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HOLISM AND INDISPENSABILITY

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Abstract

One questioned premiss in the indispensability argument of Quine and Putnam is confirmational holism. In this paper I argue for a weakened form of holism, and thus a strengthened version of the indispensability argument. The argument is based on an idea of concept formation in mathematics. Mathematical concepts are arrived at via a sequence of explications, in Carnap’s sense, of non-clear, originally empirical, concepts. I identify a deductive and an empirical component in mathematical concepts. In a test situation the use of the empirical component, but not of the deductive one, is corroborated or falsified together with the scientific theory.

1. *Introduction*

The indispensability argument of Quine and Putnam is widely recognized as the only really good argument for mathematical realism. It is founded on the theses of *indispensability*, *confirmational holism*, *naturalism*, and the principle of *ontological commitment*. According to Putnam the argument is as follows:

> ... quantification over mathematical entities is indispensible for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question ([Putnam 1971], Sec. VIII).

He also points to

> the intellectual dishonesty of denying the existence of what one daily presupposes (*ibid.*).

The four theses on which the argument is founded have each and all been subject to criticism ever since the indispensability argument first appeared.[1] The weakest part of the argument is, in my opinion, the thesis of confirmational holism. It seems that there are lots of cases where scientific theories are corroborated or falsified in test situations, and the involved mathematics is not affected at all. It also seems that there are a multitude of examples of mathematical theories, or single results, that are impossible to even regard as falsifiable. Indeed, several philosophers question the thesis. To name but a few, Penelope Maddy claims that the theses of naturalism and holism are inconsistently used by Putnam, and Elliott Sober means that contrastive, scientific hypotheses often use the same mathematics, and consequently a test cannot rule out the mathematical theories (*op. cit.*). Others are of the opinion that holism is not necessary for the indispensability argument at all ([Resnik 1995], [Dieveney 2007]). I shall argue in this paper that mathematical theories are not tested in the way proponents of confirmational holism claim. I will, furthermore, try to make clear exactly in which way mathematical theories really are tested, and in which way they are unaffected by empirical tests.

It is obvious that there are several ways in which mathematics is used in scientific applications. One can distinguish at least three levels of uses; purely instrumental, explanatory, and representative (descriptive) ones.[2] Purely instrumental uses, as e.g. numerical methods or perturbation calculus, can be of no relevance when questions of realism are at stake, and neither can explanatory uses, as when e.g. complex analysis unifies seemingly different results in real analysis. In some cases, however, mathematical and empirical theories are strongly connected to each other. Anthony Peressini (*op. cit.*) mentions the use of Hilbert spaces in quantum mechanics, and Michael Resnik states that "Fields and particles are functions from spacetime points to probabilities" [Resnik 1991]. The representative use of mathematics indicates that there are examples where mathematics is so involved in physical theories that in a test situation both will be affected. I will argue below that in some cases these mathematical theories will receive empirical support if corroborated together with scientific ones, and I will point to just what this support amounts to.

Inspired by Aristotle and Carnap, I propose an idea of concept formation in mathematics. According to Aristotle, mathematical objects are inherent in substances (individual objects), and via a thought process traits of these objects can be isolated in thought. This process of making abstractions can be

[1] See, e.g., [Field 1980], [Maddy 1992], [Sober 1993], and [Resnik 1995] for critical aspects of the argument, and [Colyvan 2001] for a defense against these attacks.

[2] See [Peressini 1997], and [Colyvan 2001].

described as a sequence of explications, in Carnap's sense, in an ontological neutral way. In this way several mathematical concepts are explications of vague, or otherwise non-clear, concepts, and these concepts are intimately connected with scientific ones. When, for example, a physical theory, together with the mathematics used, is corroborated in a test situation, this justification carries over to a justification of how the involved mathematical concepts are related to reality via, what Peressini (*op. cit.*) calls, bridge principles, and also to the mathematical sentences involving the concepts. I will use this to argue for a weaker version of the thesis of holism, thus giving a stronger indispensability argument.

The structure of the paper is as follows. In section two I briefly review this idea of concept formation in mathematics. Section three discusses how mathematical theories may contain empirical elements, and in section four I put forward a modified indispensability argument using a weakened thesis of holism. In the presentation below, I follow Carnap and describe how *concepts* are explicated ([Carnap 1950], p. 3). I use the expression "mathematical entity" without any ontological commitments, and "mathematical object" when the issues may have ontological implications.

2. *Concept Formation in Mathematics*

Two main sources for my view on concept formation put forward below are Aristotle's philosophy of mathematics, and Carnap's use of explications as a means to develop exact and fruitful concepts. Since these ideas have been presented in other contexts, I will be rather brief here.[3] According to Aristotle mathematical entities are inherent in substances (individual objects). By abstraction these entities, or traits, can be isolated in thought, but they do not have any separate existence like the Platonic forms. One difference between physics and mathematics is that the former treats accidental properties (like snub nosed) while the latter treats essential ones (like curved)(concerning the nose of Socrates).[4]

I suggest that we regard this process of making abstractions, by isolating essential traits, as a process of making explications in Carnap's sense ([Carnap 1950], ch. 1). This can be understood as a totally non-metaphysical way of formulating more precise concepts for a given purpose. In this way it is possible to replace vague, imprecise, or otherwise non-clear concepts by more exact ones. To be able to mathematize a part of 'reality', an abstract

[3] See [Sjögren 2008], [Sjögren 2010], and above all [Sjögren 2011].

[4] See [Lear 1982], and Aristotle, e.g. Phys., Book II, 193^b–194^a.

mathematical one or an empirical, sufficiently exact mathematical concepts are needed. This can be achieved as above. Not only concepts of empirical science, but also mathematical ones, have an origin in reality as suggested by Aristotle. Mathematics has in this way been able to generate concepts via explications; concepts that are fruitful in mathematizing reality. The vague and ambiguous concept *mathematical proof* has exact counterparts in proof theory. The concept *natural number* (*finite cardinal number*) can be explicated as certain sets of sets in different ways, all preserving the idea of a certain type of sequence or progression. The concept *effectively computable function* is identified with *Turing computable function*, or any of its equivalents. The concepts of *function* and *continuity* have received explications as a certain kind of pairing of elements of two sets, and within topology as preservation of open sets, respectively. The concept *infinitesimal (fluxion)* was used to explicate *instantaneous velocity* via *time derivative*. This list of explications could be made longer *ad libitum*.[5]

Some of the concepts mentioned above can be seen as far removed from empirical reality, but tracing their origins, in a kind of concept archaeology, will lead us back to a more or less non-clear empirical counterpart. The concept of continuity, for example, has to do with the idea that natural processes, as we perceive them, do not take place in jumps. This was treated geometrically in the seventeenth century, and at that time functions were regarded as (continuous) curves. In the nineteenth century the concept received a precise enough explication via the $\varepsilon - \delta$ strategy, and still later as preservation of open sets in inverse mappings. The concept of functionality has its origin in one process determining another. Perhaps it can even be seen as a mathematical counterpart to causality. With the modern set-theoretical explication this origin is lost. Of course, mathematical concepts are also defined of purely mathematical concerns, like e.g. *uniform continuity* as distinct from (pointwise) continuity. The point is that several, fundamental, mathematical concepts originally are closely related to physical reality.

Concerning at least some mathematical concepts it seems that just one explication forces itself upon us as in the paradigmatic case of *effectively computable function* with explicatum *Turing computable function*. In taking this example as paradigmatic, I disregard from abstruse mentalistic versions, and cling to the original theses of Turing and Church. One possible way to describe the difference between mathematics and physics at the conceptual level is to emphasize the idea that mathematics deals with uniquely explicable objects. Compare this with the situation in physics, where e.g. *gravitation* is one thing in Newton's mechanics, and something different in Einstein's general theory of relativity. It will perhaps be something totally

[5] These concepts and others are discussed in the references in footnote 3.

different in a future, quantized theory of gravitation. Physics has to turn to new explications as physical theories evolve.[6] This analysis of difference between concepts is also in line with the Aristotelian conception of the relation between mathematics and physics, although it is of course described in other technical terms here.

What is of importance, however, is that mathematical concepts evolve via sequences of explications. Regard once again *continuity*; this concept gets more precise via the $\varepsilon - \delta$ strategy, and gets a more fruitful formulation in topology, but there is no radical break in meaning. Mathematical concepts are reinterpretable with new explications, while concepts in physics may change meaning in a more radical sense.[7]

It is possible to identify a deductive and an empirical component in several mathematical concepts. To illustrate the idea, consider the concept *derivative*. This was introduced by Newton, and independently by Leibniz, to solve problems of movement and geometry in a general way.[8] Both Newton's method of fluxions and Leibniz's of infinitesimals were inconsistent since increments simultaneously were regarded as both zero and non-zero depending on the calculations to be made. But the developed techniques did solve the problems they were designed to solve, so it was rational for the mathematical community to accept them. Note, however, that the concept had an empirical origin as an analysis of movement. Concerning the relation between the a priori and the empirical, Carnap tried to separate them using Ramsey sentences. He divided the vocabulary into theoretical and observational terms, and managed to separate observation sentences from theoretical ones [Carnap 1952]. From our perspective it is, however, not possible to divide the vocabulary into two disjunct sets in this way.

As is well known, the infinitesimal calculus was rapidly developing in the eighteenth century, and was extremely fruitful in solving problems in, above all, physics. As is also well known, a need to find a solid foundation for the calculus grew in the early nineteenth century. This was accomplished with the *limit* concept by Cauchy, made precise by Weierstraß, and finally with the arithmetization of analysis, as Kitcher (*op. cit.*) names the developed calculus of Newton and Leibniz. In this later context the concept *derivative* has a purely mathematical status in a well connected body of mathematical concepts.

[6] See especially paper 5, jointly written with Christian Bennet, in [Sjögren 2011].

[7] See [Kitcher 1984], ch. 6:II on reinterpretation of mathematical concepts.

[8] See [Kitcher 1984], ch. 10 for a case study of the development of analysis from its origin in the calculus of Newton and Leibniz, and [Kline 1972] for a more complete exposition.

A similar history of evolution can be described for the concept *set* as an explication of *collection of objects*, with Cantor's explication as the first one. Though fruitful, it led to inconsistencies that were removed (?) in the development of axiomatic set theory, a part of mathematics that is still in progress. *Objects* are here regarded in a wide sense, noting that Cantor originally treated sets of reals, and that it was only later on that other types of objects were treated.

Here it is more fruitful to contrast mature and new mathematical theories, and not to pay so much attention to the distinction between pure and applied mathematics.[9] When a mathematical theory has grown mature, it has (often) lost the original connection with the empirical problems that inspired it. The theory thus becomes insensitive to falsification.[10] If we e.g. regard the empirical facts corroborating the general theory of relativity as falsifying Newtonian mechanics, these had no influence at all on mature analysis. But, in the time when analysis was just formulated, its success together with mechanics was a strong argument in its favour. Also, several concepts were at the same time used as both mathematical and physical.

Since several mathematical concepts have an empirical origin, and thus contain an empirical component, these ideas are relevant in understanding the applicability of mathematics too. Mathematics deals with reality in a more or less abstract way, and mathematicians construct abstract structures out of these concepts. I prefer to say that mathematical theories and concepts are *fruitful*, if they, together with some empirical theory, are applicable. Actually, there are two aspects of truth involved. One is that mathematical sentences are true, if they correctly describe adequate mathematical structures. The other is that these mathematical structures correspond to an inherent mathematical structure in Aristotle's sense. Mathematical sentences would then be true if they correctly describe *this* structure. This can be compared with the relation between sentences of physics, physical models and physical reality.[11]

[9] See [Peressini 2008] on the complicated relation between pure and applied mathematics.

[10] This is also pointed out in [Hellman 1999].

[11] Fruitfulness does not in itself imply truth. It demands further supporting principles as e.g. 'inference to the best explanation'. See [Peressini 1999] on this issue.

3. *Some Empirical Aspects of Mathematics*

In this section I discuss how mathematical concepts are related to reality, and more specifically how theories involving them can be empirically tested. This will in turn make it possible to weaken the thesis of holism, and thus to support realism via the indispensability argument.

Standard versions of the indispensability argument depend on the thesis of confirmational holism, which means that mathematical and scientific theories are tested together. In a corroboration the mathematical and empirical theories both get empirical support, and in a falsification there is something wrong with at least one of the theories. Note that the sentences of these two theories cannot be syntactically separated into two disjoint sets, since they may be extremely interconnected. Hartry Field had the ambition to nominalize away the mathematical parts of a fragment of physics, but the project has not been especially successful [Field 1980]. Quine held that changes in the two theories, when falsified, should destroy as little as possible, and this means that the mathematical parts are (almost) sacred ([Quine 1990], ch. 1, §6). In what follows we will see a way to understand the difference between testable and 'sacred' parts of mathematics.

In recent literature questions of empiricism in mathematics now and then emerge. Philip Kitcher argues for an empirically founded epistemology of mathematics [Kitcher 1984]. Imre Lakatos introduced quasi-empiricism which was discussed from the 1970's onwards.[12] There is, however, nothing empirical, in the sense of observation reports, etc., in quasi-empirical processes according to Lakatos. Thomas Tymoczko takes over Lakatos's term "quasi-empiricism", but he is more inclined to empiricism than Lakatos is.[13]

Another type of situation where questions of empiricism recently have entered into mathematics is computer-assisted proofs [Tymoczko 1979]. Tymoczko has suggested that the reliability of a computer in the running of a program is an engineering problem, thus empirical, and consequently the Four-Colour Theorem, e.g., is empirically founded and known only a posteriori.

This is, however, not a viable argument for the empirical character of results in mathematics. To see this one can point to an analogy between mathematicians doing mathematics and computers making computations according to advanced programs. Mathematicians sometimes doubt whether received results are correct or not, and then the arguments are checked again until the mathematical community is sufficiently sure that they are correct.

[12] See [Lakatos 1978], ch. 2, and [Putnam 1975], ch. 4.

[13] On "quasi-empiricism" see [Tymoczko 1998], p. *xvi*, and on empiricism [Tymoczko 1979].

Likewise, if one is in doubt whether a computer or a computer program gives the correct result, one may run the computer again, or try the program on another one, until one is pleased with the result. The adequate working of a computer may be an engineering problem, just as the sanity of a mathematician may be a psychological one, but this does not make the *result* of the mathematician's work, nor that of the running of the computer, an empirical one.

Turning to experimental mathematics, one view is that propositions that are well corroborated by numerical experiments, like Goldbach's conjecture or Riemann's hypothesis, could be incorporated as axioms into relevant theories.[14] Chaitin argues for letting in Riemann's hypothesis as an axiom on the ground that it is well corroborated. Chaitin seems to say that this corroboration is empirical [Chaitin 1974]. Especially Riemann's hypothesis (RH) has many important consequences, but instead of adding the hypothesis as a new axiom on 'empirical' grounds, one may just as well formulate theorems depending on the hypothesis as "If RH, then ...". Compare this with axiomatic set theory, where one strategy is to highlight which axioms a result depends on by writing "If the Axiom of Choice is true, then ...", "If there exists a measurable cardinal, then ...", etc. The same strategy is used in constructive mathematics in writing, e.g., "If the Law of Excluded Middle is valid, then ...".

However, there *are* cases where empirical considerations affect our trust in mathematics, and I will consider some examples in order to illustrate what testing a mathematical theory can amount to.

Euclidean geometry was thought to describe physical space, and thus being representational. With the development of non-Euclidean geometries in the nineteenth century, the geometry of physical space became a problem. Letting primitive terms of geometry be connected to physical entities, empirical tests may reveal this geometry. But note that a falsification of the idea that physical space is Euclidean will not falsify theorems in the *Elements*, since they are consequences of (the incomplete) axioms.

The introduction of fluxions by Newton (infinitesimals by Leibniz) was extremely successful in solving problems of physics as mentioned in section two. Fluxions are intimately connected to the physical concept *instantaneous velocity*; it is difficult to accept one of the concepts while rejecting the other. It is an example of a representational use.

Examples like the two mentioned above where mathematics is used representationally could be produced at will, and as mentioned in the introduction Peressini further exemplifies with the use of Hilbert spaces in quantum mechanics, and Resnik regards fields and particles as functions. To name

[14] See [Baker 2008] on experimental mathematics.

a more problematic example, consider renormalization, a technique used in quantum field theories to cancel out infinities in some cases. At this point we do not know whether this is just an instrumental use, or if it has a representational role, since we simply do not understand the mathematics involved [Jaffe 2004].

The use of mathematics in these examples, with the possible exception of renormalization, is representational. The mathematical concepts are intimately connected with physical ones, and the testing of the theories involving them affects both the mathematics and the physics. The empirical support for the theories give confidence to both the physical and the mathematical theories and the way these concepts are used in them, and also to how the concepts are related to 'reality'. The influence on mathematics is at the concept formation level. What may happen is that a search for more precise concepts as well as further developments of theories involving these concepts are motivated. Furthermore, mathematics may together with physics reveal insights into the way nature is constituted, as illustrated by e.g. the discovery of the positron.

If exact mathematical counterparts are extracted from non-clear concepts via explications as described above, there emerges a picture of mathematical concepts having an origin in empirical reality. One consequence of this idea is that the distinction between mathematical concepts and the concepts of the more theoretical parts of physics is not a distinction between kinds of concepts. The representational use of mathematics is also an indication of how tight concepts of mathematics and physics are tied together, as illustrated by the examples. The difference between mathematical and physical theories lies rather in how the concepts are used, and in our attitudes towards how they are to be used. Concepts in physics are treated with deductive methods, but axioms and theorems involving the concepts are possible to test. In mathematics we are normally content with deducing results.

There are at least two ways in which we become confident of the correctness (or fruitfulness) of explications. The first is internal to mathematics, and has nothing to do with physics at all, as when the concept *effectively computable function* is explicated as *Turing computable function*. In this case there are several ways to bring about explications, but they all pick out the same set of functions; the explication is unique. In this way we have isolated a distinct set of functions, precisely those that can be computed with algorithmic means. Carnap was of the opinion that there are no questions of right or wrong in the process of formulating explications. This position has been questioned by e.g. Kreisel and Schoenfield, and recently a proof of the correctness of the explication of *effectively computable function* as *Turing computable function* has appeared.[15]

The second way is external to mathematics, at least in the first phase of development. The infinitesimal calculus, and its use in physics, was extremely successful. This gave confidence in the mathematical concepts used, notwithstanding their shaky mathematical ground. If we accept that there are velocities, which as it seems *must* be explicated as time derivatives, we ought to accept the existence of infinitesimals. With the work of Cauchy, Weierstraß, Riemann, *et al*, the infinitesimal calculus received a secure foundation, and there was no mathematical need any longer for discussions about infinitesimals, since they 'disappeared' in the arithmetization of analysis. Non-standard analysis brought about a new understanding of infinitesimals, and made them mathematically legitimate. There has been a development from a concept with an apparent empirical origin; a concept on the border between mathematics and physics to a concept with an internal mathematical explanation. It may be regarded as a problem for realism that there are several different ways of describing, or formalising, the real numbers. But this is not necessarily the case. Different formalisations can have different models, and one formalisation can have several non-isomorphic models. All these models may, however, correspond in relevant aspects to an Aristotelian, inherent, real structure. Furthermore, one main reason to believe in continuous space and time, or space-time, is the success of the mathematical description using differentiable, or at least continuous, functions representing these structures.

Thus, mathematics sometimes develops in intimate connections with science in a wide sense. In this development mathematics and science use the same concepts. Mathematics examines how the concepts are related deductively, and the empirical testing of theories gives, in positive instances, credibility to the explications used. In negative instances new explications may have to be found.

4. *Holism and Indispensability*

As mentioned in the introduction, the use of confirmational holism in the indispensability argument has been questioned by several philosophers. The view of concept formation put forward in section two, and the impact of

[15] See [Carnap 1950], p. 4, [Kreisel 1967a], [Kreisel 1967b], [Shoenfield 1993], p. 26, and [Dershowitz & Gurevich 2008].

the empirical on mathematics in the testing of mathematical theories as discussed in section three, indicate that a weaker version of the traditional thesis of confirmational holism is possible.

The variant of the indispensability argument I prefer thus rests on a weaker version of holism. As we have seen, there is often an empirical component in a mathematical concept, and this component may relate the concept to reality via chains of explications. Mathematical concepts also have a deductive component that relates the concept to other components in a more or less formal, deductive structure. Since a concept has both these relations, sentences involving them cannot be separated into two disjoint sets. Confirmational holism, in standard versions, states that in a test situation both mathematics and science is put on trial, while I maintain that the deductive part is not tested; only the empirical part is. Mathematical theories may be rejected, but not on the ground that an empirical theory and the mathematics used in it are falsified. The mathematical concepts can come to be related to reality in new ways, but the deductive structure is almost never affected in this way. What is of relevance in the indispensability argument is, furthermore, that it is this relation to reality that may be affected since mathematical entities are abstracted or, as Aristotle puts it, separated in thought from individual objects.

Geoffrey Hellman proposes a "moderate holism", noting that what is really needed in physics are very weak fragments of mathematics.[16] From my perspective, Hellman makes a doubtful use of the indispensability argument, since the principles used in these fragments are often reformulations of mature, well established mathematics that are insensitive to empirical tests. Furthermore, these weak fragments may very well be inapplicable in scientific contexts. To be attractive, or even usable, mathematical concepts and theories must fit naturally into the purported applications.

Finally, the origin of a concept is not automatically a justification of its use. I have emphasized the empirical origin of concepts, referring to Aristotle and Carnap. This is part of an explanation of the applicability of mathematics. These concepts are then used in mathematical and empirical theories, and with bridge principles they are related to observables. These theories may be corroborated in test situations, thus giving an empirical justification for the involved sentences, and this grounding carries over to mathematics.

The stronger version of the indispensability argument is thus as follows. Mathematics is necessary for science. When scientific theories are tested, the empirical side of mathematics is also put on trial, and there is no essential difference at the concept formation level between science and mathematics. It is science that decides questions of existence, and since, as Putnam stated,

[16] See [Hellman 1999], and [Peressini 2008] for some comments on Hellman's position.

it is dishonest to deny the existence of what one presupposes in daily work, we ought to accept that mathematical objects as well as the objects of physics have an objective existence.

Summarizing, mathematical concepts evolve via sequences of explications. The explicated concepts often have an empirical origin, even though this origin may be distant, at least in mature theories. When a scientific theory is tested, the empirical component is also put on trial, but the deductive structure of the mathematics in question is not tested. We thus get a weaker version of holism and a stronger version of the indispensability argument.

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