

COMBINING INTUITIONISTIC LOGIC WITH PARACONSISTENT OPERATORS

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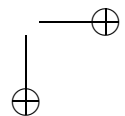
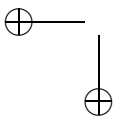
Abstract

A new propositional intuitionistic paraconsistent logic, IL_ω , is introduced as a sequent calculus combining Gentzen’s LJ with paraconsistent negation-like and involution-like operators. Completeness theorem with respect to Kripke semantics, embedding theorem into LJ, cut-elimination theorem and decidability theorem are shown for IL_ω .

1. Introduction

In this paper, a new propositional intuitionistic paraconsistent logic, IL_ω , is introduced as a cut-free and Kripke-complete Gentzen-type sequent calculus combining Gentzen’s LJ with paraconsistent negation-like and involution-like operators. The proposed paraconsistent negation-like operators are regarded as a variant of the paraconsistent negation operators of the well-known “useful” many-valued paraconsistent logics: Belnap’s and Dunn’s 4-valued logic B4 [4, 5], first-degree entailment FDE [2], Nelson’s paraconsistent logic N4 [1], Arieli-Avron’s bilattice logics [3] and Shramko-Wansing’s trilattice logics [9].

Gentzen-type sequent calculi for these many-valued paraconsistent logics have been studied by many researchers. For example, cut-free sequent calculi for some bilattice-based paraconsistent logics, which are natural extensions of N4, were studied by Gargov [6] and by Arieli and Avron [3], and a cut-free sequent calculus L16 that includes Shramko-Wansing’s logic $\text{FDE}^{t+\sim_f}$ was introduced by Kamide [7]. Since $\text{FDE}^{t+\sim_f}$ has both the negation and involution operators, L16 needed a bit complicated formalization to obtain a cut-free system. In order to simplify and refine L16, two sequent calculi L_ω and FL_ω have recently been introduced by Kamide [8] presenting a new negation operator that can simultaneously represent both paraconsistent negation-like and involution-like operators. In these logics,



the uncertainty level of the truth (or falsehood) of a proposition can be represented by a given number of nested occurrences of the new negation operator.

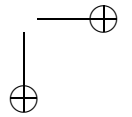
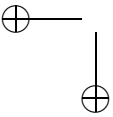
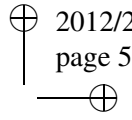
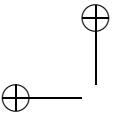
However, L_ω and FL_ω do not support intuitionistic or constructive characters such as the property of “constructible falsity” [1], since L_ω and FL_ω are based on (propositional and first-order, respectively) classical logic. The systems L_ω and FL_ω are also not appropriate for representing “partial (or incomplete) information”, i.e., the situation when $\alpha \vee \neg\alpha$ is not always true for any information α . It is known that Nelson’s N4 is useful for representing “constructible falsity” and that intuitionistic logic and N4 are suitable as a base logic for representing “partial information”. The aim of introducing IL_ω is thus to obtain an intuitionistic version of L_ω by extending LJ and modifying N4, in order to represent “constructible falsity” and “partial information”.

The contents of this paper are then summarized as follows. In Section 2, IL_ω is introduced as a Gentzen-type sequent calculus by extending LJ and modifying N4. A theorem for embedding IL_ω into LJ is shown, and by using this theorem, the cut-elimination and decidability theorems are shown for IL_ω . The properties of paraconsistency and constructible falsity for IL_ω are also derived from the cut-elimination theorem. In Section 3, a Kripke semantics for IL_ω is introduced, and the completeness theorem w.r.t. this semantics is proved. This theorem is the main result of this paper. In Section 4, some versions of IL_ω , which can include N4, are presented, and a modal version LM_ω of L_ω , which can be associated with IL_ω by the Gödel-McKinsey-Tarski translation, is presented.

2. Sequent calculus and cut-elimination

The following list of symbols is adopted for the language of the underlying logic: (countable) propositional variables p_0, p_1, \dots , constant \perp (falsity constant), logical connectives \rightarrow (implication), \wedge (conjunction), \vee (disjunction) and \sim (paraconsistent negation). The intuitionistic negation \neg can be defined by $\neg\alpha := \alpha \rightarrow \perp$. Greek lower-case letters α, β, \dots are used to denote formulas, and Greek capital letters Γ, Δ, \dots are used to represent finite (possibly empty) sets of formulas. We write $A \equiv B$ to indicate the syntactical identity between A and B . The symbol ω is used to represent the set of natural numbers. The symbols ω_e and ω_o are used to represent $\{i \in \omega \mid i \text{ is even}\}$ and $\{i \in \omega \mid i \text{ is odd}\}$, respectively. An expression $\sim^i\alpha$ for any $i \in \omega$

is used to denote $\overbrace{\sim\sim\dots\sim}^i\alpha$, which is defined inductively by $(\sim^0\alpha := \alpha)$ and $(\sim^{n+1}\alpha := \sim\sim^n\alpha)$. Lower-case letters i, j and k are used to denote any natural numbers. An expression of the form $\Gamma \Rightarrow \Delta$ where Δ is empty or



singleton is called a *sequent*. An expression $L \vdash S$ is used to denote the fact that a sequent S is provable in a sequent calculus L . A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

of R , if $L \vdash S_i$ for all i , then $L \vdash S$.

Definition 2.1: ($\mathbb{I}\mathbb{L}_\omega$) Let Δ be empty or singleton.

The initial sequents of $\mathbb{I}\mathbb{L}_\omega$ are of the form: for any propositional variable p and any $i \in \omega$,

$$\sim^i p \Rightarrow \sim^i p \quad \sim^i \perp \Rightarrow.$$

The structural inference rules of $\mathbb{I}\mathbb{L}_\omega$ are of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta} \text{ (cut)} \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (w-left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \alpha} \text{ (w-right)}.$$

The even logical inference rules of $\mathbb{I}\mathbb{L}_\omega$ are of the form: for any $i \in \omega_e$,

$$\frac{\Gamma \Rightarrow \sim^i \alpha \quad \sim^i \beta, \Sigma \Rightarrow \Delta}{\sim^i(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta} \text{ (}\rightarrow\text{left}^e) \quad \frac{\sim^i \alpha, \Gamma \Rightarrow \sim^i \beta}{\Gamma \Rightarrow \sim^i(\alpha \rightarrow \beta)} \text{ (}\rightarrow\text{right}^e)$$

$$\frac{\sim^i \alpha, \sim^i \beta, \Gamma \Rightarrow \Delta}{\sim^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left}^e) \quad \frac{\Gamma \Rightarrow \sim^i \alpha \quad \Gamma \Rightarrow \sim^i \beta}{\Gamma \Rightarrow \sim^i(\alpha \wedge \beta)} \text{ (}\wedge\text{right}^e)$$

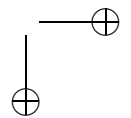
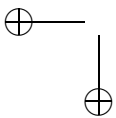
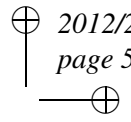
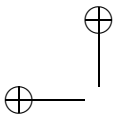
$$\frac{\sim^i \alpha, \Gamma \Rightarrow \Delta \quad \sim^i \beta, \Gamma \Rightarrow \Delta}{\sim^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \text{ (}\vee\text{left}^e)$$

$$\frac{\Gamma \Rightarrow \sim^i \alpha}{\Gamma \Rightarrow \sim^i(\alpha \vee \beta)} \text{ (}\vee\text{right1}^e) \quad \frac{\Gamma \Rightarrow \sim^i \beta}{\Gamma \Rightarrow \sim^i(\alpha \vee \beta)} \text{ (}\vee\text{right2}^e).$$

The odd logical inference rules of $\mathbb{I}\mathbb{L}_\omega$ are of the form: for any $j \in \omega_o$,

$$\frac{\sim^{j-1} \alpha, \sim^j \beta, \Gamma \Rightarrow \Delta}{\sim^j(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta} \text{ (}\rightarrow\text{left}^o) \quad \frac{\Gamma \Rightarrow \sim^{j-1} \alpha \quad \Gamma \Rightarrow \sim^j \beta}{\Gamma \Rightarrow \sim^j(\alpha \rightarrow \beta)} \text{ (}\rightarrow\text{right}^o)$$

$$\frac{\sim^j \alpha, \Gamma \Rightarrow \Delta \quad \sim^j \beta, \Gamma \Rightarrow \Delta}{\sim^j(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left}^o)$$



$$\frac{\Gamma \Rightarrow \sim^j \alpha}{\Gamma \Rightarrow \sim^j (\alpha \wedge \beta)} (\wedge\text{right}1^o) \quad \frac{\Gamma \Rightarrow \sim^j \beta}{\Gamma \Rightarrow \sim^j (\alpha \wedge \beta)} (\wedge\text{right}2^o)$$

$$\frac{\sim^j \alpha, \sim^j \beta, \Gamma \Rightarrow \Delta}{\sim^j (\alpha \vee \beta), \Gamma \Rightarrow \Delta} (\vee\text{left}^o) \quad \frac{\Gamma \Rightarrow \sim^j \alpha \quad \Gamma \Rightarrow \sim^j \beta}{\Gamma \Rightarrow \sim^j (\alpha \vee \beta)} (\vee\text{right}^o).$$

The sequents of the form $\sim^i \alpha \Rightarrow \sim^i \alpha$ for any formula α and any $i \in \omega$ are provable in cut-free \mathbb{IL}_ω . This fact can be proved by induction on the complexity of α . Hence, these sequents can also be regarded as the initial sequents of \mathbb{IL}_ω . The \perp -less fragment of \mathbb{IL}_ω with both $i = 0$ and $j = 1$ is just a sequent system for Nelson's 4-valued logic N4 [1] without the double-negation-elimination axiom: $\sim\sim\alpha \leftrightarrow \alpha$. Also, the $\{\rightarrow, \perp\}$ -less fragment of \mathbb{IL}_ω with both $i = 0$ and $j = 1$ is a sequent system for Belnap's and Dunn's 4-valued logic B4 [4, 5] without the double-negation-elimination axiom for \sim . For a detailed explanation for sequent calculi for N4 and B4, see e.g., [10].

The following proposition shows that the expressions \sim^i (i : even) and \sim^j (j : odd) are regarded as an involution-like operator and a negation-like operator, respectively.

An expression $\alpha \Leftrightarrow \beta$ is an abbreviation for the pair of sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.2: The following sequents are provable in \mathbb{IL}_ω : for any formulas α, β , any $i \in \omega_e$ and any $j \in \omega_o$,

1. $\sim^i (\alpha \circ \beta) \Leftrightarrow \sim^i \alpha \circ \sim^i \beta$ where $\circ \in \{\rightarrow, \wedge, \vee\}$,
2. $\sim^j (\alpha \rightarrow \beta) \Leftrightarrow \sim^{j-1} \alpha \wedge \sim^j \beta$ (esp., $\sim (\alpha \rightarrow \beta) \Leftrightarrow \alpha \wedge \sim \beta$),
3. $\sim^j (\alpha \wedge \beta) \Leftrightarrow \sim^j \alpha \vee \sim^j \beta$,
4. $\sim^j (\alpha \vee \beta) \Leftrightarrow \sim^j \alpha \wedge \sim^j \beta$.

Proof. Similar to the proofs of \mathbb{L}_ω in [8]. □

Note that \mathbb{IL}_ω is also an extension of the sequent calculus LJ for intuitionistic logic.

Observation 2.3: (LJ) LJ is obtained from \mathbb{IL}_ω by deleting the odd logical inference rules and replacing i in the initial sequents and the even logical inference rules by 0 (i.e., deleting every occurrence of \sim). The modified inference rules for LJ by replacing i by 0 are denoted by deleting the superscript "e".

As well-known, LJ enjoys cut-elimination.

Definition 2.4: Let $\Phi := \{p, q, r, \dots\}$ be a fixed countable non-empty set of propositional variables. Then, we define the sets $\Phi_i := \{p_i \mid p \in \Phi\}$ ($i \in \omega$) of propositional variables where $p_0 := p$, i.e., $\Phi_0 = \Phi$. The language $\mathcal{L}_{\mathbb{I}\mathbb{L}_\omega}$ of $\mathbb{I}\mathbb{L}_\omega$ is defined using Φ , \perp , \rightarrow , \wedge , \vee and \sim . The language \mathcal{L}_{LJ} of LJ is defined using $\bigcup_{i \in \omega} \Phi_i$, \perp , \rightarrow , \wedge and \vee .

A mapping f from $\mathcal{L}_{\mathbb{I}\mathbb{L}_\omega}$ to \mathcal{L}_{LJ} is defined as follows.

1. $f(\sim^i p) := p_i \in \Phi_i$ for each $p \in \Phi$ and each $i \in \omega$ (especially, $f(p) := p \in \Phi$),
2. $f(\sim^i \perp) := \perp$ for each $i \in \omega$,
3. $f(\sim^i (\alpha \circ \beta)) := f(\sim^i \alpha) \circ f(\sim^i \beta)$ ($\circ \in \{\rightarrow, \wedge, \vee\}$) for each $i \in \omega$,
4. $f(\sim^j (\alpha \rightarrow \beta)) := f(\sim^{j-1} \alpha) \wedge f(\sim^j \beta)$ for each $j \in \omega$,
5. $f(\sim^j (\alpha \wedge \beta)) := f(\sim^j \alpha) \vee f(\sim^j \beta)$ for each $j \in \omega$,
6. $f(\sim^j (\alpha \vee \beta)) := f(\sim^j \alpha) \wedge f(\sim^j \beta)$ for each $j \in \omega$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

Theorem 2.5: Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathbb{I}\mathbb{L}_\omega}$ and f be the mapping defined in Definition 2.4. Then:

1. $\mathbb{I}\mathbb{L}_\omega \vdash \Gamma \Rightarrow \Delta$ iff $\text{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. $\mathbb{I}\mathbb{L}_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$ iff $\text{LJ} - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

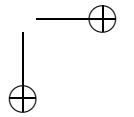
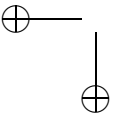
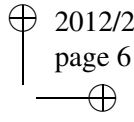
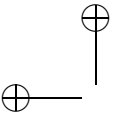
Proof. (2) immediately follows from (1). Thus, we only examine (1).

(Left-to-right): By induction on the length of the proof P of $\Gamma \Rightarrow \Delta$ in $\mathbb{I}\mathbb{L}_\omega$. We distinguish the cases according to the last inference of P . We only show the following cases.

Case $(\sim^i p \Rightarrow \sim^i p)$: The last inference of P is of the form: $\sim^i p \Rightarrow \sim^i p$. In this case, we obtain $f(\sim^i p) \Rightarrow f(\sim^i p)$, i.e., $p_i \Rightarrow p_i$ ($p_i \in \Phi_i$), which is an initial sequent of LJ.

Case $(\rightarrow\text{left}^e)$: The last inference of P is of the form:

$$\frac{\Gamma_1 \Rightarrow \sim^i \alpha \quad \sim^i \beta, \Gamma_2 \Rightarrow \Delta}{\sim^i (\alpha \rightarrow \beta), \Gamma_1, \Gamma_2 \Rightarrow \Delta} (\rightarrow\text{left}^e).$$



By induction hypothesis, we have $\text{LJ} \vdash f(\Gamma_1) \Rightarrow f(\sim^i \alpha)$ and $\text{LJ} \vdash f(\sim^i \beta), f(\Gamma_2) \Rightarrow f(\Delta)$. Then, we obtain

$$\frac{\begin{array}{c} \vdots \\ f(\Gamma_1) \Rightarrow f(\sim^i \alpha) \end{array} \quad \begin{array}{c} \vdots \\ f(\sim^i \beta), f(\Gamma_2) \Rightarrow f(\Delta) \end{array}}{f(\sim^i \alpha) \rightarrow f(\sim^i \beta), f(\Gamma_1), f(\Gamma_2) \Rightarrow f(\Delta)} \quad (\rightarrow\text{left})$$

where $f(\sim^i \alpha) \rightarrow f(\sim^i \beta)$ coincides with $f(\sim^i(\alpha \rightarrow \beta))$ by the definition of f .

Case ($\rightarrow\text{right}^o$): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \sim^{j-1} \alpha \quad \Gamma \Rightarrow \sim^j \beta}{\Gamma \Rightarrow \sim^j(\alpha \rightarrow \beta)} \quad (\rightarrow\text{right}^o).$$

By induction hypothesis, we have $\text{LJ} \vdash f(\Gamma) \Rightarrow f(\sim^{j-1} \alpha)$ and $\text{LJ} \vdash f(\Gamma) \Rightarrow f(\sim^j \beta)$. Then, we obtain

$$\frac{\begin{array}{c} \vdots \\ f(\Gamma) \Rightarrow f(\sim^{j-1} \alpha) \end{array} \quad \begin{array}{c} \vdots \\ f(\Gamma) \Rightarrow f(\sim^j \beta) \end{array}}{f(\Gamma) \Rightarrow f(\sim^{j-1} \alpha) \wedge f(\sim^j \beta)} \quad (\wedge\text{right})$$

where $f(\sim^{j-1} \alpha) \wedge f(\sim^j \beta)$ coincides with $f(\sim^j(\alpha \rightarrow \beta))$ by the definition of f .

(Right-to-left): By induction on the length of the proof Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LJ. We distinguish the cases according to the last inference of Q , and show only the case ($\wedge\text{left}$).

Subcase (1): The last inference of Q is of the form:

$$\frac{f(\sim^{j-1} \alpha), f(\sim^j \beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim^{j-1} \alpha) \wedge f(\sim^j \beta), f(\Gamma') \Rightarrow f(\Delta)} \quad (\wedge\text{left})$$

where $f(\sim^{j-1} \alpha) \wedge f(\sim^j \beta)$ coincides with $f(\sim^j(\alpha \rightarrow \beta))$ by the definition of f . By induction hypothesis, we have $\text{IL}_\omega \vdash \sim^{j-1} \alpha, \sim^j \beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c} \vdots \\ \sim^{j-1} \alpha, \sim^j \beta, \Gamma' \Rightarrow \Delta \end{array}}{\sim^j(\alpha \rightarrow \beta), \Gamma' \Rightarrow \Delta} \quad (\rightarrow\text{left}^o).$$

Subcase (2): The last inference of Q is of the form:

$$\frac{f(\sim^j \alpha), f(\sim^j \beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim^j \alpha) \wedge f(\sim^j \beta), f(\Gamma') \Rightarrow f(\Delta)} (\wedge\text{left})$$

where $f(\sim^j \alpha) \wedge f(\sim^j \beta)$ coincides with $f(\sim^j(\alpha \vee \beta))$ by the definition of f . By induction hypothesis, we have $\text{IL}_\omega \vdash \sim^j \alpha, \sim^j \beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c} \vdots \\ \sim^j \alpha, \sim^j \beta, \Gamma' \Rightarrow \Delta \end{array}}{\sim^j(\alpha \vee \beta), \Gamma' \Rightarrow \Delta} (\vee\text{left}^o).$$

Subcase (3): The last inference of Q is of the form:

$$\frac{f(\sim^i \alpha), f(\sim^i \beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim^i \alpha) \wedge f(\sim^i \beta), f(\Gamma') \Rightarrow f(\Delta)} (\wedge\text{left})$$

where $f(\sim^i \alpha) \wedge f(\sim^i \beta)$ coincides with $f(\sim^i(\alpha \wedge \beta))$ by the definition of f . By induction hypothesis, we have $\text{IL}_\omega \vdash \sim^i \alpha, \sim^i \beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c} \vdots \\ \sim^i \alpha, \sim^i \beta, \Gamma' \Rightarrow \Delta \end{array}}{\sim^i(\alpha \wedge \beta), \Gamma' \Rightarrow \Delta} (\wedge\text{left}^e).$$

□

Using Theorem 2.5, we can obtain the following theorems.

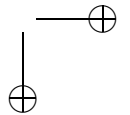
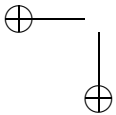
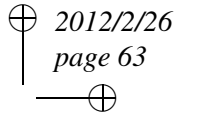
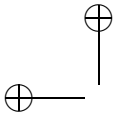
Theorem 2.6: The rule (cut) is admissible in cut-free IL_ω .

Proof. Suppose $\text{IL}_\omega \vdash \Gamma \Rightarrow \Delta$. Then, we have $\text{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.5 (1), and hence $\text{LJ} - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LJ. By Theorem 2.5 (2), we obtain $\text{IL}_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$. □

Theorem 2.7: IL_ω is decidable.

Proof. By decidability of LJ, for each α , it is possible to decide if $f(\alpha)$ is LJ-provable. Then, by Theorem 2.5, IL_ω is decidable. □

Definition 2.8: Let \sharp be a unary connective. A sequent calculus L is called explosive with respect to \sharp if for each pair of formulas α and β , the sequent



$\alpha, \sharp\alpha \Rightarrow \beta$ is provable in L . It is called *paraconsistent with respect to \sharp* if it is not explosive with respect to \sharp .

Theorem 2.9: Let \sharp be \sim^i ($i \in \omega_e$) or \sim^j ($j \in \omega_o$). Then, $\mathbb{I}L_\omega$ is paraconsistent with respect to \sharp .

Proof. Consider a sequent $p, \sharp p \Rightarrow q$ where p and q are distinct atomic formulas. Then, the unprovability of this sequent is guaranteed by using Theorem 2.6. \square

The following theorem says that $\mathbb{I}L_\omega$ has the property of constructible falsity with respect to \sim^j ($j \in \omega_o$).

Theorem 2.10: Let $j \in \omega_o$. If $\mathbb{I}L_\omega \vdash \Rightarrow \sim^j(\alpha \wedge \beta)$, then $\mathbb{I}L_\omega \vdash \Rightarrow \sim^j\alpha$ or $\mathbb{I}L_\omega \vdash \Rightarrow \sim^j\beta$.

Proof. By Theorem 2.6, it is sufficient to consider the cut-free proof P of $\Rightarrow \sim^j(\alpha \wedge \beta)$ in $\mathbb{I}L_\omega - (\text{cut})$. Then, the last inference of P is $(\wedge\text{right}^o)$ or $(\wedge\text{right}^o)$. Therefore we have the required fact. \square

3. Semantics and completeness

Definition 3.1: A Kripke frame is a structure $\langle M, N, R \rangle$ satisfying the following conditions.

1. M is a nonempty set.
2. N is the set of natural numbers.
3. R is a reflexive and transitive binary relation on M .

Definition 3.2: A valuation \models on a Kripke frame $\langle M, N, R \rangle$ is a mapping from the set Ψ of all propositional variables to the power set $2^{M \times N}$ of the direct product $M \times N$ such that for any $p \in \Psi$, any $i \in N$, and any $x, y \in M$, if $(x, i) \in \models (p)$ and xRy , then $(y, i) \in \models (p)$. We will write $(x, i) \models p$ for $(x, i) \in \models (p)$. Each valuation \models is extended to a mapping from the set Φ of all formulas to $2^{M \times N}$ by the following prescriptions: for any $i \in \omega_e$, any $j \in \omega_o$ and any $k \in \omega$,

1. $(x, k) \models \sim\alpha$ iff $(x, k+1) \not\models \alpha$,
2. $(x, k) \models \perp$ does not hold,
3. $(x, i) \models \alpha \rightarrow \beta$ iff $\forall y \in M [xRy \text{ and } (y, i) \models \alpha \text{ imply } (y, i) \models \beta]$,

4. $(x, i) \models \alpha \wedge \beta$ iff $(x, i) \models \alpha$ and $(x, i) \models \beta$,
5. $(x, i) \models \alpha \vee \beta$ iff $(x, i) \models \alpha$ or $(x, i) \models \beta$,
6. $(x, j) \models \alpha \rightarrow \beta$ iff $(x, j-1) \models \alpha$ and $(x, j) \models \beta$,
7. $(x, j) \models \alpha \wedge \beta$ iff $(x, j) \models \alpha$ or $(x, j) \models \beta$,
8. $(x, j) \models \alpha \vee \beta$ iff $(x, j) \models \alpha$ and $(x, j) \models \beta$.

Proposition 3.3: Let \models be a valuation on a Kripke frame $\langle M, N, R \rangle$. For any formula α , any $i \in N$, and any $x, y \in M$, if $(x, i) \models \alpha$ and xRy , then $(y, i) \models \alpha$.

Proof. By induction on the complexity of α . □

An expression Γ^\wedge means $\gamma_1 \wedge \gamma_2 \wedge \dots \wedge \gamma_n$ if $\Gamma \equiv \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ ($0 \leq n$). An expression Δ^* means α or \perp if $\Delta \equiv \{\alpha\}$ or \emptyset , respectively. An expression $(\Gamma \Rightarrow \Delta)^*$ means $\Gamma^\wedge \rightarrow \Delta^*$ if Γ is not empty, and means Δ^* otherwise.

Definition 3.4: A Kripke model is a structure $\langle M, N, R, \models \rangle$ such that

1. $\langle M, N, R \rangle$ is a Kripke frame, and
2. \models is a valuation on $\langle M, N, R \rangle$.

A formula α is true in a Kripke model $\langle M, N, R, \models \rangle$ if $(x, 0) \models \alpha$ for any $x \in M$, and valid in a Kripke frame $\langle M, N, R \rangle$ if it is true for any valuation \models on the Kripke frame.

A sequent $\Gamma \Rightarrow \Delta$ is true in a Kripke model $\langle M, N, R, \models \rangle$ if the formula $(\Gamma \rightarrow \Delta)^*$ is true in the Kripke model, and valid in a Kripke frame $\langle M, N, R \rangle$ if it is true for any valuation \models on the Kripke frame.

The following soundness theorem can straightforwardly be obtained.

Theorem 3.5: Let C be the class of all Kripke frames, $L := \{\Gamma \Rightarrow \Delta \mid \mathbb{IL}_\omega \vdash \Gamma \Rightarrow \Delta\}$ and $L(C) := \{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \text{ is valid in all frames of } C\}$. Then, $L \subseteq L(C)$.

Now we start to prove the completeness theorem.

Definition 3.6: Let x and y be sets of formulas. The pair (x, y) is consistent iff for any $\alpha_1, \dots, \alpha_m \in x$ and any $\beta_1, \dots, \beta_n \in y$ with $(m, n \geq 0)$, the sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1 \vee \dots \vee \beta_n$ is not provable in \mathbb{IL}_ω . The pair (x, y) is maximal consistent iff it is consistent and for every formula α , $\alpha \in x$ or $\alpha \in y$.

The following lemma can be proved using (cut).

Lemma 3.7: Let x and y be sets of formulas. If the pair (x, y) is consistent, then there is a maximal consistent pair (x', y') such that $x \subseteq x'$ and $y \subseteq y'$.

Proof. Let $\gamma_1, \gamma_2, \dots$ be an enumeration of all formulas of \mathbb{IL}_ω . Define a sequence of pairs (x_n, y_n) ($n = 0, 1, \dots$) inductively by $(x_0, y_0) := (x, y)$, and $(x_{m+1}, y_{m+1}) := (x_m, y_m \cup \{\gamma_{m+1}\})$ if $(x_m, y_m \cup \{\gamma_{m+1}\})$ is consistent, and $(x_{m+1}, y_{m+1}) := (x_m \cup \{\gamma_{m+1}\}, y_m)$ otherwise. We can obtain the fact that if (x_m, y_m) is consistent, then so is (x_{m+1}, y_{m+1}) . To verify this, suppose (x_{m+1}, y_{m+1}) is not consistent. Then, there are formulas $\alpha_1, \dots, \alpha_i, \alpha'_1, \dots, \alpha'_j \in x_m$ and $\beta_1, \dots, \beta_k, \beta'_1, \dots, \beta'_l \in y_m$ such that $\mathbb{IL}_\omega \vdash \alpha_1, \dots, \alpha_i \Rightarrow \beta_1 \vee \dots \vee \beta_k \vee \gamma_{m+1}$ and $\mathbb{IL}_\omega \vdash \alpha'_1, \dots, \alpha'_j, \gamma_{m+1} \Rightarrow \beta'_1 \vee \dots \vee \beta'_l$. By using (cut) and some other rules, we can obtain $\mathbb{IL}_\omega \vdash \alpha_1, \dots, \alpha_i, \alpha'_1, \dots, \alpha'_j \Rightarrow \beta_1 \vee \dots \vee \beta_k \vee \beta'_1 \vee \dots \vee \beta'_l$. This contradicts the consistency of (x_m, y_m) . Hence, a pair (x_k, y_k) produced is consistent for any k . We thus obtain a maximal consistent pair $(\bigcup_{n=0}^{\infty} x_n, \bigcup_{n=0}^{\infty} y_n)$. \square

We now construct a canonical model from a given unprovable sequent $\Gamma \Rightarrow \Delta$ in \mathbb{IL}_ω . Since the pair (Γ, Δ) is consistent, by Lemma 3.7, there is a maximal consistent pair (u, v) such that $\Gamma \subseteq u$ and $\Delta \subseteq v$.

Definition 3.8: Let M_L be the set of all maximal consistent pairs. A binary relation R_L on M_L is defined by $(x, w)R_L(y, z)$ iff $x \subseteq y$. A valuation $\models_L(p)$ for any propositional variable p is defined by $\{(x, w), i\} \in M_L \times N \mid \sim^i p \in x\}$.

Lemma 3.9: The structure $\langle M_L, N, R_L, \models_L \rangle$ defined is a Kripke model such that for any formula α , any $i \in N$, and any $(x, w) \in M_L$, $\sim^i \alpha \in x$ iff $((x, w), i) \models_L \alpha$.

Proof. It can be shown that (1) M_L is a nonempty set, because $(u, v) \in M_L$ by the discussion above Definition 3.8, (2) R_L is a reflexive and transitive relation on M_L , and (3) for any propositional variable p and any $(x, w), (y, z) \in M_L$, if $(x, w)R_L(y, z)$ and $((x, w), i) \models_L(p)$, then $((y, z), i) \models_L(p)$. Thus, the structure $\langle M_L, N, R_L, \models_L \rangle$ is a Kripke model.

It remains to show that in this model, for any formula α , any $i \in N$, and any $(x, w) \in M_L$, $\sim^i \alpha \in x$ iff $((x, w), i) \models_L \alpha$. This is shown by induction on the complexity of α . The base step is obvious by Definition 3.8. We now consider the induction step below.

- Case $\alpha \equiv \perp$: By the consistency of (x, w) , $\sim^i \perp \in x$ does not hold.

• Case $\alpha \equiv \sim\beta$: $\sim^i\sim\beta \in x$ iff $\sim^{i+1}\beta \in x$ iff $((x, w), i + 1) \models_L \beta$ (by the induction hypothesis) iff $((x, w), i) \models_L \sim\beta$.

• Case $\alpha \equiv \gamma \rightarrow \delta$:

Subcase ($i \in \omega_e$): Suppose $\sim^i(\gamma \rightarrow \delta) \in x$. We will show $((x, w), i) \models_L \gamma \rightarrow \delta$, i.e., $\forall (y, z) \in M_L [(x, w)R_L(y, z) \text{ and } ((y, z), i) \models_L \gamma \text{ imply } ((y, z), i) \models_L \delta]$. Suppose $(x, w)R_L(y, z)$ and $((y, z), i) \models_L \gamma$. Then, we have (*): $\sim^i(\gamma \rightarrow \delta) \in y$ by the definition of R_L , and obtain (**): $\sim^i\gamma \in y$ by the induction hypothesis. Since (*), (**) and $\text{IL}_\omega \vdash \sim^i(\gamma \rightarrow \delta), \sim^i\gamma \Rightarrow \sim^i\delta$, the fact $\sim^i\delta \in z$ contradicts the consistency of (y, z) , and hence $\sim^i\delta \notin z$. By the maximality of (y, z) , we obtain $\sim^i\delta \in y$. By the induction hypothesis, we obtain the required fact $((y, z), i) \models_L \delta$. Conversely, suppose $\sim^i(\gamma \rightarrow \delta) \notin x$. Then, $\sim^i(\gamma \rightarrow \delta) \in w$ by the maximality of (x, w) . Then, the pair $(x \cup \{\sim^i\gamma\}, \{\sim^i\delta\})$ is consistent because of the following reason. If it is not consistent, $\text{IL}_\omega \vdash \Gamma, \sim^i\gamma \Rightarrow \sim^i\delta$ for some Γ consisting of formulas in x , and hence $\text{IL}_\omega \vdash \Gamma \Rightarrow \sim^i(\gamma \rightarrow \delta)$. This fact contradicts the consistency of (x, w) . By Lemma 3.7, there is a maximal consistent pair (y, z) such that $x \cup \{\sim^i\gamma\} \subseteq y$ and $\{\sim^i\delta\} \subseteq z$ (thus, we have $\sim^i\delta \notin y$ by the consistency of (y, z)). Thus, we have $(x, w)R_L(y, z)$, $((y, z), i) \models_L \gamma$ and not- $[(y, z), i) \models_L \delta]$ by the induction hypothesis. Therefore $((x, w), i) \models_L \gamma \rightarrow \delta$ does not hold.

Subcase ($i \in \omega_o$): Suppose $\sim^i(\gamma \rightarrow \delta) \in x$. Since $\text{IL}_\omega \vdash \sim^i(\gamma \rightarrow \delta) \Rightarrow \sim^{i-1}\gamma$, the fact $\sim^{i-1}\gamma \in w$ contradicts the consistency of (x, w) , and hence $\sim^{i-1}\gamma \in x$. Similarly, we obtain $\sim^i\delta \in x$. By the induction hypothesis, we obtain $((x, w), i - 1) \models_L \gamma$ and $((x, w), i) \models_L \delta$, and hence $((x, w), i) \models_L \gamma \rightarrow \delta$. Conversely, suppose $((x, w), i) \models_L \gamma \rightarrow \delta$, i.e., $((x, w), i - 1) \models_L \gamma$ and $((x, w), i) \models_L \delta$. Then, we obtain $\sim^{i-1}\gamma \in x$ and $\sim^i\delta \in x$ by the induction hypothesis. Since $\text{IL}_\omega \vdash \sim^{i-1}\gamma, \sim^i\delta \Rightarrow \sim^i(\gamma \rightarrow \delta)$, the fact $\sim^i(\gamma \rightarrow \delta) \in w$ contradicts the consistency of (x, w) , and hence $\sim^i(\gamma \rightarrow \delta) \notin w$. By the maximality of (x, w) , we obtain $\sim^i(\gamma \rightarrow \delta) \in x$.

• Case $\alpha \equiv \gamma \wedge \delta$:

Subcase ($i \in \omega_e$): Suppose $\sim^i(\gamma \wedge \delta) \in x$. Since $\text{IL}_\omega \vdash \sim^i(\gamma \wedge \delta) \Rightarrow \sim^i\gamma$, the fact $\sim^i\gamma \in w$ contradicts the consistency of (x, w) , and hence $\sim^i\gamma \in x$. Similarly, we obtain $\sim^i\delta \in x$. By the induction hypothesis, we obtain $((x, w), i) \models_L \gamma$ and $((x, w), i) \models_L \delta$, and hence $((x, w), i) \models_L \gamma \wedge \delta$. Conversely, suppose $((x, w), i) \models_L \gamma \wedge \delta$, i.e., $((x, w), i) \models_L \gamma$ and $((x, w), i) \models_L \delta$. Then, we obtain $\sim^i\gamma \in x$ and $\sim^i\delta \in x$ by the induction hypothesis. Since $\text{IL}_\omega \vdash \sim^i\gamma, \sim^i\delta \Rightarrow \sim^i(\gamma \wedge \delta)$, the fact $\sim^i(\gamma \wedge \delta) \in w$ contradicts the consistency of (x, w) , and hence $\sim^i(\gamma \wedge \delta) \notin w$. By the maximality of (x, w) , we obtain $\sim^i(\gamma \wedge \delta) \in x$.

Subcase ($i \in \omega_o$): Suppose $\sim^i(\gamma \wedge \delta) \in x$. Since $\text{IL}_\omega \vdash \sim^i(\gamma \wedge \delta) \Rightarrow \sim^i\gamma \vee \sim^i\delta$, the fact $\sim^i\gamma, \sim^i\delta \in w$ contradicts the consistency of (x, w) , and hence

$\sim^i \gamma \notin w$ or $\sim^i \delta \notin w$. Thus, we obtain $\sim^i \gamma \in x$ or $\sim^i \delta \in x$ by the maximality of (x, w) . By the induction hypothesis, we obtain $((x, w), i) \models_L \gamma$ or $((x, w), i) \models_L \delta$, and hence $((x, w), i) \models_L \gamma \wedge \delta$. Conversely, suppose $((x, w), i) \models_L \gamma \wedge \delta$, i.e., $((x, w), i) \models_L \gamma$ or $((x, w), i) \models_L \delta$. By the induction hypothesis, we obtain $\sim^i \gamma \in x$ or $\sim^i \delta \in x$. Since $\mathbf{IL}_\omega \vdash \sim^i \gamma \Rightarrow \sim^i(\gamma \wedge \delta)$ and $\mathbf{IL}_\omega \vdash \sim^i \delta \Rightarrow \sim^i(\gamma \wedge \delta)$, the fact $\sim^i(\gamma \wedge \delta) \in w$ contradicts the consistency of (x, w) , and hence $\sim^i(\gamma \wedge \delta) \notin w$. By the maximality of (x, w) , we obtain $\sim^i(\gamma \wedge \delta) \in x$.

- Case $\alpha \equiv \gamma \vee \delta$: Similar to (Case $\alpha \equiv \gamma \wedge \delta$). \square

We then obtain the following completeness theorem.

Theorem 3.10: Let C be the class of all Kripke frames, $L := \{\Gamma \Rightarrow \Delta \mid \mathbf{IL}_\omega \vdash \Gamma \Rightarrow \Delta\}$ and $L(C) := \{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \text{ is valid in all frames of } C\}$. Then, $L(C) \subseteq L$.

Proof. It is sufficient to show that for any sequent $\Gamma \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta$ is valid in an arbitrary frame in C , then it is provable in \mathbf{IL}_ω . To show this, we show that if $\Gamma \Rightarrow \Delta$ is not provable in \mathbf{IL}_ω , then there is a frame $F = \langle M_L, N, R_L \rangle \in C$ such that $\Gamma \Rightarrow \Delta$ is not valid in F , i.e., there is a Kripke model $\langle M_L, N, R_L, \models_L \rangle$ such that $\Gamma \Rightarrow \Delta$ is not true in it.

Suppose that $\Gamma \Rightarrow \Delta$ is not provable in \mathbf{IL}_ω . Then, the pair (Γ, Δ) is consistent. By Lemma 3.7, there is a maximal consistent pair (u, v) such that $\Gamma \subseteq u$ and $\Delta \subseteq v$. Note that if $\Delta \equiv \{\alpha\}$, then $\alpha \notin u$ by the consistency of (u, v) .

Then, our goal is to show that $((u, v), 0) \models_L \Gamma \Rightarrow \Delta$ does not hold in the constructed model. Here we consider only the case $\Gamma \neq \emptyset$. We show that $((u, v), 0) \models_L \Gamma^\wedge \rightarrow \Delta^*$ does not hold, i.e., $\exists(x, z) \in M_L$ [$[(u, v)R_L(x, z)$ and $((x, z), 0) \models_L \Gamma^\wedge$] and $[(x, z), 0) \not\models_L \Delta^*$ does not hold]. Taking (u, v) for (x, z) and 0 for i , we can verify that there is $(u, v) \in M_L$ such that $[(u, v)R_L(u, v)$ and $((u, v), 0) \models_L \Gamma^\wedge$] and $[(u, v), 0) \not\models_L \Delta^*$ does not hold]. The first argument is obvious since the reflexivity of R_L and the fact $\Gamma \subseteq u$. The second argument is shown below. The case $\Delta \equiv \emptyset$ is obvious because $((u, v), 0) \models_L \perp$ does not hold. The case $\Delta \equiv \{\alpha\}$ can be proved by using Lemma 3.9 and the fact $\alpha \notin u$, because we have the fact $\alpha \notin u$ iff $[(u, v), 0) \not\models_L \alpha$ does not hold] by Lemma 3.9. \square

4. Remarks

4.1. Finite-valued version

Although IL_ω may be regarded as a kind of infinite-valued logic, a finite-valued version IL_n of IL_ω can be obtained from IL_ω by adding the inference rules of the form: for a fixed positive integer $n \geq 2$,

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim^n \alpha, \Gamma \Rightarrow \Delta} (\sim^n \text{left}) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim^n \alpha} (\sim^n \text{right})$$

where Δ is empty or singleton. In these rules, the case $n = 2$ corresponds to the double-negation-elimination axiom $\sim\sim\alpha \leftrightarrow \alpha$. The completeness, cut-elimination and embedding results for IL_n can be obtained by imposing some appropriate modifications. The embedding function f w.r.t. IL_n , which is like an embedding function presented in Definition 2.4, needs the condition:

$$f(\sim^n \alpha) := f(\alpha),$$

and the Kripke semantics for IL_n needs the following *cyclic* valuation condition instead of the condition 1 of Definition 3.2:

$$1'. (x, i) \models \sim \alpha \text{ iff } (x, i + 1) \models \alpha \text{ if } i < n - 1, \text{ and } (x, 0) \models \alpha \text{ otherwise.}$$

Note that the logic IL_2 (i.e., the case $n = 2$) without both $\sim^i \perp \Rightarrow$ and (w-right) is just Nelson's N4, since the cyclic valuations $(x, 0) \models \alpha$ and $(x, 1) \models \alpha$ respectively correspond to the well-known dual valuations $x \models^+ \alpha$ (verification) and $x \models^- \alpha$ (falsification) used in N4.

4.2. Modal version

An S4-type modal extension of L_ω [8] with the S4-type modal operator \Box can naturally be considered, and such an extension can be associated with IL_ω by (a slightly modified version of) the well-known Gödel-McKinsey-Tarski translation. A logic ML_ω is obtained from L_ω by adding the even-odd inference rules of the form: for any $i, k \in \omega$,

$$\frac{\sim^i \alpha, \Gamma \Rightarrow \Delta}{\sim^i \Box \alpha, \Gamma \Rightarrow \Delta} (\Box \text{left}^{eo}) \quad \frac{\sim^i \Box \Gamma \Rightarrow \sim^k \alpha}{\sim^i \Box \Gamma \Rightarrow \sim^k \Box \alpha} (\Box \text{right}^{eo}).$$

Then, the embedding theorem of ML_ω into a sequent calculus for S4 can be shown in a natural way, and using this theorem, the cut-elimination theorem for ML_ω can also be shown. The corresponding condition on \Box in the embedding function f is

$$f(\sim^i(\Box\alpha)) := \Box f(\sim^i\alpha) \text{ for any } i \in \omega.$$

A Kripke semantics for ML_ω is defined below. A structure $\langle M, R \rangle$ is a standard S4-type Kripke frame, i.e., M is a non-empty set and R is a transitive and reflexive binary relation on M . Valuations $\{\models_i\}_{i \in \omega}$ are mappings from the set of all formulas to the power set of M . For example, the condition on \Box is defined as follows: for any $i \in \omega$,

$$x \models_i \Box\alpha \text{ iff } \forall y \in M [xRy \text{ implies } y \models_i \alpha].$$

The validity of a formula and that of a sequent can be defined naturally, and the soundness and completeness theorems w.r.t. this semantics can be shown for ML_ω in a standard way. Obviously, ML_ω is associated with IL_ω by the Gödel-McKinsey-Tarski translation. This fact is analogous to the relationship between S4 and intuitionistic logic.

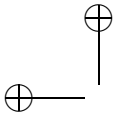
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