{N}} \{\Diamond^i A \mid A \in \Theta_i\}$.

Where $A \in \mathcal{W}^l$, let $!^i A$ abbreviate $\Diamond^i A \wedge \neg A$. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. The prioritized logic $\mathsf{K}^{\mathsf{m}}_{\sqsubset}$ is characterized by the following triple:

1. The modal logic K^+, obtained by enriching K with the checked connectives
2. The sequence of sets of abnormalities: $\langle \Omega^{\mathsf{K}}_i \rangle_{i \in \mathbb{N}}$, where for every $i \in \mathbb{N}$, $\Omega^{\mathsf{K}}_i = \{!^i A \mid A \in \mathcal{W}^l\}$
3. The Strategy: $\sqsubset$-minimal abnormality

To compare the format for prioritized logics with flat adaptive logics, it will be convenient to refer to the logics K^{m} and K^{r}, defined by (i) K^+, (ii) $\Omega^{\mathsf{K}} = \bigcup_{i \in \mathbb{N}} \Omega^{\mathsf{K}}_i$ and (iii) Minimal Abnormality, respectively Reliability.

The logic $\mathsf{K}^{\mathsf{m}}_{\sqsubset}$ allows for the defeasible inference from $\Diamond^i A$ (where $i \in \mathbb{N}$) to A. This is done by defining "A is plausible (to degree i), but false" as

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
	$p, q,$ r, s	$p, q,$ $\neg r, s$	$\neg p, q,$ r, s	$\neg p, q,$ $r, \neg s$	$\neg p, q,$ $\neg r, s$	$\neg p, q,$ $\neg r, \neg s$	$\neg p, \neg q,$ r, s	$\neg p, \neg q,$ $\neg r, s$
$!^1 p$			+	+	+	+	+	+
$!^1 \neg q$	+	+	+	+	+	+		
$!^1 r$		+			+	+		+
$!^2 s$				+		+		
$!^2 \neg s$	+	+	+		+		+	+

Table 1. A representation of the K^+-models of Γ_2. The first row shows the non-modal propositions each model validates, the second row the abnormalities of rank 1 and the third row the abnormalities of rank 2.

an abnormality (of rank i).[11] Consider the prioritized belief base $\Psi_{\mathsf{ex}} = \langle\{p \supset q, q \vee s, p \supset s\}, \{p, \neg q \wedge r\}, \{s, \neg s\}\rangle$. The translation gives us $\Psi^{\Diamond}_{\mathsf{ex}} = \{p \supset q, q \vee s, p \supset s, \Diamond p, \Diamond(\neg q \wedge r), \Diamond\Diamond s, \Diamond\Diamond\neg s\}$. To facilitate the reading, let henceforth $\Gamma_2 = \Psi^{\Diamond}_{\mathsf{ex}}$. Let us take a look at the K^+-models of Γ_2. Note that every such model validates the modal formulas $\Diamond p, \Diamond\neg q, \Diamond r, \Diamond\Diamond s$ and $\Diamond\Diamond\neg s$. Table 1 represents these models in terms of (1) the non-modal literals they validate and (2) their abnormal part. For reasons of simplicity, we restrict the scope to those propositional letters that occur in Γ_2.[12]

Figure 1 shows the partial order imposed on the models from Table 1 by the two logics $\mathsf{K^m}$ and $\mathsf{K^m_\sqsubset}$. M_1, M_4, M_7 are $\subset$-minimally abnormal. From these, M_4 is not $\sqsubset$-minimally abnormal: $Ab(M_1) \cap \Omega^{\mathsf{K}}_1 \subset Ab(M_4) \cap \Omega^{\mathsf{K}}_1$, whence $Ab(M_1) \sqsubset Ab(M_4)$. M_1 and M_7 are incommensurable in view of Ω_1, whence $Ab(M_1) \not\sqsubset Ab(M_7)$ and $Ab(M_7) \not\sqsubset Ab(M_1)$. Recall that the set of $\mathsf{AL^m_\sqsubset}$-models is always a subset of the $\mathsf{AL^m}$-models, whence in this particular case, M_1 and M_7 are the only $\sqsubset$-minimally abnormal models. As a result, s and $p \vee \neg q$ are semantic $\mathsf{K^m_\sqsubset}$-consequences of Γ_2. Note that in view of M_4, these are not semantic $\mathsf{K^m}$-consequences of Γ_2.

We can explain this outcome as follows. In view of Γ_2, both p and $\neg q$ are plausible, but one of them has to be false (although we do not know which one). So if we want to privilege our most plausible beliefs, all we can do

[11] Note that for all $i, j \in \mathbb{N}$ such that $i \neq j$, $\Omega^{\mathsf{K}}_i \cap \Omega^{\mathsf{K}}_j = \emptyset$. This is not required for a logic to fit the format of $\mathsf{AL}_\sqsubset$; all that is required is that each Ω_i is characterized by a logical form.

[12] It is provable for that (1) for every $M \in \mathcal{M}_{\mathsf{K}^+}(\Gamma_2)$, $Ab(M) \supseteq Ab(M_i)$ for a "model" M_i in the table and (2), for every "model" M_i in the table, there is a $M \in \mathcal{M}_{\mathsf{K}^+}(\Gamma_2)$ such that $Ab(M) = Ab(M_i)$. Hence it suffices to look at these limited representations, to decide which abnormalities hold in the minimal abnormal models. This allows one to derive the claims about $Cn_{\mathsf{K^m_\sqsubset}}(\Gamma_2)$ that are made in this section.

(a) (b)

Figure 1. A graphic comparison of the partial orders $\subset$ (1a) and $\sqsubset$ (1b) on the abnormal parts of the models $M_1, \ldots, M_8$.

is assume that one of both holds: $p \vee \neg q$. So all the selected models either verify p or they verify $\neg q$. Since $\Gamma_2 \cup \{p \vee \neg q\} \vdash_{\mathsf{K}^+} s$, these models also verify s. The logic K^{m} cannot achieve this result, since it considers M_1 and M_4 as incommensurable.

3.4. *An Alternative Characterization of the* $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$*-Semantics*

In Section 2.1 we pointed out that the set of AL^{m}-models of Γ can be characterized alternatively, in view of the minimal Dab-consequences of Γ. A similar characterization can be given of $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. We define a choice set as in Section 2.1. We say that φ is a $\sqsubset$-minimal choice set of Σ iff there is no choice set ψ of Σ such that $\psi \sqsubset \varphi$. Let $\Sigma(\Gamma)$ be defined as in Section 2.

Definition 9: $\Phi^{\sqsubset}(\Gamma)$ *is the set of* $\sqsubset$*-minimal choice sets of* $\Sigma(\Gamma)$.

Note that the following theorem follows immediately from Lemma 2:

Theorem 8: $\Phi^{\sqsubset}(\Gamma) \subseteq \Phi(\Gamma)$.

In the appendix, it is proven that for every Γ, $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$ — see Theorem 25. We will now show that, just as the set $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}}(\Gamma)$ can be characterized in view of $\Phi(\Gamma)$, the set $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$ can be characterized in view of $\Phi^{\sqsubset}(\Gamma)$.

Lemma 3: *Where* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$, $Ab(M)$ *is a choice set of* $\Sigma(\Gamma)$.

Proof. Suppose $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$. Let $Dab(\Delta)$ be an arbitrary minimal Dab-consequence of Γ. By the soundness of LLL^+, $\Gamma \models_{\mathsf{LLL}^+} Dab(\Delta)$. Hence $M \Vdash^+ Dab(\Delta)$, which implies that $M \Vdash^+ A$ for an $A \in \Delta$. Hence $Ab(M) \cap \Delta \neq \emptyset$. □

Lemma 4: *If* Γ *has* LLL^+*-models, then for every choice set* φ *of* $\Sigma(\Gamma)$*, there is a* LLL^+*-model* M *of* Γ *such that* $Ab(M) \subseteq \varphi$.

Proof. Suppose (†) Γ has LLL^+-models. Let φ be a choice set of $\Sigma(\Gamma)$. Suppose there is no $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ such that $Ab(M) \subseteq \varphi$. Hence $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$ has no LLL^+-models. By the compactness of LLL^+, there is a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq \Omega - \varphi$ such that $\Gamma' \cup \Delta^{\check{\neg}}$ has no LLL^+-models. However, by (†) and the monotonicity of LLL^+, Γ' has LLL^+-models, whence $\Delta \neq \emptyset$. By CL-properties, $\Gamma' \vdash_{\mathsf{LLL}^+} Dab(\Delta)$, whence by the monotonicity of LLL^+, $\Gamma \vdash_{\mathsf{LLL}^+} Dab(\Delta)$. Note that there is a minimal non-empty $\Delta' \subseteq \Delta$ such that $\Gamma \vdash_{\mathsf{LLL}^+} Dab(\Delta')$, and also $\Delta' \cap \varphi = \emptyset$. Hence φ is not a choice set of $\Sigma(\Gamma)$ — a contradiction. □

Theorem 9: $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$ *iff* $(M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ *and* $Ab(M) \in \Phi^{\sqsubset}(\Gamma))$.

Proof. (⇒) Suppose $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Definition 8, $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$. Suppose (†) $Ab(M) \notin \Phi^{\sqsubset}(\Gamma)$, and let $Ab(M) = \varphi$. By Lemma 3, $Ab(M)$ is a choice set of $\Sigma(\Gamma)$, whence by (†), there is a choice set ψ of $\Sigma(\Gamma)$ such that $\psi \sqsubset \varphi$. By Lemma 4, there is a LLL^+-model M' of Γ such that $Ab(M') \subseteq \psi$.
Case 1: $Ab(M') = \psi$. Hence $Ab(M') \sqsubset \varphi$.
Case 2: $Ab(M') \subset \psi$. Hence $Ab(M') \sqsubset \psi$ in view of Lemma 2. By the transitivity of $\sqsubset$, $Ab(M') \sqsubset \varphi$.

Hence in either case, there is a LLL^+-model M' of Γ such that $Ab(M') \sqsubset Ab(M)$, which contradicts the fact that $M \in \mathcal{M}_{\mathsf{AL}_{\sqsubset}{}^{\mathsf{m}}}(\Gamma)$.

(⇐) Suppose $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$, but $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. Then there is a $M' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) : Ab(M') \sqsubset Ab(M)$. By Lemma 3, $Ab(M')$ is a choice set of $\Sigma(\Gamma)$, whence in view of Definition 9, $Ab(M) \notin \Phi^{\sqsubset}(\Gamma)$. □

Note that the above theorem nicely parallels Theorem 1. The theorem below states that whenever Γ has LLL^+-models, then we can also go in the opposite direction: the set $\Phi^{\sqsubset}(\Gamma)$ can be defined in view of $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$.

Theorem 10: *If* Γ *has* LLL^+*-models, then* $\Phi^{\sqsubset}(\Gamma) = \{Ab(M) \mid M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)\}$.

Proof. Suppose Γ has LLL^+-models. That $\{Ab(M) \mid M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)\} \subseteq \Phi^{\sqsubset}(\Gamma)$ is immediate in view of Theorem 9. Let $\varphi \in \Phi^{\sqsubset}(\Gamma)$. By Lemma 4, there is a $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ such that $Ab(M) = \varphi$. By Theorem 9, $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. □

Below we will see that $\Phi^{\sqsubset}(\Gamma)$ has a proof-theoretic counterpart, $\Phi^{\sqsubset}_s(\Gamma)$, that determines the marking of lines of a proof at stage s. Hence Theorems 9 and 10 function as a bridge between the proof theory and semantics of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$.

3.5. *The* $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$*-semantics*

Recall that $U(\Gamma) = \bigcup \Phi(\Gamma)$ — see page 607 —, where $U(\Gamma)$ is associated with AL^{r} and $\Phi(\Gamma)$ with AL^{m}. In view of Theorem 1, this implies that an abnormality is unreliable iff it is verified by a $\subset$-minimally abnormal model: $U(\Gamma) = \{A \in \Omega \mid M \Vdash^{+} A$ for a $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}}(\Gamma)\}$.

Let us now take a look at $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$. Just as $U(\Gamma)$, the set of $\sqsubset$-unreliable abnormalities can be characterized in two equivalent ways: (i) syntactically, as the union of all the members of $\Phi^{\sqsubset}(\Gamma)$ and (ii) semantically, as the set of those abnormalities that are verified by a $\sqsubset$-minimally abnormal model. To simplify the meta-theory and to stay as close as possible to the standard format, we will use (i) as the official definition of the set of $\sqsubset$-unreliable abnormalities:

Definition 10: $U^{\sqsubset}(\Gamma) = \bigcup \Phi^{\sqsubset}(\Gamma)$

By Theorem 9, $U^{\sqsubset}(\Gamma) = \{A \in \Omega \mid M \Vdash^{+} A$ for a $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)\}$. We can now define the set of $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-models of Γ as we did for $\mathcal{M}_{\mathsf{AL}^{\mathsf{r}}}(\Gamma)$:

Definition 11: $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$ *iff* $(M \in \mathcal{M}_{\mathsf{LLL}}(\Gamma)$ *and* $Ab(M) \subseteq U^{\sqsubset}(\Gamma))$

In view of Theorem 8, the fact that $U(\Gamma) = \bigcup \Phi(\Gamma)$ and Definition 10, we obtain:

Theorem 11: $U^{\sqsubset}(\Gamma) \subseteq U(\Gamma)$

Theorem 12: *Every* $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$*-model of* Γ *is a* AL^{r}*-model of* Γ.

Let us reconsider the example from Section 3.3 from the viewpoint of the $\mathsf{K}^{\mathsf{r}}_{\sqsubset}$-semantics. In view of the above definitions, it is required that we first look at the minimal Dab-consequences of a set Γ, to find the set of $\sqsubset$-unreliable formulas. The set $\Gamma_2 = \{p \supset q, q \vee s, p \supset s, \Diamond p, \Diamond(\neg q \wedge r), \Diamond\Diamond s, \Diamond\Diamond\neg s\}$ has four minimal Dab-consequences: $!^1 p \,\check{\vee}\, !^1\neg q$, $!^1 p \,\check{\vee}\, !^2\neg s$, $!^1\neg q \,\check{\vee}\, !^2\neg s$, and $!^2 s \,\check{\vee}\, !^2\neg s$.

Hence $\Sigma(\Gamma_2) = \{\{!^1 p, !^1\neg q\}, \{!^1 p, !^2\neg s\}, \{!^1\neg q, !^2\neg s\}, \{!^2 s, !^2\neg s\}\}$. The set of $\sqsubset$-minimal choice sets of $\Sigma(\Gamma_2)$ is $\Phi^{\sqsubset}(\Gamma_2) = \{\{!^1\neg q, !^2\neg s\}, \{!^1 p, !^2\neg s\}\}$. Remark that these sets correspond to the $\sqsubset$-minimal abnormal models M_1 and M_7 depicted in Table 1. As a result, $U^{\sqsubset}(\Gamma_2) = \{!^1 p, !^1\neg q, !^2\neg s\}$.

This means that all $\sqsubset$-reliable models falsify $!^2 s$, whence in view of Γ_2, they verify s. Hence s is also a semantic $\mathsf{K}^{\mathsf{r}}_{\sqsubset}$-consequence of Γ_2. Note however that $\mathcal{M}_{\mathsf{K}^{\mathsf{r}}_{\sqsubset}}(\Gamma_2) \neq \mathcal{M}_{\mathsf{K}^{\mathsf{m}}_{\sqsubset}}(\Gamma_2)$: for the model M_3 represented in Section 3.3, we have that $M_3 \in \mathcal{M}_{\mathsf{K}^{\mathsf{r}}_{\sqsubset}}(\Gamma_2) - \mathcal{M}_{\mathsf{K}^{\mathsf{m}}_{\sqsubset}}(\Gamma_2)$. This implies that $p \vee \neg q$ is not a semantic $\mathsf{K}^{\mathsf{r}}_{\sqsubset}$-consequence of Γ_2.

4. *A Proof Theory for* $\mathsf{AL}_{\sqsubset}$

4.1. *The Generic Proof Theory for* $\mathsf{AL}_{\sqsubset}$

One of the merits of the standard format is that it provides every logic in this format with a sound and complete proof theory. This proof theory explicates the defeasible reasoning methods the logics were developed for. In what follows, we will present a proof theory that does the same for logics in the $\mathsf{AL}_{\sqsubset}$-format.

The inference rules of a $\mathsf{AL}_{\sqsubset}$-proof are identical to those of a AL-proof — see page 607. The concept of a line, a stage, a proof and an extension of a proof in the $\mathsf{AL}_{\sqsubset}$-format are also inherited from the standard format. As a result, Lemma 1 holds also for $\mathsf{AL}_{\sqsubset}$-proofs. This implies that apart from the marks, every AL-proof is a $\mathsf{AL}_{\sqsubset}$-proof and vice versa.

The distinctive feature of an $\mathsf{AL}_{\sqsubset}$-proof lies in its marking definition. Let $\Sigma_s(\Gamma)$ be defined as in Section 2.

Definition 12: $\Phi^{\sqsubset}_s(\Gamma)$ *is the set of* $\sqsubset$*-minimal choice sets of* $\Sigma_s(\Gamma)$.

In the appendix, we prove that for every Γ and at every stage s of a $\mathsf{AL}_{\sqsubset}$-proof from Γ, $\Phi^{\sqsubset}_s(\Gamma) \neq \emptyset$. Of course, it may be the case that $\Phi^{\sqsubset}_s(\Gamma) = \{\emptyset\}$, i.e. whenever $\Sigma_s(\Gamma) = \emptyset$. Marking in view of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$ is now done in the same way as for AL^{m}, replacing $\Phi_s(\Gamma)$ by $\Phi^{\sqsubset}_s(\Gamma)$:

Definition 13: $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$*-Marking: a line* l *with formula* A *is marked at stage* s *iff, where its condition is* Δ*: (i) no* $\varphi \in \Phi^{\sqsubset}_s(\Gamma)$ *is such that* $\varphi \cap \Delta = \emptyset$*, or (ii) for a* $\varphi \in \Phi^{\sqsubset}_s(\Gamma)$*, there is no line on which* A *is derived on a condition* Θ *for which* $\Theta \cap \varphi = \emptyset$.

The set of $\sqsubset$-unreliable formulas at stage s is defined as the union of the members of $\Phi^{\sqsubset}_s(\Gamma)$:

Definition 14: $U^{\sqsubset}_s(\Gamma) = \bigcup \Phi^{\sqsubset}_s(\Gamma)$

Definition 15: $\mathsf{AL}^r_{\sqsubset}$*-Marking: a line* l *with formula* A *is marked at stage* s *iff, where its condition is* Δ, $\Delta \cap U^{\sqsubset}_s(\Gamma) \neq \emptyset$.

Derivability at a stage and final derivability are defined as for AL — see Definitions 5 and 6. This gives us the relation $\Gamma \vdash_{\mathsf{AL}_{\sqsubset}} A$.

The following is an immediate consequence of Lemma 2:

Fact 2: *At every stage* s *of a proof from* Γ, $\Phi^{\sqsubset}_s(\Gamma) \subseteq \Phi_s(\Gamma)$.

This fact implies that at every stage s of a proof from Γ, we can first check which choice sets of $\Sigma_s(\Gamma)$ are $\subset$-minimal, and only afterwards select the subset of $\sqsubset$-minimal choice sets from these. Also, from Fact 2, the fact that at every stage s, $U_s(\Gamma) = \bigcup \Phi_s(\Gamma)$ and Definition 14, we can derive:

Fact 3: *At every stage* s *of a proof from* Γ, $U^{\sqsubset}_s(\Gamma) \subseteq U_s(\Gamma)$.

Facts 2 and 3 imply that whenever a line is unmarked in an AL^{x}-proof (where $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$), it is unmarked in an $\mathsf{AL}^{\mathsf{x}}_{\sqsubset}$-proof as well — recall that apart from the marking, these proofs are interchangeable. Hence if something is (finally) derived in an AL^{x}-proof, then it is finally derived in an $\mathsf{AL}^{\mathsf{x}}_{\sqsubset}$-proof as well. This allows us to safely infer:

Theorem 13: *Where* $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$*:* $Cn_{\mathsf{AL}^{\mathsf{x}}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{x}}_{\sqsubset}}(\Gamma)$.

4.2. *Example of a* $\mathsf{K}_{\sqsubset}$*-proof*

$\sqsubset$*-Minimal Abnormality.* To illustrate the new marking definitions, let us take a look at a particular $\mathsf{K}^{\mathsf{m}}_{\sqsubset}$-proof from $\Gamma_2 = \{p \supset q, q \vee s, p \supset s, \Diamond p, \Diamond (\neg q \wedge r), \Diamond\Diamond s, \Diamond\Diamond\neg s\}$:

1	$q \vee s$	PREM	$\emptyset$
2	$\Diamond(\neg q \wedge r)$	PREM	$\emptyset$
3	$\Diamond\neg q$	2;RU	$\emptyset$
4	$\neg q$	3;RC	$\{!^1\neg q\}$
5	s	1,4;RU	$\{!^1\neg q\}$
6	$\Diamond\Diamond\neg s$	PREM	$\emptyset$
7	$!^1\neg q \,\check{\vee}\, !^2\neg s$	1,3,6;RU	$\emptyset$

Note that $\Sigma_7(\Gamma_2) = \{\{!^1\neg q, !^2\neg s\}\}$. This implies that the set of $\sqsubset$-minimal choice sets at stage 7, $\Phi^{\sqsubset}_7(\Gamma_2)$ only contains one member, i.e. $\{!^2\neg s\}$

— note that $\{!^2\neg s\} \sqsubset \{!^1\neg q\}$. Since the condition of line 5 has an empty intersection with this set, line 5 is unmarked.

Suppose we extend the proof as follows (we repeat from line 5 on):

5	s	1,4;RU	$\{!^1\neg q\}$ $\checkmark^{10}$
6	$\Diamond\Diamond\neg s$	PREM	$\emptyset$
7	$!^1\neg q \,\check{\vee}\, !^2\neg s$	1,3,6;RU	$\emptyset$
8	$p \supset q$	PREM	$\emptyset$
9	$\Diamond p$	PREM	$\emptyset$
10	$!^1 p \,\check{\vee}\, !^1\neg q$	3,8,9;RU	$\emptyset$

$\Sigma_{10}(\Gamma_2) = \{\{!^1\neg q, !^2\neg s\}, \{!^1 p, !^1\neg q\}\}$, whence there are two $\sqsubset$-minimal choice sets at this stage: $\Phi_{10}(\Gamma_2) = \{\{!^1\neg q\}, \{!^1 p, !^2\neg s\}\}$. In view of the first choice set, line 5 is marked. We can however further extend the proof such that line 5 is again unmarked:

5	s	1,4;RU	$\{!^1\neg q\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	$!^1 p \,\check{\vee}\, !^1\neg q$	3,8,9;RU	$\emptyset$
11	p	9;RC	$\{!^1 p\}$
12	$p \supset s$	PREM	$\emptyset$
13	s	11,12;RU	$\{!^1 p\}$

Note that since no new Dab-formula has been derived, $\Phi_{13}(\Gamma_2) = \Phi_{10}(\Gamma_2)$. However, s is now also derived on a condition that has an empty intersection with $\{!^1\neg q\}$. As a result, lines 5 and 13 are unmarked.

$\sqsubset$*-Reliability.* If the marking definition for $\sqsubset$-Reliability is applied, the above proof does not suffice to finally derive s. That is, $U^{\sqsubset}_{13}(\Gamma_2) = \bigcup \Phi^{\sqsubset}_{13}(\Gamma_2) = \{!^1 p, !^1\neg q, !^2\neg s\}$. As a result, both line 5 and line 13 are marked.

Nevertheless, s is finally derivable in a $\mathsf{K}^{\mathsf{r}}_{\sqsubset}$-proof from Γ. To show how, let us recapitulate lines 5–15 from the above proof, but now mark lines according to Definition 15:

5	s	1,4;RU	$\{!^1\neg q\}$ $\checkmark^{15}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
11	p	9;RC	$\{!^1 p\}$ $\checkmark^{15}$
12	$p \supset s$	PREM	$\emptyset$
13	s	11,12;RU	$\{!^1 p\}$ $\checkmark^{15}$

14	$\Diamond\Diamond s$	PREM	$\emptyset$
15	s	14;RC	$\{!^2 s\}$

Note that this time, lines 5 and 13 are marked. However, we have derived s on a condition that is not $\sqsubset$-unreliable at stage 15. As we explained in Section 3, $!^2 s$ is not contained in any $\sqsubset$-minimal choice set of $\Sigma(\Gamma_2)$. This warrants that s is finally derived in the proof. To explain why, let us look at an extension of the proof:

15	s	14;RC	$\{!^2 s\} \checkmark^{16}$
16	$!^2 s \check{\vee} !^2 \neg s$	6,14;RU	$\emptyset$

$\Sigma_{16}(\Gamma_2) = \{\{!^1\neg q, !^2\neg s\}, \{!^1 p, !^1\neg q\}, \{!^2 s, !^2\neg s\}\}$, whence $\Phi^{\sqsubset}_{16}(\Gamma_2) = \{\{!^1 p, !^2\neg s\}, \{!^1\neg q, !^2 s\}, \{!^1\neg q, !^2\neg s\}\}$. As a result, $U^{\sqsubset}_{16}(\Gamma_2) = \{!^1 p, !^1\neg q, !^2 s, !^2\neg s\}$. However, it suffices to derive the fourth minimal Dab-consequence of Γ_2 (see page 620) to undo the marking of line 15:

15	s	14;RC	$\{!^2 s\}$
16	$!^2 s \check{\vee} !^2 \neg s$	6,14;RU	$\emptyset$
17	$!^1 p \check{\vee} !^2 \neg s$	6,9,12;RU	$\emptyset$

At stage 17, all minimal Dab-consequences of Γ_2 have been derived, whence $U^{\sqsubset}_{17}(\Gamma_2) = U^{\sqsubset}(\Gamma_2) = \{!^1 p, !^1\neg q, !^2\neg s\}$ — see Section 3.5. As a result, line 15 is unmarked again and will remain unmarked in every further extension of this proof.

4.3. *The Standard Format as a Border Case*

In the introduction, we mentioned that the standard format of is a border case of the format for prioritized ALs we introduced above. Let us briefly spell out why this holds. Consider the sequence of sets of abnormalities: $S = \langle \Omega_i \rangle_{i \in I}$, where $\Omega_i = \Omega_j$ for every $i, j \in I$. Note that this is the case e.g. whenever $I = \{1\}$, i.e. whenever there is only one set in the sequence. As before, let $\Omega = \bigcup_{i \in I} \Omega_i$. We leave it to the reader to prove that in this case (†) $\Delta \sqsubset \Delta'$ iff $\Delta \subset \Delta'$.

For the sake of clarity, let us use the name $\mathsf{BAL}^{\mathsf{x}}_{\sqsubset}$ for the border case logic defined by (i) LLL, (ii) S and (iii) a strategy $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$. By (†) and Definitions 2 and 8, we immediately have that $\mathcal{M}_{\mathsf{BAL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}}(\Gamma)$. Also, since in this case $\Phi^{\sqsubset}(\Gamma) = \Phi(\Gamma)$, we have by Definition 10 that $U^{\sqsubset}(\Gamma) = U(\Gamma)$. This implies by Definitions 1 and 11 that $\mathcal{M}_{\mathsf{BAL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}}(\Gamma)$.

Similar results can be established for the proof theory. In view of Definition 9, it easy to see that by (†), for every stage s of a proof from Γ, $\Phi_s^{\sqsubset}(\Gamma) = \Phi_s(\Gamma)$. From this and Definition 14, it follows that $U_s^{\sqsubset}(\Gamma) = U_s(\Gamma)$. Hence, where $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$, a line is unmarked in a AL^{x}-proof, iff it is unmarked in a $\mathsf{BAL}_{\sqsubset}^{\mathsf{x}}$-proof. This implies that where $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$, $Cn_{\mathsf{BAL}_{\sqsubset}^{\mathsf{x}}}(\Gamma) = Cn_{\mathsf{AL}^{\mathsf{x}}}(\Gamma)$.

So every AL in standard format is equivalent to a logic in the new format. Remark that the equivalence is not restricted to the respective consequence sets, but to all the crucial concepts in the semantics and proof theory of both logics. This implies that all the meta-theoretic properties of $\mathsf{AL}_{\sqsubset}$ hold for AL as well.

5. *Meta-Theory of* $\mathsf{AL}_{\sqsubset}$

In this section, we show that all the meta-theoretic properties discussed in Section 2 hold for $\mathsf{AL}_{\sqsubset}$ as well. Since we already discussed the meaning and importance of these properties, we will simply state them here. We refer to the second appendix for their proofs — some of these are variations of proofs from the meta-theory of the standard format (see [7] for their most recent formulation).

Theorem 14: *If* $\Gamma \vdash_{\mathsf{AL}_{\sqsubset}} A$, *then* $\Gamma \models_{\mathsf{AL}_{\sqsubset}} A$. *(Soundness)*

Theorem 15: *Where* $\Gamma \subseteq \mathcal{W}$: *if* $\Gamma \models_{\mathsf{AL}_{\sqsubset}} A$, *then* $\Gamma \vdash_{\mathsf{AL}_{\sqsubset}} A$. *($\mathcal{L}$-Completeness)*

Theorem 16: *If* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}_{\sqsubset}}(\Gamma)$, *then there is an* $M' \in \mathcal{M}_{\mathsf{AL}_{\sqsubset}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$. *(Strong Reassurance)*

Theorem 17: $\Gamma \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$ *(Reflexivity)*

Theorem 18: *Where* $\Gamma \subseteq \mathcal{W}$: *if* $\Gamma' \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$, *then* $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma \cup \Gamma')$ *(Cautious Indifference)*

Theorem 19: *Where* $\Gamma \subseteq \mathcal{W}$: $Cn_{\mathsf{AL}_{\sqsubset}}(Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$. *(Fixed Point / Idempotence)*

Theorem 20: *Where* $\Gamma \subseteq \mathcal{W}$: *if* $\Gamma \cup \{A\} \vdash_{\mathsf{AL}_{\sqsubset}^{\mathsf{m}}} B$, *then* $\Gamma \vdash_{\mathsf{AL}_{\sqsubset}^{\mathsf{m}}} \check{\neg} A \check{\vee} B$. *(Deduction Theorem for* $\mathsf{AL}_{\sqsubset}^{\mathsf{m}}$*)*[13]

[13] The Deduction Theorem does not hold for $\mathsf{AL}_{\sqsubset}^{\mathsf{r}}$. This follows immediately in view of the fact that it does not hold for AL^{r} — see [4, Theorem 13.3] — and the fact that every logic AL^{r} is a logic in the extended format as well — see Section 4.3.

Theorem 21: *Each of the following holds:*

1. $Cn_{\mathsf{LLL}^+}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{ULL}}(\Gamma)$
2. $Cn_{\mathsf{LLL}^+}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{ULL}}(\Gamma)$
3. $Cn_{\mathsf{LLL}^+}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{ULL}}(\Gamma)$

Theorem 22: *If* Γ *is normal, then* $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{ULL}}(\Gamma)$.

Theorem 23: *Where* $\Gamma, \Gamma' \subseteq \mathcal{W}$, $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma')$ *if one of the following holds:*

(C1) $\Gamma' \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$ *and* $\Gamma \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma')$
(C2) Where L *is a Tarski-logic weaker than or identical to* $\mathsf{AL}_{\sqsubset}$*:* $Cn_{\mathsf{L}}(\Gamma) = Cn_{\mathsf{L}}(\Gamma')$
(C3) Where L *is a Tarski-logic and for every* $\Theta \subseteq \mathcal{W}$, $Cn_{\mathsf{AL}_{\sqsubset}}(\Theta) = Cn_{\mathsf{L}}(Cn_{\mathsf{AL}_{\sqsubset}}(\Theta))$*:* $Cn_{\mathsf{L}}(\Gamma) = Cn_{\mathsf{L}}(\Gamma')$

6. *Conclusion and Outlook*

Let us briefly summarize our main results. We have developed a new format for prioritized ALs, that includes the standard format as a border case. We proved that the most central properties of the standard format hold for the new format as well. Many of these, notably soundness and completeness, were proven independently of previous results. Apart from that, the new format offers all the advantages that make (flat) ALs so attractive, e.g. their straightforward semantics and a proof theory that mirrors the dynamic aspects of human reasoning.

In our paper, we only presented one particular logic $\mathsf{K}_{\sqsubset}$. This logic can be used to explicate reasoning with prioritized belief bases or background knowledge. However, the $\mathsf{AL}_{\sqsubset}$-format can be applied in a broad variety of other contexts: hierarchies of imperatives, prioritized combinations of deontic and doxastic logics, abduction and inductive generalization, etc. As a result, a huge range of defeasible reasoning forms can be studied from the viewpoint of this unifying framework.

Many issues still require our consideration, such as computational complexity [28], decision procedures for final derivability [2], proof heuristics [10], and so on. Although the proof of the pudding will be in the eating, it is likely that $\mathsf{AL}_{\sqsubset}$ will resemble AL in these respects, in view of their structural similarity.

APPENDIX

A. $\Phi^{\sqsubset}_s(\Gamma) \neq \emptyset$ *and* $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$

As promised, we prove here that $\Phi^{\sqsubset}_s(\Gamma) \neq \emptyset$ at every stage s of a proof from Γ. From this, it follows almost immediately that $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$ — see Theorem 25. The latter property is called upon in the proof of Lemma 9 below.

Lemma 5: *Where (1)* $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ *is a set of sets and (2)* φ *is a choice set of* Σ*: (3) for every* $A \in \varphi$*, there is a* $\Delta \in \Sigma$ *for which* $\Delta \cap \varphi = \{A\}$ *iff (4)* φ *is a minimal choice set of* Σ*.*

Proof. Suppose (1) and (2) hold. ($\Rightarrow$) Suppose (3) holds, and consider a $\varphi' \subset \varphi$ and a $B \in \varphi, B \notin \varphi'$. By (3), there is a $\Delta \in \Sigma$ for which $\Delta \cap \varphi = \{B\}$ and hence $\Delta \cap \varphi' = \emptyset$. This implies that φ' is not a choice set of Σ. As a result, φ is a minimal choice set of Σ. ($\Leftarrow$) Suppose (3) is false, whence there is a $B \in \varphi$ such that, for no $\Delta \in \Sigma$, $\varphi \cap \Delta = \{B\}$. In that case for every Δ for which $B \in \Delta$, there is a $C \in \varphi - \{B\}$ such that $C \in \Delta$. Hence $\varphi - \{B\}$ is a choice set of Σ, hence φ is not a minimal choice set of Σ. $\square$

Let $\overline{\Omega}_1 = \Omega_1$. For all $i \in I, i > 1$, let $\overline{\Omega}_i = \Omega_i - (\Omega_1 \cup \ldots \cup \Omega_{i-1})$. Where $i \in I$, a Dab_i-formula is the classical disjunction of the members of $\Delta \subset (\Omega_1 \cup \ldots \cup \Omega_i)$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab$_i$-formulas at stage s of a proof of from Γ, let $\Sigma^i_s(\Gamma) = \{\Delta_1, \Delta_2, \ldots\}$. Note that $\Sigma^1_s(\Gamma) \subseteq \Sigma^2_s(\Gamma) \subseteq \ldots$. Let $\Phi^i_s(\Gamma)$ be the set of minimal choice sets of $\Sigma^i_s(\Gamma)$.

Note that for each $\Delta \in \Sigma^{i+1}_s(\Gamma) - \Sigma^i_s(\Gamma)$, $\Delta \cap \overline{\Omega}_{i+1} \neq \emptyset$. Where $\varphi \in \Phi^i_s(\Gamma)$, let $\overline{\Phi}^{i+1}_{\varphi,s}(\Gamma)$ be the set of minimal choice sets of $\{\Delta \cap \overline{\Omega}_{i+1} \mid \Delta \in \Sigma^{i+1}_s(\Gamma), \varphi \cap \Delta = \emptyset\}$. As before, if the latter set is empty we have that $\overline{\Phi}^{i+1}_{\varphi,s}(\Gamma) = \{\emptyset\}$.

Lemma 6: *For all* $\varphi \in \Phi^i_s(\Gamma)$ *and all* $\varphi' \in \overline{\Phi}^{i+1}_{\varphi,s}(\Gamma)$, $\varphi \cup \varphi' \in \Phi^{i+1}_s(\Gamma)$.

Proof. Let $\varphi \in \Phi^i_s(\Gamma)$ and consider an arbitrary $\varphi' \in \overline{\Phi}^{i+1}_{\varphi,s}(\Gamma)$. Suppose $\Delta \cap \varphi = \emptyset$ for a $\Delta \in \Sigma^{i+1}_s(\Gamma)$. Then $\Delta \notin \Sigma^i_s(\Gamma)$ since $\varphi \in \Phi^i_s(\Gamma)$. In this case $\Delta \cap \overline{\Omega}^{i+1} \neq \emptyset$. Hence $\varphi' \cap \Delta \neq \emptyset$, since $\varphi' \in \overline{\Phi}^{i+1}_{\varphi,s}$. Hence $\varphi \cup \varphi'$ is a choice set of $\Sigma^{i+1}_s(\Gamma)$.

By the right-left direction of Lemma 5 and the fact that $\varphi \in \Phi^i_s(\Gamma)$, for every $A \in \varphi$ there is a $\Delta \in \Sigma^i_s(\Gamma)$ such that $\Delta \cap \varphi = \{A\}$. Moreover, for all these Δ, $\varphi' \cap \Delta = \emptyset$, since $\varphi' \subseteq \overline{\Omega}_{i+1}$. Finally, $\Sigma^i_s(\Gamma) \subseteq \Sigma^{i+1}_s(\Gamma)$, which gives us:

(1) for every $A \in \varphi$ there is a $\Delta \in \Sigma_s^{i+1}(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$.

From the right-left direction of Lemma 5: for every $A \in \varphi'$, there is a $\Theta \in \overline{\Phi}^{i+1}_{\varphi,s}$ such that $\Theta \cap \varphi' = \{A\}$, where $\Theta = \Delta \cap \overline{\Omega}_{i+1}$ for a $\Delta \in \Sigma_s^{i+1}(\Gamma)$. Since $\varphi' \subseteq \overline{\Omega}_{i+1}$, $\Delta \cap \varphi' = \{A\}$. Moreover, in view of the definition of $\overline{\Phi}^{i+1}_{\varphi,s}$, $\Delta \cap \varphi = \emptyset$. Hence we have:

(2) for every $A \in \varphi'$, there is a $\Delta \in \Sigma_s^{i+1}(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$.

By (1) and (2): for every $A \in \varphi \cup \varphi'$, there is a $\Delta \in \Sigma_s^{i+1}(\Gamma)$ such that $\Delta \cap (\varphi \cup \varphi') = \{A\}$. By the left-right direction of Lemma 5, $\varphi \cup \varphi'$ is a minimal choice set of $\Sigma_s^{i+1}(\Gamma)$, hence $\varphi \cup \varphi' \in \Phi_s^{i+1}(\Gamma)$. ☐

Theorem 24: *For every stage s of a proof from Γ, $\Phi_s^{\sqsubset}(\Gamma) \neq \emptyset$.*

Proof. Note that at every stage s of a proof, $\Sigma_s^1(\Gamma)$ is a set of finite sets. By Fact 1, $\Phi_s^1(\Gamma) \neq \emptyset$. Let $\varphi_1 \in \Phi_s^1(\Gamma)$, and for all $i > 1$, let φ_j be some arbitrary element in $\overline{\Phi}^{i}_{\varphi_{i-1},s}$. Consider $\varphi^{\oplus} = \varphi_1 \cup \varphi_2 \cup \ldots$. Note that for every $i \in I$, $\varphi_i \subseteq \overline{\Omega}_i$. As a result, for every $i \in I$, $\varphi^{\oplus} \cap (\Omega_1 \cup \ldots \cup \Omega_i) = \varphi_1 \cup \ldots \cup \varphi_i$, whence by Lemma 6, (†) $\varphi^{\oplus} \cap (\Omega_1 \cup \ldots \cup \Omega_i) \in \Phi_s^i(\Gamma)$.

Let $\Delta \in \Sigma_s(\Gamma)$. Then there is an $i \in I$ such that $\Delta \subseteq \Omega_i$. It follows immediately by (†) that $\varphi^{\oplus} \cap \Delta \neq \emptyset$. Hence $\varphi^{\oplus}$ is a choice set of $\Sigma_s(\Gamma)$.

Suppose $\varphi^{\oplus} \notin \Phi_s^{\sqsubset}(\Gamma)$. Hence there is a choice set of $\Sigma_s(\Gamma)$, say ψ, such that for an $i \in I$, $\psi \cap \Omega_j = \varphi^{\oplus} \cap \Omega_j$ for all $j < i$ and $\psi \cap \Omega_i \subset \varphi^{\oplus} \cap \Omega_i$. Note that since $\Sigma_s^i(\Gamma) \subseteq \Sigma_s(\Gamma)$, ψ is a choice set of $\Sigma_s^i(\Gamma)$, whence also $\psi \cap (\Omega_1 \cup \ldots \cup \Omega_i)$ is a choice set of $\Sigma_s^i(\Gamma)$. This however implies that $\varphi^{\oplus} \cap (\Omega_1 \cup \ldots \cup \Omega_i)$ is not a minimal choice set of $\Sigma_s^i(\Gamma)$, which contradicts (†). ☐

Theorem 25: *For every Γ, $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$.*

Proof. Consider a $\mathsf{AL}_{\sqsubset}$-proof from Γ in which every minimal Dab-consequence of Γ has been derived at stage s. Note that $\Sigma_s(\Gamma) = \Sigma(\Gamma)$. By Definitions 9 and 12, $\Phi_s^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma)$. By Theorem 24, $\Phi^{\sqsubset}(\Gamma) \neq \emptyset$. ☐

B. Meta-theory of $\mathsf{AL}_{\sqsubset}$

B.1. Soundness and Completeness

We first prove soundness and restricted completeness for $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$, next we prove these two properties for $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$.

B.1.1. Minimal Abnormality

Lemma 7: *For every $\Gamma \subseteq \mathcal{W}$: if $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{\sqsubset}(\Gamma)$, then there is a finite $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-proof from Γ in which A is derived on the condition Δ at an unmarked line.*

Proof. Suppose the antecedent holds. Due to the compactness of LLL^+, there is a $\Gamma' = \{A_1, \ldots, A_n\} \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. Let the adaptive proof P be constructed as follows. At line 1 we introduce the premise A_1 by PREM, …, and at line n we introduce the premise A_n by PREM. At line $n+1$ we derive A by RC on the condition Δ. Let s be the stage consisting of lines 1 up to $n+1$. Since $\Gamma' \subseteq \Gamma \subseteq \mathcal{W}$, all Dab-formulas $B_1, \ldots, B_m$ that have been derived at stage s (if any) are members of Ω. Hence $\Phi^{\sqsubset}_s(\Gamma') = \{\{B_1, \ldots, B_m\}\}$. Due to the monotonicity of LLL^+, also $\Gamma \vdash_{\mathsf{LLL}^+} B_i$ for all these abnormalities B_i. Then $\{B_1, \ldots, B_m\} \subseteq \psi$ for all $\psi \in \Phi^{\sqsubset}(\Gamma)$. Since $\varphi \cap \Delta = \emptyset$ and $\varphi \in \Phi^{\sqsubset}(\Gamma)$, also $\Delta \cap \{B_1, \ldots, B_m\} = \emptyset$. Thus, line $n+1$ is unmarked. □

Lemma 8: *If $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$, then each of the following holds:*

1. *A is derivable on a line l of a finite $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-proof from Γ, on a condition Δ such that $\Delta \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{\sqsubset}(\Gamma)$*
2. *For every $\varphi \in \Phi^{\sqsubset}(\Gamma)$, there is a finite $\Delta \subseteq \Omega - \varphi$ such that $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$.*

Proof. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. By Definition 5, there is a finite $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$- proof P from Γ, such that (i) A is derived in this proof on an unmarked line l with a condition Δ, (ii) every extension of the proof in which line l is marked can be further extended such that line l is unmarked again. We now extend P to a stage s such that all minimal Dab-consequences are derived on the empty condition. Note $\Phi^{\sqsubset}_s(\Gamma) = \Phi^{\sqsubset}(\Gamma)$ and that at every later stage s', $\Phi^{\sqsubset}_{s'}(\Gamma) = \Phi^{\sqsubset}_s(\Gamma)$.
Ad 1. Suppose there is no $\varphi \in \Phi^{\sqsubset}(\Gamma)$ such that $\Delta \cap \varphi = \emptyset$. By Definition 13, line l is marked at stage s and at every later stage s', which contradicts (ii).

Ad 2. Suppose there is a $\varphi \in \Phi^{\sqsubset}(\Gamma)$ for which there is no $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$. By Definition 13 line l is marked at stage s, and we cannot further extend the proof such that line l is unmarked — this again contradicts (ii). □

Lemma 9: *Where* $\Gamma \subseteq \mathcal{W}$*: if for every* $\varphi \in \Phi^{\sqsubset}(\Gamma)$*, there is a finite* $\Delta \subseteq \Omega - \varphi$ *such that* $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$*, then* $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$*.*

Proof. Suppose that for every $\varphi \in \Phi^{\sqsubset}(\Gamma)$ there is a finite $\Delta_\varphi \subseteq \Omega - \varphi$ for which $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta_\varphi)$. Due to Lemma 7, for every such Δ_φ there is a finite $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-proof from Γ in which A is derived on the condition Δ_φ at an unmarked line l. Let P be any such proof (since $\Phi^{\sqsubset}(\Gamma)$ is non-empty by Theorem 25, there is at least one). Suppose the proof is extended to a stage s in which line l is marked. We extend the proof further to a stage s' in which (i) all minimal Dab-formulas have been derived on the empty condition, and (ii) for all $\varphi \in \Phi(\Gamma)$, A has been derived on the condition Δ_φ. By Definition 13, line l is unmarked at stage s'. □

Theorem 26: *If* $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$*, then* $\Gamma \models_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$*. (Soundness)*

Proof. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. If $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) = \emptyset$, the theorem follows immediately. Suppose $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) \neq \emptyset$. Let $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, whence $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$. By Theorem 9, $Ab(M) \in \Phi^{\sqsubset}(\Gamma)$. By Lemma 8.2, there is a $\Delta \subseteq \Omega$ such that $Ab(M) \cap \Delta = \emptyset$ and $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. By the soundness of LLL^+, $\Gamma \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. Since $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ and $M \Vdash^+ \check{\neg} Dab(\Delta)$, $M \Vdash^+ A$. □

Definition 16: *Where* $\varphi \in \Phi(\Gamma)$*:* $\mathcal{M}^\varphi = \{M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) \mid Ab(M) = \varphi\}$

Lemma 10: *Where* $\varphi \in \Phi(\Gamma)$*: if* M *is a* LLL^+*-model of* $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$*, then* $M \in \mathcal{M}^\varphi$*.*

Proof. Suppose (†) $\varphi \in \Phi(\Gamma)$ and M is a LLL^+-model of $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$. Hence (1) $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$. Note that $Ab(M) \subseteq \varphi$. By Lemma 3, $Ab(M)$ is a choice set of $\Sigma(\Gamma)$, whence by (†), $Ab(M) \not\subset \varphi$. Hence (2) $Ab(M) = \varphi$. By (1) and (2), $M \in \mathcal{M}^\varphi$. □

Lemma 11: *Where* $\varphi \in \Phi(\Gamma)$*: if all members of* $\mathcal{M}^\varphi$ *verify* A*, then* $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ *for a* $\Delta \subseteq \Omega - \varphi$*.*

Proof. Suppose all members of $\mathcal{M}^\varphi$ verify A. By Lemma 10, all LLL^+-models of $\Gamma \cup (\Omega - \varphi)^{\check{\neg}}$ verify A. This implies by the completeness of LLL^+:

$\Gamma \cup (\Omega - \varphi)^{\check{\neg}} \vdash_{\mathsf{LLL}^+} A$. By the compactness of LLL^+, $\Gamma' \cup \Delta^{\check{\neg}} \vdash_{\mathsf{LLL}^+} A$, for a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq \Omega - \varphi$. By the Deduction Theorem, $\Gamma' \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$, and by the monotonicity of LLL^+, $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. □

Theorem 27: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma \models_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$*, then* $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$*. ($\mathcal{L}$-Completeness)*

Proof. Suppose (†) $\Gamma \models_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. Consider a $\varphi \in \Phi^{\sqsubset}(\Gamma)$. By Theorem 8, $\varphi \in \Phi(\Gamma)$. By Theorem 9, we have that for every $M \in \mathcal{M}^{\varphi}$, $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. In view of (†), it follows that for every $M \in \mathcal{M}^{\varphi}$, $M \Vdash^{+} A$. By Lemma 11, $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega - \varphi$. Since this holds for all $\varphi \in \Phi^{\sqsubset}(\Gamma)$, we obtain by Lemma 9 that $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. □

B.1.2. Reliability

Lemma 12: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$*, then each of the following holds:*

1. *There is a finite* $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$*-proof from* Γ *in which* A *is derived on the condition* Δ *at an unmarked line*
2. $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$

Proof. Ad 1. The proof proceeds analogous to the proof for Lemma 7. We again construct the proof P as above. Note that since $\Gamma \vdash_{\mathsf{LLL}^+} B_i$ for all the derived abnormalities B_i, $U^{\sqsubset}_{s}(\Gamma') = \{B_1, \ldots, B_m\} \subseteq U^{\sqsubset}(\Gamma)$. Since $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$, also $\Delta \cap U^{\sqsubset}_{s}(\Gamma') = \emptyset$. Thus, line $n+1$ is unmarked.
Ad 2. Suppose that there is a finite $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By item 1, there is a finite proof from Γ such that A is derived on the condition Δ, on an unmarked line l. Suppose the proof is extended such that line l becomes marked. In that case, we can further extend the proof, deriving every minimal Dab-consequence of Γ. Then where s' is the stage of the second extension, $U^{\sqsubset}_{s'}(\Gamma) = U^{\sqsubset}(\Gamma)$, whence line l is unmarked again. □

Lemma 13: *If* $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$*, then* A *is derivable in a* $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$*-proof* P *from* Γ *on line* l *with condition* Δ *such that* $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$*.*

Proof. Suppose that $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. So A is finally derived on line l of a $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-proof from Γ. Let Δ be the condition of line l. Suppose that $\Delta \cap U^{\sqsubset}(\Gamma) \neq \emptyset$. In that case, we can extend P to a stage s such that every minimal Dab-consequence of Γ is derived in it. We have that $U^{\sqsubset}_{s}(\Gamma) = U^{\sqsubset}(\Gamma)$ and for all later stages s', $U^{\sqsubset}_{s'}(\Gamma) = U^{\sqsubset}_{s}(\Gamma)$. As a result, line l is marked at stage s and remains marked in every further extension of the proof, which contradicts the antecedent in view of Definition 5. □

Theorem 28: $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$ *iff* $\Gamma \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ *for a finite* Δ *such that* $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$.

Proof. (⇒) Suppose that $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$, whence for every $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, $M \Vdash^+ A$. By Definition 11, for every $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ such that $Ab(M) \subseteq U^{\sqsubset}(\Gamma)$, $M \Vdash^+ A$. Then $\Gamma \cup (\Omega - U^{\sqsubset}(\Gamma))^{\check{\neg}} \models_{\mathsf{LLL}^+} A$. As LLL^+ is compact, $\Gamma' \cup (\Delta)^{\check{\neg}} \models_{\mathsf{LLL}^+} A$ for a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq (\Omega - U^{\sqsubset}(\Gamma))$. Hence $\Gamma' \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. So, as LLL^+ is monotonic, $\Gamma \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$.
(⇐) Suppose there is a finite $\Delta \subseteq \Omega$ such that $\Gamma \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Note that by Definition 11, for every $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, $M \Vdash^+ \check{\neg} Dab(\Delta)$. This implies that $M \Vdash^+ A$ and we are done. □

Theorem 29: *If* $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$*, then* $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$*. (Soundness)*

Proof. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. By Lemma 13, A is derivable in a $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-proof P from Γ on line l with condition Δ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By Lemma 1 $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. By the soundness of LLL^+, $\Gamma \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. By Theorem 28, $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. □

Theorem 30: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$*, then* $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$*. ($\mathcal{L}$-Completeness)*

Proof. Suppose $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. By Theorem 28, $\Gamma \models_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ for a Δ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. By the completeness of LLL^+, $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$. By Lemma 12.2, $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. □

B.2. Strong Reassurance

As in the previous secion, we first prove the property for $\sqsubset$-Minimal Abnormality, and next for $\sqsubset$-Reliability. Where $i \in I$, let the flat adaptive logic $\mathsf{AL}^{\mathsf{m}}_{\mathsf{i}}$ be defined by (i) LLL, (ii) Ω_i and (iii) Minimal Abnormality. The proof of Strong Reassurance for $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$ relies on the Strong Reassurance property of each of these flat adaptive logics.

Theorem 31: *If* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$*, then there is an* $M' \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$.

Proof. Suppose $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. Let $\mathcal{M}$ be the set of all $M' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ such that Ab$(M') \sqsubset$ Ab(M). $\mathcal{M} \neq \emptyset$ since $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Definition 7, there is an $i_{M'} \in I$ for each $M' \in \mathcal{M}$ such that for all $j < i_{M'}$, Ab$(M) \cap \Omega_j =$ Ab$(M') \cap \Omega_j$, and Ab$(M') \cap \Omega_{i_{M'}} \subset$ Ab$(M) \cap \Omega_{i_{M'}}$. Let $k = \min(\{i_{M'} \mid M' \in \mathcal{M}\})$ and $M'' \in \mathcal{M}$ be such that $i_{M''} = k$.

If $k = 1$ let $M_k \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_1}(\Gamma)$ such that Ab$(M_k) \cap \Omega_1 \subseteq$ Ab$(M'') \cap \Omega_1$.

If $k > 1$, let for every $i < k$, $\Delta_i = (\Omega_i - \text{Ab}(M_i))^{\check{\neg}}$ and $M_i = M$. Moreover, let $M_k \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_k}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k-1})$ be such that Ab$(M_k) \cap \Omega_k \subseteq$ Ab$(M'') \cap \Omega_k$.

For every $i \in I, i \geq k$ let $\Delta_i = (\Omega_i - \text{Ab}(M_i))^{\check{\neg}}$, where for all $j \in I, j > k$, M_j is an arbitrary model in $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_j}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{j-1})$.

We show now by induction that for each $i \in I$, M_i and hence also Δ_i are well-defined. If $k > 1$, this is trivially so for all $i < k$.

"$i = k$": Suppose first $k = 1$. M_k exists due to the strong reassurance property that holds for $\mathsf{AL}^{\mathsf{m}}_1$. Suppose now $k > 1$. By the construction, $M'' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{k-1})$. M_k exists due to the strong reassurance property that holds for $\mathsf{AL}^{\mathsf{m}}_k$.

"$i \Rightarrow i{+}1$": By the induction hypothesis there is an $M_i \in \mathcal{M}_{AL^m_i}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{i-1})$. Hence $M_i \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i)$. Thus, $\mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i) \neq \emptyset$. Hence, by the reassurance property of $\mathsf{AL}^{\mathsf{m}}_{i+1}$, $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{i+1}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i) \neq \emptyset$. Let $M_{i+1} \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{i+1}}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_i)$ and $\Delta_{i+1} = (\Omega_{i+1} - \text{Ab}(M_{i+1}))^{\check{\neg}}$.

For every finite subset Γ' of $\Gamma \cup \bigcup_{i \in I} \Delta_i$ there is a j for which $\Gamma' \subseteq \Gamma \cup \Delta_1 \cup \ldots \cup \Delta_j$. Since $M_{j+1} \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_j)$, $\mathcal{M}_{\mathsf{LLL}^+}(\Gamma') \neq \emptyset$. Then, by the compactness of LLL^+, $\mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \bigcup_{i \in I} \Delta_i) \neq \emptyset$. Let $M_\star \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \bigcup_{i \in I} \Delta_i)$. We will now show that (1) Ab$(M_\star) \sqsubset$ Ab(M) and that (2) $M_\star \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$.

(1) By the construction, for all $i \in I$, Ab$(M_\star) \cap \Omega_i \subseteq$ Ab$(M_i) \cap \Omega_i$. Suppose there is an $i \in I$ for which Ab$(M_\star) \cap \Omega_i \subset$ Ab$(M_i) \cap \Omega_i$. Suppose first that $i < k$. In view of the construction, for all $j < k$, $Ab(M_j) \cap \Omega_j = Ab(M) \cap \Omega_j$, whence Ab$(M_\star) \cap \Omega_j \subseteq$ Ab$(M) \cap \Omega_j$. But then $M_\star \in \mathcal{M}$ which is a contradiction to the minimality of k. Suppose hence that $i \geq k$. Suppose $i = 1$. Since $M_1 \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_1}(\Gamma)$ and $M_\star \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ this is a contradiction. Suppose now that $i > 1$. Note that $M_i \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_i}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{i-1})$ and $M_\star \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{i-1})$, — a contradiction. Hence (‡) for all $i \in I$, Ab$(M_\star) \cap \Omega_i =$ Ab$(M_i) \cap \Omega_i$.

Since for all $i < k$, Ab$(M_\star) \cap \Omega_i =$ Ab$(M) \cap \Omega_i$, and Ab$(M_\star) \cap \Omega_k =$ Ab$(M_k) \cap \Omega_k \subseteq$ Ab$(M'') \cap \Omega_k \subset$ Ab$(M) \cap \Omega_k$, we have Ab$(M_\star) \sqsubset$ Ab(M) by Definition 7.

(2) Suppose there is an $M''' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ such that $\text{Ab}(M''') \sqsubset \text{Ab}(M_\star)$. Then there is an $l \in I$ such that (i) for all $m < l$, $\text{Ab}(M''') \cap \Omega_m = \text{Ab}(M_\star) \cap \Omega_m$ and (ii) $\text{Ab}(M''') \cap \Omega_l \subset \text{Ab}(M_\star) \cap \Omega_l$. By the transitivity of $\sqsubset$, $\text{Ab}(M''') \sqsubset \text{Ab}(M)$ and hence $M''' \in \mathcal{M}$. Thus, due to the minimality of k, $l \geq k$. Suppose $l = 1 = k$. Due to (ii) and (‡), $\text{Ab}(M''') \cap \Omega_1 \subset \text{Ab}(M_\star) \cap \Omega_1 = \text{Ab}(M_1) \cap \Omega_1$. This is a contradiction, since $M_1 \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_1}(\Gamma)$. Suppose now $l > 1$. Note that due to (i) and (‡), for all $m < l$, $\text{Ab}(M''') \cap \Omega_m = \text{Ab}(M_\star) \cap \Omega_m = \text{Ab}(M_m) \cap \Omega_m$. Thus, $M''' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{l-1})$. Due to (ii) and (‡), $\text{Ab}(M''') \cap \Omega_l \subset \text{Ab}(M_\star) \cap \Omega_l = \text{Ab}(M_l) \cap \Omega_l$. This is a contradiction, since $M_l \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_l}(\Gamma \cup \Delta_1 \cup \ldots \cup \Delta_{l-1})$. Hence $M_\star \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. □

Lemma 14: $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) \subseteq \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$.

Proof. Suppose $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 9, $Ab(M) = \varphi$ for some $\varphi \in \Phi^{\sqsubset}(\Gamma)$. Hence $Ab(M) \subseteq \bigcup \Phi^{\sqsubset}(\Gamma)$, whence by Definitions 10 and 11, $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. □

Theorem 32: *If* $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, *then there is an* $M' \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$ *such that* $Ab(M') \sqsubset Ab(M)$.

Proof. Suppose $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. By Lemma 14, $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma) - \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 31, there is a $M' \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$. By Lemma 14, $M' \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. □

B.3. Reflexivity

Reflexivity follows immediately from the following property:

Theorem 33: $Cn_{\mathsf{LLL}^+}(\Gamma) \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$ *(*LLL^+ *is weaker than or identical to* $\mathsf{AL}_{\sqsubset}$*)*

Proof. Suppose $\Gamma \vdash_{\mathsf{LLL}^+} A$. By the compactness of LLL^+, there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathsf{LLL}^+} A$. We may introduce all the elements of Γ' as premises in a $\mathsf{AL}_{\sqsubset}$-proof from Γ, using the rule PREM. We apply the rule RU to these premises to derive A, say on a line l. Note that the condition of line l is $\emptyset$, whence in view of marking definitions 13 and 15, line l is unmarked and remains unmarked in every extension of the proof. □

B.4. Cautious Indifference

We will first prove that Cautious Indifference holds for $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$, and then that it also holds for $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$. In what follows, Γ' is an arbitrary subset of $\mathcal{W}^{+}$.

Theorem 34: *If* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, *then* $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$.

Proof. Suppose (†) $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. Consider a $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By Definition 8, $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Gamma')$ and hence $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$. Suppose that $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 31, there is a $M' \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$ such that $Ab(M') \sqsubset Ab(M)$. However, in view of (†), $M' \Vdash^{+} A$ for every $A \in \Gamma'$, whence also $M' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Gamma')$. By Definition 8, $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$, which contradicts the supposition.

Consider a $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By (†), $M \Vdash^{+} A$ for every $A \in \Gamma'$. By Definition 8, M is a LLL^{+}-model of Γ. We thus obtain that M is a LLL^{+}-model of $\Gamma \cup \Gamma'$. Suppose $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By Theorem 31, there is a $M' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Gamma')$: $Ab(M') \sqsubset Ab(M)$. Hence $M' \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$. By Definition 8, $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, whence we have obtained a contradiction. $\square$

Lemma 15: *If* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, *then* $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$.

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. If Γ has no LLL^{+}-models, then Γ and $\Gamma \cup \Gamma'$ are LLL^{+}-trivial, whence $\Phi^{\sqsubset}(\Gamma) = \{\Omega\} = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$.

If (1) Γ has LLL^{+}-models, then in view of the reassurance of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$, there is a $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Theorem 34, $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$, whence also (2) $\Gamma \cup \Gamma'$ has LLL^{+}-models. By Theorem 34, $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By (1), (2) and Theorem 10, this means that $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$. $\square$

Theorem 35: *If* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, *then* $Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. *(Cautious Monotonicity)*

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, whence by Lemma 15, (†) $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. By Lemma 8.2 and (†), we have that (‡) for every $\varphi \in \Phi^{\sqsubset}(\Gamma \cup \Gamma')$, $\Gamma \vdash_{\mathsf{LLL}^+} A \check{\vee} Dab(\Delta)$ for a Δ such that $\varphi \cap \Delta = \emptyset$. By Lemma 8.1 and (‡), there is a finite $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-proof P from Γ such that A is derived at an unmarked line l with condition Δ, and $\Delta \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{\sqsubset}(\Gamma \cup \Gamma')$. Note that P is also a proof from $\Gamma \cup \Gamma'$.

Suppose line l is marked in an extension of P. We may extend this extension further such that (a) all minimal Dab-consequences of $\Gamma \cup \Gamma'$ are derived on the empty condition and (b) for every $\varphi \in \Phi^{\sqsubset}(\Gamma \cup \Gamma')$, A is derived on a condition Δ such that $\Delta \cap \varphi = \emptyset$ — this is possible in view of (‡). Let s be the stage of this second extension of P.

Note that by (a), for every later stage s', $\Phi_{s'}(\Gamma \cup \Gamma') = \Phi(\Gamma \cup \Gamma')$. By (b), at every later stage s', for every $\varphi \in \Phi_{s'}(\Gamma \cup \Gamma')$, A is derived on a condition Δ such that $\Delta \cap \varphi = \emptyset$. By Definition 13, line l is unmarked at every such stage s', whence A is finally derived in the proof. Hence $\Gamma \cup \Gamma' \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. $\square$

Theorem 36: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, *then* $Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. *(Cumulative Transitivity)*

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, whence by Theorem 34, (†) $\mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma')$. Suppose $\Gamma \cup \Gamma' \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. By the soundness of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$, $\Gamma \cup \Gamma' \models_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. By (†), $\Gamma \models_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. By the $\mathcal{L}$-completeness of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$, $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. $\square$

Corollary 1: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$, *then* $Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma \cup \Gamma') = Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. *(Cautious Indifference)*

Theorem 37: $Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$.

Proof. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. By Lemma 13, there is a finite $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-proof P from Γ, in which A occurs on an unmarked line l with condition Δ, and $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Let s be the stage of this proof. Since line l is unmarked, we have that (†) $\Delta \cap U^{\sqsubset}_s(\Gamma) = \emptyset$. Since $U^{\sqsubset}(\Gamma) = \bigcup \Phi^{\sqsubset}(\Gamma)$, we can derive that (‡) $\Delta \cap \varphi = \emptyset$ for every $\varphi \in \Phi^{\sqsubset}(\Gamma)$.

Note that P is also an $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-proof from Γ. By (†) and the fact that $U^{\sqsubset}_s(\Gamma) = \bigcup \Phi^{\sqsubset}_s(\Gamma)$, we can derive that $\Delta \cap \varphi = \emptyset$ for every $\varphi \in \Phi^{\sqsubset}_s(\Gamma)$. Hence line l is also unmarked in P if the strategy is $\sqsubset$-Minimal Abnormality.

Suppose line l is $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-marked in a further extension of the proof. We then extend the proof further to a stage s', such that every minimal Dab-consequence of Γ is derived at stage s'. Note that $\Phi^{\sqsubset}_{s'}(\Gamma) = \Phi^{\sqsubset}(\Gamma)$. By (‡) and Definition 13, line l is unmarked at stage s'. $\square$

Lemma 16: *If* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, *then* $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$.

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. By Theorem 37, $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$. By Lemma 15, $\Phi^{\sqsubset}(\Gamma) = \Phi^{\sqsubset}(\Gamma \cup \Gamma')$, whence by Definition 10, $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$. $\square$

Theorem 38: *If* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, *then* $\mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$.

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. By Lemma 16, (†) $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$. Suppose $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. By the supposition and the soundness of $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$, $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Gamma')$. By (†) and Definition 11, $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$.

Suppose $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$. By Definition 11, $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma \cup \Gamma')$ and $Ab(M) \subseteq U^{\sqsubset}(\Gamma \cup \Gamma')$. Then by the monotonicity of LLL^+ and (†), $M \in \mathcal{M}_{\mathsf{LLL}^+}(\Gamma)$ and $Ab(M) \subseteq U^{\sqsubset}(\Gamma)$. By Definition 11, $M \in \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. □

Theorem 39: *If* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$*, then* $Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$. *(Cautious Monotonicity)*

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, whence by Lemma 16, (†) $U^{\sqsubset}(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma)$. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$, whence by Lemma 13, A is derivable in an $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-proof P from Γ on line l with condition Δ such that $\Delta \cap U^{\sqsubset}(\Gamma) = \emptyset$. Note that P is a $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$-proof from $\Gamma \cup \Gamma'$ as well.

Suppose that line l is marked in an extension of P. We may then further extend the extension, such every minimal Dab-consequence of $\Gamma \cup \Gamma'$ is derived in it on the empty condition. Where the stage of the second extension is s, we have that $U^{\sqsubset}_s(\Gamma \cup \Gamma') = U^{\sqsubset}(\Gamma \cup \Gamma')$. By (†), $\Delta \cap U^{\sqsubset}(\Gamma \cup \Gamma') = \emptyset$. As a result, line l is unmarked at stage s. □

Theorem 40: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$*, then* $Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. *(Cumulative Transitivity)*

Proof. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$, whence by Theorem 38, (†) $\mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma) = \mathcal{M}_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma')$. Now suppose $\Gamma \cup \Gamma' \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. By the soundness of $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$, $\Gamma \cup \Gamma' \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. By (†), $\Gamma \models_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. By the $\mathcal{L}$-completeness of $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$, $\Gamma \vdash_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}} A$. □

Corollary 2: *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma' \subseteq Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$*, then* $Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma \cup \Gamma') = Cn_{\mathsf{AL}^{\mathsf{r}}_{\sqsubset}}(\Gamma)$. *(Cautious Indifference)*

B.5. Idempotence

Let $\Gamma' = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$. By Cautious Indifference, $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma \cup \Gamma')$. Moreover, by the reflexivity of $\mathsf{AL}_{\sqsubset}$, $\Gamma \subseteq \Gamma'$, whence $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma \cup \Gamma') = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma')$. The rest follows immediately.

B.6. Deduction Theorem for $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$

Suppose $\Gamma \cup \{A\} \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} B$, whence by the soundness of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$: (†) every $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-model of $\Gamma \cup \{A\}$ verifies B. Suppose $\Gamma \nvdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} \check{\neg} A \check{\vee} B$ — we derive a contradiction. By the $\mathcal{L}$-completeness of $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$, there is a $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-model M of Γ such that $M \Vdash^{+} A \check{\wedge} \check{\neg} B$. Note that M is a LLL^{+}-model of $\Gamma \cup \{A\}$. In view of (†), M is not a $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$-model of $\Gamma \cup \{A\}$, whence there is a LLL^{+}-model M' of $\Gamma \cup \{A\}$ such that $Ab(M') \sqsubset Ab(M)$. However, by the monotonicity of LLL^{+}, M' is a LLL^{+}-model of Γ. By Definition 8, $M \notin \mathcal{M}_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma)$.

B.7. Hierarchies within the New Format

Theorem 41: $Cn_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}}(\Gamma) \subseteq Cn_{\mathsf{ULL}}(\Gamma)$.

Proof. Suppose $\Gamma \vdash_{\mathsf{AL}^{\mathsf{m}}_{\sqsubset}} A$. By Lemma 8.2, $\Gamma \vdash_{\mathsf{LLL}^{+}} A \check{\vee} Dab(\Delta)$ for a $\Delta \subseteq \Omega$. By CL-properties, $\Gamma \cup \Omega^{\check{\neg}} \vdash_{\mathsf{LLL}^{+}} A$, whence in view of the definition of ULL, $\Gamma \vdash_{\mathsf{ULL}} A$. □

In view of Theorems 33, 37, 41 and 13, we can immediately derive Theorem 21 as a corollary. By Theorem 5, we can also derive Theorem 22.

B.8. Criteria for $\mathsf{AL}_{\sqsubset}$*-Equivalence*

Theorem 42: *Where* $\Gamma \subseteq \mathcal{W}$, $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(Cn_{\mathsf{LLL}^{+}}(\Gamma))$. *(*$\mathsf{LLL}^{+}$ *is conservative with respect to* $\mathsf{AL}_{\sqsubset}$*)*

Proof. By Theorem 33, (‡) $Cn_{\mathsf{LLL}^{+}}(\Gamma) \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$. By (‡) and Cautious Indifference, $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma \cup Cn_{\mathsf{LLL}^{+}}(\Gamma)) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$. Since LLL^{+} is reflexive, $\Gamma \subseteq Cn_{\mathsf{LLL}^{+}}(\Gamma)$, whence $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma \cup Cn_{\mathsf{LLL}^{+}}(\Gamma)) = Cn_{\mathsf{AL}_{\sqsubset}}(Cn_{\mathsf{LLL}^{+}}(\Gamma))$ and we are done. □

Theorem 43: *Where* $\Gamma, \Gamma' \subseteq \mathcal{W}$*: if* Γ *and* Γ' *are* LLL^{+}*-equivalent, then they are* $\mathsf{AL}_{\sqsubset}$*-equivalent.*

Proof. Suppose Γ and Γ' are LLL^{+}-equivalent, whence $Cn_{\mathsf{LLL}^{+}}(\Gamma) = Cn_{\mathsf{LLL}^{+}}(\Gamma')$. By Theorem 42, $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(Cn_{\mathsf{LLL}^{+}}(\Gamma))$ and $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma') = Cn_{\mathsf{AL}_{\sqsubset}}(Cn_{\mathsf{LLL}^{+}}(\Gamma'))$. Hence $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma')$. □

Theorem 44: *Every monotonic logic that is weaker than or identical to* $\mathsf{AL}_{\sqsubset}$ *is weaker than or identical to* LLL^{+}*. (Maximality of* LLL^{+}*)*

Proof. ($\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$*)* This follows immediately in view of (i) Lemma 8.2 from the current paper, (ii) the proof of Theorem 10 in [11] — replace $\Phi(\Gamma \cup \Gamma')$ in that proof by $\Phi^{\sqsubset}(\Gamma \cup \Gamma')$ and Theorem 4 in that proof by Lemma 8.2 from the current paper.
($\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$*)* This follows immediately in view of the fact that $\mathsf{AL}^{\mathsf{m}}_{\sqsubset}$ is stronger than $\mathsf{AL}^{\mathsf{r}}_{\sqsubset}$ — see Theorem 37 — and item 1. □

Fact 4: *Where* L *is a Tarski-logic weaker than or identical to* LLL^+*: if* Γ *and* Γ' *are* L*-equivalent, then they are* LLL^+*-equivalent.*

Proof of Theorem 23. In view of the preceding lemmas and theorems, the proof of Theorem 23 is fairly straightforward:

Proof. Ad 1. Suppose $\Gamma' \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma)$ and $\Gamma \subseteq Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma')$. By Cautious Indifference, $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma \cup \Gamma')$ and $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma') = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma' \cup \Gamma)$, hence $Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma) = Cn_{\mathsf{AL}_{\sqsubset}}(\Gamma')$.
Ad 2. and 3. It was proven in [11] that (C2) and (C3) are coextensive whenever (i) $\mathsf{AL}_{\sqsubset}$ is reflexive and has the fixed point property, and (ii) L is monotonic. Hence in view of the reflexivity and idempotence of $\mathsf{AL}_{\sqsubset}$, it suffices to prove item 2.

Suppose L is a Tarski-logic weaker than or identical to $\mathsf{AL}_{\sqsubset}$. By Theorem 44, L is weaker than or identical to LLL^+. Now suppose Γ and Γ' are L-equivalent. By Fact 4, Γ and Γ' are LLL^+-equivalent. By Theorem 43, Γ and Γ' are $\mathsf{AL}_{\sqsubset}$-equivalent. □

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