Logique & Analyse **217** (2012), 57–71

COMBINING INTUITIONISTIC LOGIC WITH PARACONSISTENT OPERATORS

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Abstract
A new propositional intuitionistic paraconsistent logic, IL_ω, is introduced as a sequent calculus combining Gentzen's LJ with paraconsistent negation-like and involution-like operators. Completeness theorem with respect to Kripke semantics, embedding theorem into LJ, cut-elimination theorem and decidability theorem are shown for IL_ω.

1. *Introduction*

In this paper, a new propositional intuitionistic paraconsistent logic, IL_ω, is introduced as a cut-free and Kripke-complete Gentzen-type sequent calculus combining Gentzen's LJ with paraconsistent negation-like and involution-like operators. The proposed paraconsistent negation-like operators are regarded as a variant of the paraconsistent negation operators of the well-known "useful" many-valued paraconsistent logics: Belnap's and Dunn's 4-valued logic B4 [4, 5], first-degree entailment FDE [2], Nelson's paraconsistent logic N4 [1], Arieli-Avron's bilattice logics [3] and Shramko-Wansing's trilattice logics [9].

Gentzen-type sequent calculi for these many-valued paraconsistent logics have been studied by many researchers. For example, cut-free sequent calculi for some bilattice-based paraconsistent logics, which are natural extensions of N4, were studied by Gargov [6] and by Arieli and Avron [3], and a cut-free sequent calculus L16 that includes Shramko-Wansing's logic $\mathrm{FDE}^{t+\sim_f}$ was introduced by Kamide [7]. Since $\mathrm{FDE}^{t+\sim_f}$ has both the negation and involution operators, L16 needed a bit complicated formalization to obtain a cut-free system. In order to simplify and refine L16, two sequent calculi L_ω and FL_ω have recently been introduced by Kamide [8] presenting a new negation operator that can simultaneously represent both paraconsistent negation-like and involution-like operators. In these logics, the uncertainty level of the truth (or falsehood) of a proposition can be represented by a given number of nested occurrences of the new negation operator.

However, L_ω and FL_ω do not support intuitionistic or constructive characters such as the property of "constructible falsity" [1], since L_ω and FL_ω are based on (propositional and first-order, respectively) classical logic. The systems L_ω and FL_ω are also not appropriate for representing "partial (or incomplete) information", i.e., the situation when $\alpha \vee \neg\alpha$ is not always true for any information α. It is known that Nelson's N4 is useful for representing "constructible falsity" and that intuitionistic logic and N4 are suitable as a base logic for representing "partial information". The aim of introducing IL_ω is thus to obtain an intuitionistic version of L_ω by extending LJ and modifying N4, in order to represent "constructible falsity" and "partial information".

The contents of this paper are then summarized as follows. In Section 2, IL_ω is introduced as a Gentzen-type sequent calculus by extending LJ and modifying N4. A theorem for embedding IL_ω into LJ is shown, and by using this theorem, the cut-elimination and decidability theorems are shown for IL_ω. The properties of paraconsistency and constructible falsity for IL_ω are also derived from the cut-elimination theorem. In Section 3, a Kripke semantics for IL_ω is introduced, and the completeness theorem w.r.t. this semantics is proved. This theorem is the main result of this paper. In Section 4, some versions of IL_ω, which can include N4, are presented, and a modal version LM_ω of L_ω, which can be associated with IL_ω by the Gödel-McKinsey-Tarski translation, is presented.

2. *Sequent calculus and cut-elimination*

The following list of symbols is adopted for the language of the underlying logic: (countable) propositional variables $p_0, p_1, ...$, constant $\bot$ (falsity constant), logical connectives $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\sim$ (paraconsistent negation). The intuitionistic negation $\neg$ can be defined by $\neg\alpha := \alpha\rightarrow\bot$. Greek lower-case letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sets of formulas. We write $A \equiv B$ to indicate the syntactical identity between A and B. The symbol ω is used to represent the set of natural numbers. The symbols ω_e and ω_o are used to represent $\{i \in \omega \mid i$ is even$\}$ and $\{i \in \omega \mid i$ is odd$\}$, respectively. An expression $\sim^i\alpha$ for any $i \in \omega$ is used to denote $\overbrace{\sim\sim\cdots\sim}^{i}\alpha$, which is defined inductively by $(\sim^0\alpha := \alpha)$ and $(\sim^{n+1}\alpha := \sim\sim^n\alpha)$. Lower-case letters i, j and k are used to denote any natural numbers. An expression of the form $\Gamma \Rightarrow \Delta$ where Δ is empty or singleton is called a *sequent*. An expression $L \vdash S$ is used to denote the fact that a sequent S is provable in a sequent calculus L. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Definition 2.1: (IL_ω) *Let* Δ *be empty or singleton.*

The initial sequents of IL_ω *are of the form: for any propositional variable* p *and any* $i \in \omega$,

$$\sim^i p \Rightarrow \sim^i p \qquad\qquad \sim^i \bot \Rightarrow .$$

The structural inference rules of IL_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta}\ (\mathrm{cut}) \qquad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{w-left}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{w-right}).$$

The even *logical inference rules of* IL_ω *are of the form: for any* $i \in \omega_e$,

$$\frac{\Gamma \Rightarrow \sim^i \alpha \quad \sim^i \beta, \Sigma \Rightarrow \Delta}{\sim^i(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta}\ (\rightarrow\text{left}^e) \qquad \frac{\sim^i \alpha, \Gamma \Rightarrow \sim^i \beta}{\Gamma \Rightarrow \sim^i(\alpha \rightarrow \beta)}\ (\rightarrow\text{right}^e)$$

$$\frac{\sim^i \alpha, \sim^i \beta, \Gamma \Rightarrow \Delta}{\sim^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}^e) \qquad \frac{\Gamma \Rightarrow \sim^i \alpha \quad \Gamma \Rightarrow \sim^i \beta}{\Gamma \Rightarrow \sim^i(\alpha \wedge \beta)}\ (\wedge\text{right}^e)$$

$$\frac{\sim^i \alpha, \Gamma \Rightarrow \Delta \quad \sim^i \beta, \Gamma \Rightarrow \Delta}{\sim^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}^e)$$

$$\frac{\Gamma \Rightarrow \sim^i \alpha}{\Gamma \Rightarrow \sim^i(\alpha \vee \beta)}\ (\vee\text{right1}^e) \qquad \frac{\Gamma \Rightarrow \sim^i \beta}{\Gamma \Rightarrow \sim^i(\alpha \vee \beta)}\ (\vee\text{right2}^e).$$

The odd *logical inference rules of* IL_ω *are of the form: for any* $j \in \omega_o$,

$$\frac{\sim^{j-1} \alpha, \sim^j \beta, \Gamma \Rightarrow \Delta}{\sim^j(\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta}\ (\rightarrow\text{left}^o) \qquad \frac{\Gamma \Rightarrow \sim^{j-1} \alpha \quad \Gamma \Rightarrow \sim^j \beta}{\Gamma \Rightarrow \sim^j(\alpha \rightarrow \beta)}\ (\rightarrow\text{right}^o)$$

$$\frac{\sim^j \alpha, \Gamma \Rightarrow \Delta \quad \sim^j \beta, \Gamma \Rightarrow \Delta}{\sim^j(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}^o)$$

$$\frac{\Gamma \Rightarrow {\sim}^j\alpha}{\Gamma \Rightarrow {\sim}^j(\alpha \wedge \beta)}\ (\wedge\text{right}1^o) \qquad \frac{\Gamma \Rightarrow {\sim}^j\beta}{\Gamma \Rightarrow {\sim}^j(\alpha \wedge \beta)}\ (\wedge\text{right}2^o)$$

$$\frac{{\sim}^j\alpha, {\sim}^j\beta, \Gamma \Rightarrow \Delta}{{\sim}^j(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}^o) \qquad \frac{\Gamma \Rightarrow {\sim}^j\alpha \quad \Gamma \Rightarrow {\sim}^j\beta}{\Gamma \Rightarrow {\sim}^j(\alpha \vee \beta)}\ (\vee\text{right}^o).$$

The sequents of the form ${\sim}^i\alpha \Rightarrow {\sim}^i\alpha$ for any formula α and any $i \in \omega$ are provable in cut-free IL_ω. This fact can be proved by induction on the complexity of α. Hence, these sequents can also be regarded as the initial sequents of IL_ω. The $\perp$-less fragment of IL_ω with both $i = 0$ and $j = 1$ is just a sequent system for Nelson's 4-valued logic N4 [1] without the double-negation-elimination axiom: ${\sim}{\sim}\alpha \leftrightarrow \alpha$. Also, the $\{\rightarrow, \perp\}$-less fragment of IL_ω with both $i = 0$ and $j = 1$ is a sequent system for Belnap's and Dunn's 4-valued logic B4 [4, 5] without the double-negation-elimination axiom for $\sim$. For a detailed explanation for sequent calculi for N4 and B4, see e.g., [10].

The following proposition shows that the expressions ${\sim}^i$ (i: even) and ${\sim}^j$ (j: odd) are regarded as an involution-like operator and a negation-like operator, respectively.

An expression $\alpha \Leftrightarrow \beta$ is an abbreviation for the pair of sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.2: *The following sequents are provable in* IL_ω*: for any formulas* α, β*, any* $i \in \omega_e$ *and any* $j \in \omega_o$*,*

1. ${\sim}^i(\alpha \circ \beta) \Leftrightarrow {\sim}^i\alpha \circ {\sim}^i\beta$ *where* $\circ \in \{\rightarrow, \wedge, \vee\}$,
2. ${\sim}^j(\alpha{\rightarrow}\beta) \Leftrightarrow {\sim}^{j-1}\alpha \wedge {\sim}^j\beta$ *(esp.,* ${\sim}(\alpha{\rightarrow}\beta) \Leftrightarrow \alpha \wedge {\sim}\beta$*),*
3. ${\sim}^j(\alpha \wedge \beta) \Leftrightarrow {\sim}^j\alpha \vee {\sim}^j\beta$,
4. ${\sim}^j(\alpha \vee \beta) \Leftrightarrow {\sim}^j\alpha \wedge {\sim}^j\beta$.

Proof. Similar to the proofs of L_ω in [8]. □

Note that IL_ω is also an extension of the sequent calculus LJ for intuitionistic logic.

Observation 2.3: (LJ) LJ *is obtained from* IL_ω *by deleting the odd logical inference rules and replacing* i *in the initial sequents and the even logical inference rules by* 0 *(i.e., deleting every occurrence of* $\sim$*). The modified inference rules for* LJ *by replacing* i *by* 0 *are denoted by deleting the superscript "e".*

As well-known, LJ enjoys cut-elimination.

Definition 2.4: *Let $\Phi := \{p, q, r, ...\}$ be a fixed countable non-empty set of propositional variables. Then, we define the sets $\Phi_i := \{p_i \mid p \in \Phi\}$ $(i \in \omega)$ of propositional variables where $p_0 := p$, i.e., $\Phi_0 = \Phi$. The language $\mathcal{L}_{\mathrm{IL}_\omega}$ of IL_ω is defined using Φ, $\bot, \rightarrow, \wedge, \vee$ and $\sim$. The language $\mathcal{L}_{\mathrm{LJ}}$ of LJ is defined using $\bigcup_{i \in \omega} \Phi_i$, $\bot, \rightarrow$, $\wedge$ and $\vee$.*

A mapping f from $\mathcal{L}_{\mathrm{IL}_\omega}$ to $\mathcal{L}_{\mathrm{LJ}}$ is defined as follows.

1. *$f(\sim^i p) := p_i \in \Phi_i$ for each $p \in \Phi$ and each $i \in \omega$ (especially, $f(p) := p \in \Phi$),*
2. *$f(\sim^i \bot) := \bot$ for each $i \in \omega$,*
3. *$f(\sim^i(\alpha \circ \beta)) := f(\sim^i \alpha) \circ f(\sim^i \beta)$ $(\circ \in \{\rightarrow, \wedge, \vee\})$ for each $i \in \omega_e$,*
4. *$f(\sim^j(\alpha \rightarrow \beta)) := f(\sim^{j-1} \alpha) \wedge f(\sim^j \beta)$ for each $j \in \omega_o$,*
5. *$f(\sim^j(\alpha \wedge \beta)) := f(\sim^j \alpha) \vee f(\sim^j \beta)$ for each $j \in \omega_o$,*
6. *$f(\sim^j(\alpha \vee \beta)) := f(\sim^j \alpha) \wedge f(\sim^j \beta)$ for each $j \in \omega_o$.*

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

Theorem 2.5: *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{IL}_\omega}$ and f be the mapping defined in Definition 2.4. Then:*

1. $\mathrm{IL}_\omega \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. $\mathrm{IL}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LJ} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. (2) immediately follows from (1). Thus, we only examine (1).

(Left-to-right): By induction on the length of the proof P of $\Gamma \Rightarrow \Delta$ in IL_ω. We distinguish the cases according to the last inference of P. We only show the following cases.

Case ($\sim^i p \Rightarrow \sim^i p$): The last inference of P is of the form: $\sim^i p \Rightarrow \sim^i p$. In this case, we obtain $f(\sim^i p) \Rightarrow f(\sim^i p)$, i.e., $p_i \Rightarrow p_i$ $(p_i \in \Phi_i)$, which is an initial sequent of LJ.

Case ($\rightarrow$left^e): The last inference of P is of the form:

$$\frac{\Gamma_1 \Rightarrow \sim^i \alpha \quad \sim^i \beta, \Gamma_2 \Rightarrow \Delta}{\sim^i(\alpha \rightarrow \beta), \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\rightarrow\mathrm{left}^e).$$

By induction hypothesis, we have LJ $\vdash f(\Gamma_1) \Rightarrow f({\sim}^i\alpha)$ and LJ $\vdash f({\sim}^i\beta), f(\Gamma_2) \Rightarrow f(\Delta)$. Then, we obtain

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma_1) \Rightarrow f({\sim}^i\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f({\sim}^i\beta), f(\Gamma_2) \Rightarrow f(\Delta)\end{array}}{f({\sim}^i\alpha){\rightarrow}f({\sim}^i\beta), f(\Gamma_1), f(\Gamma_2) \Rightarrow f(\Delta)}\ ({\rightarrow}\mathrm{left})$$

where $f({\sim}^i\alpha){\rightarrow}f({\sim}^i\beta)$ coincides with $f({\sim}^i(\alpha{\rightarrow}\beta))$ by the definition of f.

Case (→right^o): The last inference of P is of the form:

$$\dfrac{\Gamma \Rightarrow {\sim}^{j-1}\alpha \quad \Gamma \Rightarrow {\sim}^j\beta}{\Gamma \Rightarrow {\sim}^j(\alpha{\rightarrow}\beta)}\ ({\rightarrow}\mathrm{right}^o).$$

By induction hypothesis, we have LJ $\vdash f(\Gamma) \Rightarrow f({\sim}^{j-1}\alpha)$ and LJ $\vdash f(\Gamma) \Rightarrow f({\sim}^j\beta)$. Then, we obtain

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f({\sim}^{j-1}\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f({\sim}^j\beta)\end{array}}{f(\Gamma) \Rightarrow f({\sim}^{j-1}\alpha) \wedge f({\sim}^j\beta)}\ (\wedge\mathrm{right})$$

where $f({\sim}^{j-1}\alpha) \wedge f({\sim}^j\beta)$ coincides with $f({\sim}^j(\alpha{\rightarrow}\beta))$ by the definition of f.

(Right-to-left): By induction on the length of the proof Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LJ. We distinguish the cases according to the last inference of Q, and show only the case (∧left).

Subcase (1): The last inference of Q is of the form:

$$\dfrac{f({\sim}^{j-1}\alpha), f({\sim}^j\beta), f(\Gamma') \Rightarrow f(\Delta)}{f({\sim}^{j-1}\alpha) \wedge f({\sim}^j\beta), f(\Gamma') \Rightarrow f(\Delta)}\ (\wedge\mathrm{left})$$

where $f({\sim}^{j-1}\alpha) \wedge f({\sim}^j\beta)$ coincides with $f({\sim}^j(\alpha{\rightarrow}\beta))$ by the definition of f. By induction hypothesis, we have $\mathrm{IL}_\omega \vdash {\sim}^{j-1}\alpha, {\sim}^j\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\dfrac{\begin{array}{c}\vdots\\ {\sim}^{j-1}\alpha, {\sim}^j\beta, \Gamma' \Rightarrow \Delta\end{array}}{{\sim}^j(\alpha{\rightarrow}\beta), \Gamma' \Rightarrow \Delta}\ ({\rightarrow}\mathrm{left}^o).$$

Subcase (2): The last inference of Q is of the form:

$$\frac{f(\sim^j\alpha), f(\sim^j\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim^j\alpha) \wedge f(\sim^j\beta), f(\Gamma') \Rightarrow f(\Delta)} \ (\wedge\text{left})$$

where $f(\sim^j\alpha) \wedge f(\sim^j\beta)$ coincides with $f(\sim^j(\alpha \vee \beta))$ by the definition of f. By induction hypothesis, we have $\mathrm{IL}_\omega \vdash \sim^j\alpha, \sim^j\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c}\vdots\\ \sim^j\alpha, \sim^j\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim^j(\alpha \vee \beta), \Gamma' \Rightarrow \Delta} \ (\vee\text{left}^o).$$

Subcase (3): The last inference of Q is of the form:

$$\frac{f(\sim^i\alpha), f(\sim^i\beta), f(\Gamma') \Rightarrow f(\Delta)}{f(\sim^i\alpha) \wedge f(\sim^i\beta), f(\Gamma') \Rightarrow f(\Delta)} \ (\wedge\text{left})$$

where $f(\sim^i\alpha) \wedge f(\sim^i\beta)$ coincides with $f(\sim^i(\alpha \wedge \beta))$ by the definition of f. By induction hypothesis, we have $\mathrm{IL}_\omega \vdash \sim^i\alpha, \sim^i\beta, \Gamma' \Rightarrow \Delta$, and hence obtain:

$$\frac{\begin{array}{c}\vdots\\ \sim^i\alpha, \sim^i\beta, \Gamma' \Rightarrow \Delta\end{array}}{\sim^i(\alpha \wedge \beta), \Gamma' \Rightarrow \Delta} \ (\wedge\text{left}^e).$$

□

Using Theorem 2.5, we can obtain the following theorems.

Theorem 2.6: *The rule* (cut) *is admissible in cut-free* IL_ω.

Proof. Suppose $\mathrm{IL}_\omega \vdash \Gamma \Rightarrow \Delta$. Then, we have $\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.5 (1), and hence $\mathrm{LJ} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LJ. By Theorem 2.5 (2), we obtain $\mathrm{IL}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. □

Theorem 2.7: IL_ω *is decidable.*

Proof. By decidability of LJ, for each α, it is possible to decide if $f(\alpha)$ is LJ-provable. Then, by Theorem 2.5, IL_ω is decidable. □

Definition 2.8: *Let* $\sharp$ *be a unary connective. A sequent calculus L is called* explosive *with respect to* $\sharp$ *if for each pair of formulas* α *and* β*, the sequent*

$\alpha, \sharp\alpha \Rightarrow \beta$ *is provable in* L. *It is called* paraconsistent *with respect to* $\sharp$ *if it is not explosive with respect to* $\sharp$.

Theorem 2.9: *Let* $\sharp$ *be* $\sim^i$ $(i \in \omega_e)$ *or* $\sim^j$ $(j \in \omega_o)$. *Then,* IL_ω *is paraconsistent with respect to* $\sharp$.

Proof. Consider a sequent $p, \sharp p \Rightarrow q$ where p and q are distinct atomic formulas. Then, the unprovability of this sequent is guaranteed by using Theorem 2.6. □

The following theorem says that IL_ω has the property of constructible falsity with respect to $\sim^j$ $(j \in \omega_o)$.

Theorem 2.10: *Let* $j \in \omega_o$. *If* $\mathrm{IL}_\omega \vdash \Rightarrow \sim^j(\alpha \wedge \beta)$, *then* $\mathrm{IL}_\omega \vdash \Rightarrow \sim^j\alpha$ *or* $\mathrm{IL}_\omega \vdash \Rightarrow \sim^j\beta$.

Proof. By Theorem 2.6, it is sufficient to consider the cut-free proof P of $\Rightarrow \sim^j(\alpha \wedge \beta)$ in $\mathrm{IL}_\omega -$ (cut). Then, the last inference of P is ($\wedge$right^o) or ($\wedge$right^o). Therefore we have the required fact. □

3. *Semantics and completeness*

Definition 3.1: *A* Kripke frame *is a structure* $\langle M, N, R\rangle$ *satisfying the following conditions.*

1. M *is a nonempty set.*
2. N *is the set of natural numbers.*
3. R *is a reflexive and transitive binary relation on* M.

Definition 3.2: *A* valuation $\models$ *on a Kripke frame* $\langle M, N, R\rangle$ *is a mapping from the set* Ψ *of all propositional variables to the power set* $2^{M\times N}$ *of the direct product* $M \times N$ *such that for any* $p \in \Psi$, *any* $i \in N$, *and any* $x, y \in M$, *if* $(x, i) \in \models (p)$ *and* xRy, *then* $(y, i) \in \models (p)$. *We will write* $(x, i) \models p$ *for* $(x, i) \in \models (p)$. *Each valuation* $\models$ *is extended to a mapping from the set* Φ *of all formulas to* $2^{M\times N}$ *by the following prescriptions: for any* $i \in \omega_e$, *any* $j \in \omega_o$ *and any* $k \in \omega$,

1. $(x, k) \models \sim\alpha$ *iff* $(x, k+1) \models \alpha$,
2. $(x, k) \models \bot$ *does not hold,*
3. $(x, i) \models \alpha\rightarrow\beta$ *iff* $\forall y \in M$ $[xRy$ *and* $(y, i) \models \alpha$ *imply* $(y, i) \models \beta]$,

4. $(x,i) \models \alpha \wedge \beta$ *iff* $(x,i) \models \alpha$ *and* $(x,i) \models \beta$,
5. $(x,i) \models \alpha \vee \beta$ *iff* $(x,i) \models \alpha$ *or* $(x,i) \models \beta$,
6. $(x,j) \models \alpha{\to}\beta$ *iff* $(x,j-1) \models \alpha$ *and* $(x,j) \models \beta$,
7. $(x,j) \models \alpha \wedge \beta$ *iff* $(x,j) \models \alpha$ *or* $(x,j) \models \beta$,
8. $(x,j) \models \alpha \vee \beta$ *iff* $(x,j) \models \alpha$ *and* $(x,j) \models \beta$.

Proposition 3.3: *Let* $\models$ *be a valuation on a Kripke frame* $\langle M, N, R\rangle$. *For any formula* α, *any* $i \in N$, *and any* $x, y \in M$, *if* $(x,i) \models \alpha$ *and* xRy, *then* $(y,i) \models \alpha$.

Proof. By induction on the complexity of α. □

An expression $\Gamma^{\wedge}$ means $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$ if $\Gamma \equiv \{\gamma_1, \gamma_2, ..., \gamma_n\}$ $(0 \leq n)$. An expression Δ^* means α or $\perp$ if $\Delta \equiv \{\alpha\}$ or $\emptyset$, respectively. An expression $(\Gamma \Rightarrow \Delta)^*$ means $\Gamma^{\wedge}{\to}\Delta^*$ if Γ is not empty, and means Δ^* otherwise.

Definition 3.4: *A* Kripke model *is a structure* $\langle M, N, R, \models\rangle$ *such that*

1. $\langle M, N, R\rangle$ *is a Kripke frame, and*
2. $\models$ *is a valuation on* $\langle M, N, R\rangle$.

A formula α *is* true *in a Kripke model* $\langle M, N, R, \models\rangle$ *if* $(x, 0) \models \alpha$ *for any* $x \in M$, *and* valid *in a Kripke frame* $\langle M, N, R\rangle$ *if it is true for any valuation* $\models$ *on the Kripke frame.*

A sequent $\Gamma \Rightarrow \Delta$ *is true in a Kripke model* $\langle M, N, R, \models\rangle$ *if the formula* $(\Gamma{\to}\Delta)^*$ *is true in the Kripke model, and valid in a Kripke frame* $\langle M, N, R\rangle$ *if it is true for any valuation* $\models$ *on the Kripke frame.*

The following soundness theorem can straightforwardly be obtained.

Theorem 3.5: *Let C be the class of all Kripke frames,* $L := \{\Gamma \Rightarrow \Delta \mid \mathrm{IL}_\omega \vdash \Gamma \Rightarrow \Delta\}$ *and* $L(C) := \{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta$ *is valid in all frames of* $C\}$. *Then,* $L \subseteq L(C)$.

Now we start to prove the completeness theorem.

Definition 3.6: *Let* x *and* y *be sets of formulas. The pair* (x, y) *is* consistent *iff for any* $\alpha_1, ..., \alpha_m \in x$ *and any* $\beta_1, ..., \beta_n \in y$ *with* $(m, n \geq 0)$, *the sequent* $\alpha_1, ..., \alpha_m \Rightarrow \beta_1 \vee \cdots \vee \beta_n$ *is not provable in* IL_ω. *The pair* (x, y) *is* maximal consistent *iff it is consistent and for every formula* α, $\alpha \in x$ *or* $\alpha \in y$.

The following lemma can be proved using (cut).

Lemma 3.7: *Let x and y be sets of formulas. If the pair (x, y) is consistent, then there is a maximal consistent pair (x', y') such that $x \subseteq x'$ and $y \subseteq y'$.*

Proof. Let $\gamma_1, \gamma_2, ...$ be an enumeration of all formulas of IL_ω. Define a sequence of pairs (x_n, y_n) $(n = 0, 1, ...)$ inductively by $(x_0, y_0) := (x, y)$, and $(x_{m+1}, y_{m+1}) := (x_m, y_m \cup \{\gamma_{m+1}\})$ if $(x_m, y_m \cup \{\gamma_{m+1}\})$ is consistent, and $(x_{m+1}, y_{m+1}) := (x_m \cup \{\gamma_{m+1}\}, y_m)$ otherwise. We can obtain the fact that if (x_m, y_m) is consistent, then so is (x_{m+1}, y_{m+1}). To verify this, suppose (x_{m+1}, y_{m+1}) is not consistent. Then, there are formulas $\alpha_1, ..., \alpha_i, \alpha'_1, ..., \alpha'_j \in x_m$ and $\beta_1, ..., \beta_k, \beta'_1, ..., \beta'_l \in y_m$ such that $\mathrm{IL}_\omega \vdash \alpha_1, ..., \alpha_i \Rightarrow \beta_1 \vee \cdots \vee \beta_k \vee \gamma_{m+1}$ and $\mathrm{IL}_\omega \vdash \alpha'_1, ..., \alpha'_j, \gamma_{m+1} \Rightarrow \beta'_1 \vee \cdots \vee \beta'_l$. By using (cut) and some other rules, we can obtain $\mathrm{IL}_\omega \vdash \alpha_1, ..., \alpha_i, \alpha'_1, ..., \alpha'_j \Rightarrow \beta_1 \vee \cdots \vee \beta_k \vee \beta'_1 \vee \cdots \vee \beta'_l$. This contradicts the consistency of (x_m, y_m). Hence, a pair (x_k, y_k) produced is consistent for any k. We thus obtain a maximal consistent pair $(\bigcup_{n=0}^{\infty} x_n, \bigcup_{n=0}^{\infty} y_n)$. □

We now construct a canonical model from a given unprovable sequent $\Gamma \Rightarrow \Delta$ in IL_ω. Since the pair (Γ, Δ) is consistent, by Lemma 3.7, there is a maximal consistent pair (u, v) such that $\Gamma \subseteq u$ and $\Delta \subseteq v$.

Definition 3.8: *Let M_L be the set of all maximal consistent pairs. A binary relation R_L on M_L is defined by $(x, w)R_L(y, z)$ iff $x \subseteq y$. A valuation $\models_L (p)$ for any propositional variable p is defined by $\{((x, w), i) \in M_L \times N \mid {\sim}^i p \in x\}$.*

Lemma 3.9: *The structure $\langle M_L, N, R_L, \models_L \rangle$ defined is a Kripke model such that for any formula α, any $i \in N$, and any $(x, w) \in M_L$, ${\sim}^i \alpha \in x$ iff $((x, w), i) \models_L \alpha$.*

Proof. It can be shown that (1) M_L is a nonempty set, because $(u, v) \in M_L$ by the discussion above Definition 3.8, (2) R_L is a reflexive and transitive relation on M_L, and (3) for any propositional variable p and any $(x, w), (y, z) \in M_L$, if $(x, w)R_L(y, z)$ and $((x, w), i) \models_L (p)$, then $((y, z), i) \models_L (p)$. Thus, the structure $\langle M_L, N, R_L, \models_L \rangle$ is a Kripke model.

It remains to show that in this model, for any formula α, any $i \in N$, and any $(x, w) \in M_L$, ${\sim}^i \alpha \in x$ iff $((x, w), i) \models_L \alpha$. This is shown by induction on the complexity of α. The base step is obvious by Definition 3.8. We now consider the induction step below.

- Case $\alpha \equiv \bot$: By the consistency of (x, w), ${\sim}^i \bot \in x$ does not hold.

• Case $\alpha \equiv {\sim}\beta$: ${\sim}^i{\sim}\beta \in x$ iff ${\sim}^{i+1}\beta \in x$ iff $((x,w), i+1) \models_L \beta$ (by the induction hypothesis) iff $((x,w), i) \models_L {\sim}\beta$.

• Case $\alpha \equiv \gamma{\rightarrow}\delta$:

Subcase ($i \in \omega_e$): Suppose ${\sim}^i(\gamma{\rightarrow}\delta) \in x$. We will show $((x,w), i) \models_L \gamma{\rightarrow}\delta$, i.e., $\forall (y,z) \in M_L$ $[(x,w)R_L(y,z)$ and $((y,z), i) \models_L \gamma$ imply $((y,z), i) \models_L \delta]$. Suppose $(x,w)R_L(y,z)$ and $((y,z), i) \models_L \gamma$. Then, we have (*): ${\sim}^i(\gamma{\rightarrow}\delta) \in y$ by the definition of R_L, and obtain (**): ${\sim}^i\gamma \in y$ by the induction hypothesis. Since (*), (**) and $\text{IL}_\omega \vdash {\sim}^i(\gamma{\rightarrow}\delta), {\sim}^i\gamma \Rightarrow {\sim}^i\delta$, the fact ${\sim}^i\delta \in z$ contradicts the consistency of (y,z), and hence ${\sim}^i\delta \notin z$. By the maximality of (y,z), we obtain ${\sim}^i\delta \in y$. By the induction hypothesis, we obtain the required fact $((y,z), i) \models_L \delta$. Conversely, suppose ${\sim}^i(\gamma{\rightarrow}\delta) \notin x$. Then, ${\sim}^i(\gamma{\rightarrow}\delta) \in w$ by the maximality of (x,w). Then, the pair $(x \cup \{{\sim}^i\gamma\}, \{{\sim}^i\delta\})$ is consistent because of the following reason. If it is not consistent, $\text{IL}_\omega \vdash \Gamma, {\sim}^i\gamma \Rightarrow {\sim}^i\delta$ for some Γ consisting of formulas in x, and hence $\text{IL}_\omega \vdash \Gamma \Rightarrow {\sim}^i(\gamma{\rightarrow}\delta)$. This fact contradicts the consistency of (x,w). By Lemma 3.7, there is a maximal consistent pair (y,z) such that $x \cup \{{\sim}^i\gamma\} \subseteq y$ and $\{{\sim}^i\delta\} \subseteq z$ (thus, we have ${\sim}^i\delta \notin y$ by the consistency of (y,z)). Thus, we have $(x,w)R_L(y,z)$, $((y,z), i) \models_L \gamma$ and not-$[((y,z), i) \models_L \delta]$ by the induction hypothesis. Therefore $((x,w), i) \models_L \gamma{\rightarrow}\delta$ does not hold.

Subcase ($i \in \omega_o$): Suppose ${\sim}^i(\gamma{\rightarrow}\delta) \in x$. Since $\text{IL}_\omega \vdash {\sim}^i(\gamma{\rightarrow}\delta) \Rightarrow {\sim}^{i-1}\gamma$, the fact ${\sim}^{i-1}\gamma \in w$ contradicts the consistency of (x,w), and hence ${\sim}^{i-1}\gamma \in x$. Similarly, we obtain ${\sim}^i\delta \in x$. By the induction hypothesis, we obtain $((x,w), i-1) \models_L \gamma$ and $((x,w), i) \models_L \delta$, and hence $((x,w), i) \models_L \gamma{\rightarrow}\delta$. Conversely, suppose $((x,w), i) \models_L \gamma{\rightarrow}\delta$, i.e., $((x,w), i-1) \models_L \gamma$ and $((x,w), i) \models_L \delta$. Then, we obtain ${\sim}^{i-1}\gamma \in x$ and ${\sim}^i\delta \in x$ by the induction hypothesis. Since $\text{IL}_\omega \vdash {\sim}^{i-1}\gamma, {\sim}^i\delta \Rightarrow {\sim}^i(\gamma{\rightarrow}\delta)$, the fact ${\sim}^i(\gamma{\rightarrow}\delta) \in w$ contradicts the consistency of (x,w), and hence ${\sim}^i(\gamma{\rightarrow}\delta) \notin w$. By the maximality of (x,w), we obtain ${\sim}^i(\gamma{\rightarrow}\delta) \in x$.

• Case $\alpha \equiv \gamma \wedge \delta$:

Subcase ($i \in \omega_e$): Suppose ${\sim}^i(\gamma \wedge \delta) \in x$. Since $\text{IL}_\omega \vdash {\sim}^i(\gamma \wedge \delta) \Rightarrow {\sim}^i\gamma$, the fact ${\sim}^i\gamma \in w$ contradicts the consistency of (x,w), and hence ${\sim}^i\gamma \in x$. Similarly, we obtain ${\sim}^i\delta \in x$. By the induction hypothesis, we obtain $((x,w), i) \models_L \gamma$ and $((x,w), i) \models_L \delta$, and hence $((x,w), i) \models_L \gamma \wedge \delta$. Conversely, suppose $((x,w), i) \models_L \gamma \wedge \delta$, i.e., $((x,w), i) \models_L \gamma$ and $((x,w), i) \models_L \delta$. Then, we obtain ${\sim}^i\gamma \in x$ and ${\sim}^i\delta \in x$ by the induction hypothesis. Since $\text{IL}_\omega \vdash {\sim}^i\gamma, {\sim}^i\delta \Rightarrow {\sim}^i(\gamma \wedge \delta)$, the fact ${\sim}^i(\gamma \wedge \delta) \in w$ contradicts the consistency of (x,w), and hence ${\sim}^i(\gamma \wedge \delta) \notin w$. By the maximality of (x,w), we obtain ${\sim}^i(\gamma \wedge \delta) \in x$.

Subcase ($i \in \omega_o$): Suppose ${\sim}^i(\gamma \wedge \delta) \in x$. Since $\text{IL}_\omega \vdash {\sim}^i(\gamma \wedge \delta) \Rightarrow {\sim}^i\gamma \vee {\sim}^i\delta$, the fact ${\sim}^i\gamma, {\sim}^i\delta \in w$ contradicts the consistency of (x,w), and hence

$\sim^i\gamma \notin w$ or $\sim^i\delta \notin w$. Thus, we obtain $\sim^i\gamma \in x$ or $\sim^i\delta \in x$ by the maximality of (x, w). By the induction hypothesis, we obtain $((x, w), i) \models_L \gamma$ or $((x, w), i) \models_L \delta$, and hence $((x, w), i) \models_L \gamma \wedge \delta$. Conversely, suppose $((x, w), i) \models_L \gamma \wedge \delta$, i.e., $((x, w), i) \models_L \gamma$ or $((x, w), i) \models_L \delta$. By the induction hypothesis, we obtain $\sim^i\gamma \in x$ or $\sim^i\delta \in x$. Since $\mathrm{IL}_\omega \vdash \sim^i\gamma \Rightarrow \sim^i(\gamma \wedge \delta)$ and $\mathrm{IL}_\omega \vdash \sim^i\delta \Rightarrow \sim^i(\gamma \wedge \delta)$, the fact $\sim^i(\gamma \wedge \delta) \in w$ contradicts the consistency of (x, w), and hence $\sim^i(\gamma \wedge \delta) \notin w$. By the maximality of (x, w), we obtain $\sim^i(\gamma \wedge \delta) \in x$.

• Case $\alpha \equiv \gamma \vee \delta$: Similar to (Case $\alpha \equiv \gamma \wedge \delta$). □

We then obtain the following completeness theorem.

Theorem 3.10: *Let C be the class of all Kripke frames,* $L := \{\Gamma \Rightarrow \Delta \mid \mathrm{IL}_\omega \vdash \Gamma \Rightarrow \Delta\}$ *and* $L(C) := \{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta$ *is valid in all frames of* $C\}$. *Then,* $L(C) \subseteq L$.

Proof. It is sufficient to show that for any sequent $\Gamma \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta$ is valid in an arbitrary frame in C, then it is provable in IL_ω. To show this, we show that if $\Gamma \Rightarrow \Delta$ is not provable in IL_ω, then there is a frame $F = \langle M_L, N, R_L\rangle \in C$ such that $\Gamma \Rightarrow \Delta$ is not valid in F, i.e., there is a Kripke model $\langle M_L, N, R_L, \models_L\rangle$ such that $\Gamma \Rightarrow \Delta$ is not true in it.

Suppose that $\Gamma \Rightarrow \Delta$ is not provable in IL_ω. Then, the pair (Γ, Δ) is consistent. By Lemma 3.7, there is a maximal consistent pair (u, v) such that $\Gamma \subseteq u$ and $\Delta \subseteq v$. Note that if $\Delta \equiv \{\alpha\}$, then $\alpha \notin u$ by the consistency of (u, v).

Then, our goal is to show that $((u, v), 0) \models_L \Gamma \Rightarrow \Delta$ does not hold in the constructed model. Here we consider only the case $\Gamma \neq \emptyset$. We show that $((u, v), 0) \models_L \Gamma^\wedge \to \Delta^*$ does not hold, i.e., $\exists (x, z) \in M_L$ $[[(u, v)R_L(x, z)$ and $((x, z), 0) \models_L \Gamma^\wedge]$ and $[((x, z), 0) \models_L \Delta^*$ does not hold $]]$. Taking (u, v) for (x, z) and 0 for i, we can verify that there is $(u, v) \in M_L$ such that $[(u, v)R_L(u, v)$ and $((u, v), 0) \models_L \Gamma^\wedge]$ and $[((u, v), 0) \models_L \Delta^*$ does not hold$]$. The first argument is obvious since the reflexivity of R_L and the fact $\Gamma \subseteq u$. The second argument is shown below. The case $\Delta \equiv \emptyset$ is obvious because $((u, v), 0) \models_L \bot$ does not hold. The case $\Delta \equiv \{\alpha\}$ can be proved by using Lemma 3.9 and the fact $\alpha \notin u$, because we have the fact $\alpha \notin u$ iff $[((u, v), 0) \models_L \alpha$ does not hold$]$ by Lemma 3.9. □

4. *Remarks*

4.1. *Finite-valued version*

Although IL_ω may be regarded as a kind of infinite-valued logic, a finite-valued version IL_n of IL_ω can be obtained from IL_ω by adding the inference rules of the form: for a fixed positive integer $n \geq 2$,

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim^n \alpha, \Gamma \Rightarrow \Delta} \ (\sim^n \mathrm{left}) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim^n \alpha} \ (\sim^n \mathrm{right})$$

where Δ is empty or singleton. In these rules, the case $n = 2$ corresponds to the double-negation-elimination axiom $\sim\sim\alpha \leftrightarrow \alpha$. The completeness, cut-elimination and embedding results for IL_n can be obtained by imposing some appropriate modifications. The embedding function f w.r.t. IL_n, which is like an embedding function presented in Definition 2.4, needs the condition:

$f(\sim^n \alpha) := f(\alpha)$,

and the Kripke semantics for IL_n needs the following *cyclic* valuation condition instead of the condition 1 of Definition 3.2:

$1'$. $(x, i) \models \sim\alpha$ iff $(x, i+1) \models \alpha$ if $i < n-1$, and $(x, 0) \models \alpha$ otherwise.

Note that the logic IL_2 (i.e., the case $n = 2$) without both $\sim^i \bot \Rightarrow$ and (w-right) is just Nelson's N4, since the cyclic valuations $(x, 0) \models \alpha$ and $(x, 1) \models \alpha$ respectively correspond to the well-known dual valuations $x \models^+ \alpha$ (verification) and $x \models^- \alpha$ (falsification) used in N4.

4.2. *Modal version*

An S4-type modal extension of L_ω [8] with the S4-type modal operator $\Box$ can naturally be considered, and such an extension can be associated with IL_ω by (a slightly modified version of) the well-known Gödel-McKinsey-Tarski translation. A logic ML_ω is obtained from L_ω by adding the even-odd inference rules of the form: for any $i, k \in \omega$,

$$\frac{\sim^i \alpha, \Gamma \Rightarrow \Delta}{\sim^i \Box \alpha, \Gamma \Rightarrow \Delta} \ (\Box \mathrm{left}^{eo}) \qquad \frac{\sim^i \Box \Gamma \Rightarrow \sim^k \alpha}{\sim^i \Box \Gamma \Rightarrow \sim^k \Box \alpha} \ (\Box \mathrm{right}^{eo}).$$

Then, the embedding theorem of ML_ω into a sequent calculus for S4 can be shown in a natural way, and using this theorem, the cut-elimination theorem for ML_ω can also be shown. The corresponding condition on $\Box$ in the embedding function f is

$$f(\sim^i(\Box\alpha)) := \Box f(\sim^i\alpha) \text{ for any } i \in \omega.$$

A Kripke semantics for ML_ω is defined below. A structure $\langle M, R\rangle$ is a standard S4-type Kripke frame, i.e., M is a non-empty set and R is a transitive and reflexive binary relation on M. Valuations $\{\models_i\}_{i\in\omega}$ are mappings from the set of all formulas to the power set of M. For example, the condition on $\Box$ is defined as follows: for any $i \in \omega$,

$$x \models_i \Box\alpha \text{ iff } \forall y \in M\ [xRy \text{ implies } y \models_i \alpha].$$

The validity of a formula and that of a sequent can be defined naturally, and the soundness and completeness theorems w.r.t. this semantics can be shown for ML_ω in a standard way. Obviously, ML_ω is associated with IL_ω by the Gödel-McKinsey-Tarski translation. This fact is analogous to the relationship between S4 and intuitionistic logic.

ACKNOWLEDGEMENTS

I would like to thank the anonymous referees for their valuable comments. This research was supported by the Alexander von Humboldt Foundation and the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Young Scientists, 20700015.

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