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CONSTRUCTIVE DISCURSIVE LOGIC WITH STRONG NEGATION

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Abstract

Jaskowski's discursive logic (or discussive logic) is the first formal paraconsistent logic which is classified as a non-adjunctive system. It is now recognized that discursive logic is not generally appropriate for paraconsistent reasoning. To improve it in a constructive setting, we propose a constructive discursive logic with strong negation $CDLSN$ based on Nelson's constructive logic N^-. In $CDLSN$, discursive negation is defined similar to intuitionistic negation and discursive implication is defined as material implication using discursive negation. We give an axiomatic system and Kripke semantics with a completeness proof. We also discuss some advantages of the proposed system over other paraconsistent systems.

1. *Introduction*

Jaskowski's *discursive logic* (or *discussive logic*) is the first formal *paraconsistent logic* which is classified as a *non-adjunctive system*; see Jaskowski [3]. Discursive logic can be motivated by the nature of our ordinary discourse. That is, in a discourse, several *participants* exist and have some information, beliefs, and others.

In this regard, truth is formalized by means of the sum of opinions supplied by participants. Even if each participant has consistent information, some participant could be inconsistent with other participants.

This amounts to supposing that $A \wedge \sim A$ does not hold while both A and $\sim A$ do. This means that the so-called *adjunction*, i.e. from $\vdash A, \vdash B$ to $\vdash A \wedge B$ is invalid. Jaskowski modeled the idea founded on modal logic S5 and reached the discursive logic in which adjunction and *modus ponens* cannot hold. In addition, Jaskowski introduced discursive implication $A \rightarrow_d B$ as $\Diamond A \rightarrow B$ satisfying *modus ponens*.

The rest of this paper is as follows. Section 2 is devoted to an exposition Jaskowski's discursive logic. In section 3, we introduce constructive discursive logic with strong negation $CDLSN$ with an axiomatic system. Section

4 outlines a Kripke semantics. We establish the completeness theorem. The final section gives some conclusions.

2. *Jaskowski's Discursive Logic*

Discursive Logic was proposed by a Polish logician S. Jaskowski [3] in 1948. It was a formal system J satisfying the conditions: (a) from two contradictory propositions, it should not be possible to deduce any proposition; (b) most of the classical theses compatible with (a) should be valid; (c) J should have an intuitive interpretation.

Such a calculus has, among others, the following intuitive properties remarked by Jaskowski himself: suppose that one desires to systematize in only one deductive system all theses defended in a discussion. In general, the participants do not confer the same meaning to some of the symbols. One would have then as theses of a deductive system that formalize such a discussion, an assertion and its negation, so both are "true" since it has a variation in the sense given to the symbols. It is thus possible to regard discursive logic as one of the so-called *paraconsistent logics*.

Jaskowski's D_2 contains propositional formulas built from logical symbols of classical logic. In addition, possibility operator $\Diamond$ in S5 is added. Based on the possibility operator, three discursive logical symbols can be defined as follows:

discursive implication: $A \rightarrow_d B =_{def} \Diamond A \rightarrow B$
discursive conjunction: $A \wedge_d B =_{def} \Diamond A \wedge B$
discursive equivalence: $A \leftrightarrow_d B =_{def} (A \rightarrow_d B) \wedge_d (B \rightarrow_d A)$

Additionally, we can define discursive negation $\neg_d A$ as $A \rightarrow_d false$. Jaskowski's original formulation of D_2 in [3] used the logical symbols: $\rightarrow_d$, $\leftrightarrow_d, \vee, \wedge, \neg$, and he later defined $\wedge_d$ in [4].

The following axiomatization due to Kotas [5] has the following axioms and the rules of inference.

Axioms
(A1) $\Box(A \rightarrow (\neg A \rightarrow B))$
(A2) $\Box((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
(A3) $\Box((\neg A \rightarrow A) \rightarrow A)$
(A4) $\Box(\Box A \rightarrow A)$
(A5) $\Box(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))$
(A6) $\Box(\neg\Box A \rightarrow \Box\neg\Box A)$

Rules of Inference
(R1) substitution rule

(R2) $\Box A, \Box(A \to B)/\Box B$
(R3) $\Box A/\Box\Box A$
(R4) $\Box A/A$
(R5) $\neg\Box\neg\Box A/A$

There are other axiomatizations of D_2, but we omit the details here.

3. *Constructive Discursive Logic with Strong Negation*

The gist of discursive logic is to use the modal logic S5 to define discursive logical connectives which can formalize a non-adjunctive system. It follows that discursive logic can be seen as a paraconsistent logic, which does not satisfy *explosion* of the form: $\{A, \neg A\} \models B$ for any A and B, where $\models$ is a consequence relation. We say that a system is *trivial* iff all the formulas are provable. Therefore, paraconsistent logic is useful to formalize inconsistent but *non-trivial* systems.

A question arises. Most works on discursive logic utilize classical logic and S5 as a basis. However, we do not think that these are essential. For instance, an intuitionist hopes to have a discursive system in a constructive setting. This is a topic explored in this paper.

To make the idea formal, it is worth considering Nelson's constructive logic with strong negation N^- of Almukdad and Nelson [1]. In N^-, $\sim$ denotes *strong negation* satisfying the following axioms:

(N1) $\sim\sim A \leftrightarrow A$
(N2) $\sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
(N3) $\sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
(N4) $\sim (A \to B) \leftrightarrow (A \wedge \sim B)$

and the axiomatization of the intuitionistic positive logic Int^+ with *modus ponens* (MP), i.e. $A, A \to B/B$ as the rule of inference.

Note here that N^- is paraconsistent in the sense that $\sim (A \wedge \sim A)$ and $(A \wedge \sim A) \to B$ do not hold.

If we add (N0) to N^-, we have N of Nelson [6].

(N0) $(A \wedge \sim A) \to B$

In N, *intuitionistic negation* $\neg$ can be defined as follows:

$\neg A =_{def} A \to \sim A$

If we add the law of *excluded middle*: $A \vee \sim A$ to N, the resulting system is classical logic.

Indeed, N^- is itself a paraconsistent logic, but can also be accommodated as a version of discursive logic.

Now, we introduce the *constructive discursive logic with strong negation* $CDLSN$. It diverges in two ways from D_2: (1) it does not take classical logic as its starting point; and (2) it does not use the possibility operator $\diamondsuit$ as a modality, but a negation with modal operators.

$CDLSN$ can be defined in two ways. One is to extend N^- with discursive negation $\neg_d$. The other is to weaken intuitionistic negation in N^-. We adopt the first approach.

Here, we fix the language of the logics which we use in this paper. The language of Int^+ is defined as the set of propositional variables and logical symbols: $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication). The language of Int is the extension of that of Int^+ with $\neg$ (intuitionistic negation). The language of N^- is the extension of that of Int^+ with $\sim$ (strong negation). The language of $CDLSN$ is the extension of N^- with $\neg_d$ (discursive negation). Additionally, we use the logical constant $false$ as the abbreviation of $\sim (A \rightarrow A)$.

We believe that $CDLSN$ is (constructive) improvement of D_2. First, $CDLSN$ uses Int^+ rather than classical logic as the base. Second, $CDLSN$ simulates modality in D_2 by negations, although D_2 needs the possibility operator.

$\neg_d$ is similar to $\neg$, but these are not equivalent. The motivation of introducing $\neg_d$ is to interpret discursive negation as the negation used by an intuitionist in the discursive context. Unfortunately, intuitionistic negation is not a discursive negation. And we need to re-interpret it as $\neg_d$. Based on $\neg_d$, we can define $\rightarrow_d$ and $\wedge_d$.

Discursive implication $\rightarrow_d$ and discursive conjunction $\wedge_d$ can be respectively introduced by definition as follows.

$$A \rightarrow_d B =_{def} \neg_d A \vee B$$
$$A \wedge_d B =_{def} \sim \neg_d A \wedge B$$

Observe that $A \rightarrow (\sim A \rightarrow B)$ is not a theorem in $CDLSN$ while $A \rightarrow (\neg_d A \rightarrow B)$ is a theorem in $CDLSN$. The axiomatization of $CDLSN$ is that of N^- with the following three axioms.

(CDLSN1) $\neg_d A \rightarrow (A \rightarrow B)$
(CDLSN2) $(A \rightarrow B) \rightarrow ((A \rightarrow \neg_d B) \rightarrow \neg_d A)$
(CDLSN3) $A \rightarrow \sim \neg_d A$

Here, an explanation of these axioms may be in order. (CDLSN1) and (CDLSN2) describe basic properties of intuitionistic negation. By (CDLSN3), we show the connection of $\sim$ and $\neg_d$. The intuitive interpretation of $\sim \neg_d$ is like possibility under our semantics developed below.

$\neg_d$ is weaker than $\neg$. Vorob'ev [8] proposed a constructive logic having both strong and intuitionistic negation. It extends N with the following two axioms:

$\sim \neg A \leftrightarrow A$
$\sim A \rightarrow \neg A$, where A is atomic

If we replace (CDLSN3) by the axiom of the form $\sim \neg_d A \leftrightarrow A$ and add the axiom $\sim A \rightarrow \neg_d A$, then $\neg_d$ agrees with $\neg$. Thus, it is not possible to identify $\neg$ and $\neg_d$ in our axiomatization.

We use $\vdash A$ to mean that A is a theorem in $CDLSN$. Here, the notion of a proof is defined as usual. Let $\Gamma = \{B_1, ..., B_n\}$ be a set of formulas and A be a formula. Then, $\Gamma \vdash A$ iff $\vdash \Gamma \rightarrow A$.

Notice that $\neg_d$ has some similarities with $\neg$, as the following lemma indicates.

Lemma 1: The following formulas are provable in $CDLSN$.

(1) $\vdash A \rightarrow \neg_d \neg_d A$
(2) $\vdash (A \rightarrow B) \rightarrow (\neg_d B \rightarrow \neg_d A)$
(3) $\vdash (A \wedge \neg_d A) \rightarrow B$
(4) $\vdash \neg_d (A \wedge \neg_d A)$
(5) $\vdash (A \rightarrow \neg_d A) \rightarrow \neg_d A$

Proof. Ad(1): From (CDSLN1) and Int^+ (i.e. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))$, we have (i).

(i) $\vdash A \rightarrow (\neg_d A \rightarrow A)$

(ii) is an instance of (CDLSN2).

(ii) $\vdash (\neg_d A \rightarrow A) \rightarrow ((\neg_d A \rightarrow \neg_d A) \rightarrow \neg_d \neg_d A)$

(iii) is a theorem of Int^+ (i.e. $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))$

(iii) $\vdash (A \rightarrow (\neg_d A \rightarrow A)) \rightarrow (((\neg_d A \rightarrow A) \rightarrow ((\neg_d A \rightarrow \neg_d A) \rightarrow \neg_d \neg_d A)) \rightarrow (A \rightarrow ((\neg_d A \rightarrow \neg_d A) \rightarrow \neg_d \neg_d A)))$

From (i) and (iii) by (MP), we have (iv).

(iv) $\vdash (((\neg_d A \rightarrow A) \rightarrow ((\neg_d A \rightarrow \neg_d A) \rightarrow \neg_d \neg_d A)) \rightarrow (A \rightarrow ((\neg_d A \rightarrow \neg_d A) \rightarrow \neg_d \neg_d A)))$

From (ii) and (iv) by (MP), we have (v).

(v) $\vdash A \rightarrow ((\neg_d A \rightarrow \neg_d A) \rightarrow \neg_d \neg_d A))$

By $\vdash (A \to (B \to C)) \to (B \to (A \to C))$, we can derive (vi).

(vi) $\vdash (\neg_d A \to \neg_d A) \to (A \to \neg_d \neg_d A)$

Since $\vdash A \to A$, we have (vii).

(vii) $\vdash \neg_d A \to \neg_d A$

From (vi) and (vii) by (MP), we can finally obtain (viii).

(viii) $\vdash A \to \neg_d \neg_d A$

Ad(2): By (CDLSN2), we have (i).

(i) $\vdash (A \to B) \to ((A \to \neg_d B) \to \neg_d A)$

(ii) is a theorem of Int^+.

(ii) $\vdash (\neg_d B \to (A \to \neg_d B)) \to (((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A))$

(iii) is an instance of $A \to (B \to A)$, which is the axiom of Int^+.

(iii) $\vdash \neg_d B \to (A \to \neg_d B)$

From (ii) and (iii) by (MP), (iv) is obtained.

(iv) $\vdash ((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A)$

(v) is a theorem of Int^+.

(v) $\vdash ((A \to B) \to ((A \to \neg_d B) \to \neg_d A) \to ((((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A)) \to ((A \to B) \to (\neg_d B \to \neg_d A)))$

From (i) and (v) by (MP), (vi) can be proved.

(vi) $\vdash ((A \to \neg_d B) \to \neg_d A) \to (\neg_d B \to \neg_d A)) \to ((A \to B) \to (\neg_d B \to \neg_d A))$

From (iv) and (vi) by (MP), we can reach (vii).

(vii) $\vdash (A \to B) \to (\neg_d B \to \neg_d A)$

Ad(3): By (CDLSN1), we have (i).

(i) $\vdash \neg_d A \to (A \to B)$

From $\vdash (A \to (B \to C)) \to (B \to (A \to C))$, we can derive (ii).

(ii) $\vdash A \to (\neg_d A \to B)$

Since $\vdash (A \to (B \to C)) \to ((A \wedge B) \to C)$, we have (iii).

(iii) $\vdash (A \to (\neg_d A \to B)) \to ((A \wedge \neg_d A) \to B)$

From (ii) and (iii) by (MP), we can obtain (iv).

(iv) $\vdash (A \wedge \neg_d A) \to B$

Ad(4): By (3), we have (i) and (ii).

(i) $\vdash (A \wedge \neg_d A) \to B$
(ii) $\vdash (A \wedge \neg_d A) \to \neg_d B$

From (CDLSN2), (iii) holds.

(iii) $((A \wedge \neg_d A) \to B) \to (((A \wedge \neg_d A) \to \neg_d B) \to \neg_d (A \wedge \neg_d A))$

From (i) and (iii) by (MP), we have (iv).

(iv) $((A \wedge \neg_d A) \to \neg_d B) \to \neg_d (A \wedge \neg_d A)$

From (ii) and (iv) by (MP), we can derive (v).

(v) $\vdash \neg_d (A \wedge \neg_d A)$

Ad(5): By (CDLSN2), we have (i).

(i) $\vdash (A \to A) \to ((A \to \neg_d A) \to \neg_d A)$

(ii) is a theorem of Int^+.

(ii) $\vdash A \to A$

From (i) and (ii) by (MP), we can obtain (iii).

(iii) $(A \to \neg_d A) \to \neg_d A$

It should be, however, pointed out that the following formulas are not provable in $CDLSN$.

$\nvdash \sim (A \wedge \sim A)$
$\nvdash A \vee \sim A$
$\nvdash (A \to B) \to (\sim B \to \sim A)$
$\nvdash \neg_d \neg_d A \to A$
$\nvdash A \vee \neg_d A$
$\nvdash (\neg_d A \to A) \to A$

$\not\vdash \sim \neg_d A \rightarrow A$
$\not\vdash A \rightarrow_d A$

4. *Kripke Semantics*

It is possible to give a Kripke semantics for $CDLSN$ which is a discursive modification of that for N provided by Thomason [7]. Let PV be a set of propositional variables and p be a propositional variable, and For be a set of formulas. A $CDLSN$-*model* is a tuple $\langle W, w_0, R, V\rangle$, where $W \neq \emptyset$ is a set of *worlds*, $w_0 \in W$ satisfying $\forall w(w_0 R w)$, $R \subseteq W \times W$ is a reflexive and transitive relation, and $V : PV \times W \rightarrow \{0, 1\}$ is a partial *valuation* satisfying:

$V(p, w) = 1$ and $wRv \Rightarrow V(p, v) = 1$
$V(p, w) = 0$ and $wRv \Rightarrow V(p, v) = 0$

for any formula $p \in PV$ and $w, v \in W$. Here, $V(p, w) = 1$ is read "p is true at w" and $V(p, w) = 0$ is read "p is false at w", respectively. Both truth and falsity are independent statuses given by a constructive setting.

We can now extend V for any formula A, B in a tandem way as follows.

$V(\sim A, w) = 1$	iff	$V(A, w) = 0$.
$V(A \wedge B, w) = 1$	iff	$V(A, w) = 1$ and $V(B, w) = 1$.
$V(A \vee B, w) = 1$	iff	$V(A, w) = 1$ or $V(B, w) = 1$
$V(A \rightarrow B, w) = 1$	iff	$\forall v(wRv$ and $V(A, v) = 1 \Rightarrow V(B, v) = 1)$
$V(\neg_d A, w) = 1$	iff	$\forall v(wRv \Rightarrow V(A, v) = 0)$
$V(\sim A, w) = 0$	iff	$V(A, w) = 1$
$V(A \wedge B, w) = 0$	iff	$V(A, w) = 0$ or $V(B, w) = 0$
$V(A \vee B, w) = 0$	iff	$V(A, w) = 0$ and $V(B, w) = 0$
$V(A \rightarrow B, w) = 0$	iff	$V(A, w) = 1$ and $V(B, w) = 0$
$V(\neg_d A, w) = 0$	iff	$\exists v(wRv$ and $V(A, v) = 1)$

Additionally, we need the following condition:

$V(A \wedge \sim A, w) = 1$ for some A and some w.

This condition is used to invalidate $(A \wedge \sim A) \rightarrow B$, and guarantees the paraconsistency of $\sim$ in $CDSLN$.

Here, observe that truth and falsity conditions for $\sim \neg_d A$ are implicit in the above clauses from the equivalences such that $V(\sim \neg_d A, w) = 1$ iff $V(\neg_d A, w) = 0$, and $V(\sim \neg_d A, w) = 0$ iff $V(\neg_d A, w) = 1$. One can claim that $\sim \neg_d$ behaves as a modality. In this regard, we do not need to introduce a possibility operator into $CDLSN$ as a primitive.

We say that A is *valid*, written $\models A$, iff $V(A, w_0) = 1$ in all $CDLSN$ models. Let $\Gamma = \{B_1, ..., B_n\}$ be a set of formulas. Then, we say that Γ *entails* A, written $\Gamma \models A$, iff $\Gamma \rightarrow A$ is valid.

Lemma 2 states the monotonicity of valuation in a Kripke model.

Lemma 2: The following hold for any formula A which is not of the form $\sim \neg_d B$, and any worlds $w, v \in W$.

$$V(A, w) = 1 \text{ and } wRv \;\Rightarrow\; V(A, v) = 1,$$
$$V(A, w) = 0 \text{ and } wRv \;\Rightarrow\; V(A, v) = 0.$$

Proof. By induction on A.
ad($\sim$): Suppose $V(\sim A, w) = 1$ and wRv. Then, we have that $V(A, w) = 0$ and wRv. By induction hypothesis (IH), we have that $V(A, v) = 0$, i.e. $V(\sim A, v) = 1$.

Suppose $V(\sim A, w) = 0$ and wRv. Then, we have that $V(A, w) = 1$ and wRv. By (IH), we have that $V(A, v) = 1$, i.e. $V(\sim A, v) = 0$.

Ad($\wedge$): Suppose $V(A \wedge B, w) = 1$ and wRv. Then, we have $V(A, w) = 1$ and $V(B, w) = 1$. By (IH), $V(A, v) = 1$ and $V(B, v) = 1$, i.e. $V(A \wedge B, v) = 1$.

Suppose $V(A \wedge B, w) = 0$ and wRv. Then, we have $V(A, w) = 0$ or $V(B, w) = 0$. By (IH), $V(A, v) = 0$ or $V(B, v) = 0$, i.e. $V(A \wedge B, v) = 0$.

Ad($\vee$): Suppose $V(A \vee B, w) = 1$ and wRv. Then, we have $V(A, w) = 1$ or $V(B, w) = 1$. By (IH), $V(A, v) = 1$ or $V(B, v) = 1$, i.e. $V(A \vee B, v) = 1$.

Suppose $V(A \vee B, w) = 0$ and wRv. Then, we have $V(A, w) = 0$ and $V(B, w) = 0$. By (IH), $V(A, v) = 0$ and $V(B, v) = 0$, i.e. $V(A \vee B, v) = 0$.

Ad($\rightarrow$): Suppose $V(A \rightarrow B, w) = 1$ and wRv. Then, we have $\forall v(wRv$ and $V(A, v) = 1 \Rightarrow V(B, v) = 1)$. By (IH) and the transitivity of R, $\forall z(vRz$ and $V(A, z) = 1 \Rightarrow V(B, z) = 1)$, i.e. $V(A \rightarrow B, v) = 1$.

Suppose $V(A \rightarrow B, w) = 0$ and wRv. Then, we have $V(A, w) = 1$ and $V(B, w) = 0$. By (IH), $V(A, v) = 1$ and $V(B, v) = 0$, i.e. $V(A \rightarrow B, v) = 0$.

Lemma 2 does not hold for the formula of the form $\sim \neg_d A$. We can easily construct a counter model. We only treat the case of $V(\sim \neg_d A, w) = 1$. The case of $V(\sim \neg_d A, w) = 0$ is similar. Assume that $V(\sim \neg_d A, w) = 1$ and wRv. Then, $V(\neg_d A, w) = 0$ iff $\exists u(wRu$ and $V(A, u) = 1)$. Now, suppose that there exists a world t distinct from u such that vRt and a valuation such that $V(A, t) = 0$. This means that $V(\sim \neg_d A, v) = 0$. Thus, $V(\sim \neg_d A, w) = 1$ and wRv, but $V(\sim \neg_d A, v) = 0$.

We think that the fact is intuitive because $\sim \neg_d A$ behaves as possibility. There are no reasons for possibility in discourse to satisfy the monotonicity.

Next, we present a soundness theorem.

Theorem 3: (soundness) $\vdash A \ \Rightarrow \models A$.

Proof. It suffices to check that (CDLSN1), (CDLSN2) and (CDLSN3) are valid and (MP) preserves validity. The proof of preservation of validity under (MP) is well-known in constructive and intuitionistic logic. Thus, we here prove the validity of three axioms.

Ad(CDLSN1): Suppose it is not valid. Then, $V(\neg_d A, w_0) = 1$ and $V(A \rightarrow B, w_0) \neq 1$. From the first conjunct, $\forall v(w_0 Rv \Rightarrow V(A, v) \neq 1)$ holds. From the second conjunct, $\exists v(w_0 Rv$ and $V(A, v) = 1$ and $V(B, v) \neq 1)$. However, $V(A, v) = 1$ and $V(A, v) \neq 1$ are contradictory.

Ad(CDLSN2): Suppose it is not valid. Then, $V(A \rightarrow B, w_0) = 1$ and $V(A \rightarrow \neg_d B, w_0) = 1$ and $V(\neg_d A, w_0) \neq 1$. From the first conjunct, $\forall v(w_0 Rv$ and $V(A, v) = 1 \Rightarrow V(B, v) = 1)$ holds. From the second conjunct, $\forall v(w_0 Rv$ and $V(A, v) = 1 \Rightarrow V(\neg_d B, v) = 1)$ iff $\forall v(wRv$ and $V(A, v) = 1 \Rightarrow \forall z(vRz \Rightarrow V(A, z) \neq 1)$. From the third conjunct, $\exists v(w_0 Rv$ and $V(A, v) = 1$ holds. However, $V(A, v) = 1$ and $V(A, z) \neq 1$ for any z such that vRz are contradictory.

Ad(CDLSN3): Suppose it is not valid. Then, $V(A, w_0) = 1$ and $V(\sim \neg_d A, w_0) \neq 1$. From the second conjunct, we have $V(\neg_d A, w_0) \neq 0$ iff $\forall v(w_0 Rv \Rightarrow V(A, v) \neq 1)$. However, $V(A, w_0) = 1$ and $V(A, v) \neq 1$ for any v such that $w_0 Rv$ are contradictory.

Theorem 3 can be generalized as a strong form, i.e. $\Gamma \vdash A \ \Rightarrow \Gamma \models A$.

Now, we give a completeness proof. We say that a set of formulas Γ^* is a *maximal non-trivial discursive theory* (mntdt) iff (1) Γ^* is a theory, (2) Γ^* is *non-trivial*, i.e. $\Gamma^* \not\vdash B$ for some B, (3) Γ^* is *maximal*, i.e. $A \in \Gamma^*$ or $A \notin \Gamma^*$, (4) Γ^* is *discursive*, i.e. $\neg_d A \notin \Gamma^*$ iff $\sim \neg_d A \in \Gamma^*$. Here, discursiveness is needed to capture the property of discursive negation.

Lemma 4: For any mntdt Γ and any formula A, B, the following hold:

(1) $A \wedge B \in \Gamma$ iff $A \in \Gamma$ and $B \in \Gamma$
(2) $A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$
(3) $A \rightarrow B \in \Gamma$ iff $\forall \Delta(\Gamma \subseteq \Delta$ and $A \in \Delta \Rightarrow B \in \Delta)$

(4) $\neg_d A \in \Gamma$ iff $\forall\Delta(\Gamma \subseteq \Delta \Rightarrow A \notin \Delta)$
(5) $\sim (A \wedge B) \in \Gamma$ iff $\sim A \in \Gamma$ or $\sim B \in \Gamma$
(6) $\sim (A \vee B) \in \Gamma$ iff $\sim A \in \Gamma$ and $\sim B \in \Gamma$
(7) $\sim (A \rightarrow B) \in \Gamma$ iff $A \in \Gamma$ and $\sim B \in \Gamma$
(8) $\sim\sim A \in \Gamma$ iff $A \in \Gamma$
(9) $\sim \neg_d A \in \Gamma$ iff $\exists\Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta)$.

Proof. We only prove (4) and (9). Other cases are similarly justified from the literature on constructive logic (cf. Thomason [7]).

Ad(4): $\neg_d A \in \Gamma$ iff (by axiom (CDLSN1)) $A \rightarrow B \in \Gamma$ iff (by Lemma 4 (3)) $\forall\Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta \Rightarrow B \in \Delta)$. Since Γ is non-trivial, $B \notin \Gamma$ for some B. Thus, $B \in \Delta$ does not always hold, i.e. $\forall\Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta \Rightarrow false)$ iff $\forall\Delta(\Gamma \subseteq \Delta \Rightarrow A \notin \Delta)$.

Ad(9): We prove it by contraposition from (4). Contraposition can derive $\exists\Delta(\Gamma \subseteq \Delta \text{ and } A \in \Delta)$ by negating the left and right sides of (4). Then, it is shown to be equivalent to $\neg_d A \notin \Gamma$. By (discursiveness), $\neg_d A \notin \Gamma$ iff $\sim \neg_d A \in \Gamma$.

Based on the maximal non-trivial discursive theory, we can define a canonical model $(\Gamma, \subseteq, V)$ such that Γ is a mntdt, $\subseteq$ is the subset relation, and V is a valuation satisfying the conditions that $V(p, \Gamma) = 1$ iff $p \in \Gamma$ and that $V(p, \Gamma) = 0$ iff $\sim p \in \Gamma$.

Next lemma is a truth lemma.

Lemma 5: (truth lemma) For any mntdt Γ and any A, we have the following:

$V(A, \Gamma) = 1$ iff $A \in \Gamma$
$V(A, \Gamma) = 0$ iff $\sim A \in \Gamma$

Proof. It suffices to check the case $A = \neg_d B$.

$V(\neg_d B, \Gamma) = 1$	iff	$\forall\Delta \in \Gamma^*(\Gamma \subseteq \Delta \Rightarrow V(B, \Delta) \neq 1)$
(IH)	iff	$\forall\Delta \in \Gamma^*(\Gamma \subseteq \Delta \Rightarrow B \notin \Delta)$
(Lemma 4 (4))	iff	$\neg_d B \in \Gamma$
$V(\neg_d B, \Gamma) = 0$	iff	$\exists\Delta \in \Gamma^*(\Gamma \subseteq \Delta \text{ and } V(B, \Delta) = 1)$
(IH)	iff	$\exists\Delta \in \Gamma^*(\Gamma \subseteq \Delta \text{ and } B \in \Delta)$
(Lemma 4 (9))	iff	$\sim \neg_d B \in \Gamma$

Then, we can state the (strong) completeness of $CDLSN$ as follows:

Theorem 6: (completeness) $\Gamma \models A \;\Rightarrow\; \Gamma \vdash A$.

Proof. Assume $\Gamma \not\vdash A$. Then, by Lindenbaum lemma, there is a mntdt Γ such that $A \notin \Gamma$. By using a canonical model defined above, we have $V(A,\Gamma) \neq 1$ by Lemma 5. Consequently, completeness follows.

Finally, we justify the formal properties of $CDLSN$ as a discursive logic. It is extremely important because we can understand the differences of $CDLSN$ and standard discursive logics like D_2. As mentioned in section 1, Jaskowski suggested three conditions of discursive logics. We check them here.

$CDLSN$ is *discursive*. First, $\sim (A \wedge \sim A)$ does not hold. The explosion also fails, i.e. $A, \sim A \not\vdash B$. But, these hold for $\neg_d$ (cf. Lemma 1), and it is not a problem because explosion should be valid for plausible discourses.

Note that the adjunction of the form $\vdash A, \vdash B \Rightarrow \vdash A \wedge_d B$ does not hold in $CDLSN$. But, it holds for $\wedge$.

Second, in $CDLSN$, most of the theses of constructive logic are valid. Since $CDLSN$ has a constructive base, it is different from D_2 whose base is classical logic.

Third, we can give an intuitive interpretation for $CDLSN$ by means of Kripke models as discussed below.

$CDLSN$ is *constructive* because the law of excluded middle, which is a non-constructive principle, does not hold. As discussed above, N^- is a constructive logic, and the fact is not surprising.

From our Kripke semantics given above, we can give an intuitive interpretation of $CDLSN$. The interpretations of the logical symbols of N^- are obvious, and we concentrate on discursive logical symbols.

Here, it may be helpful to explain the interpretation by a brief example. Consider a *discourse* which consists of several persons who are interested in some subjects. Each person has knowledge about subjects, and a discourse is plausibly expanded by adding other persons.

In this setting, a world in our semantics could be identified with a discourse just given. So, the logical symbols can be interpreted with reference to a discourse.

Since the interpretations of $\neg_d$ are crucial, we begin with it, namely

$\neg_d A$ is true iff A is false in all plausible growing discourses,
$\neg_d A$ is false iff A is true in some plausible growing discourse.

Here, the second clause corresponds to the possibility used in discursive logic. Note here that the plausible growth of discourse implies the increase of information (or knowledge) in view of constructive setting.

Other discursive logical symbols can be read as follows:

$A \wedge_d B$ is true iff A is true in one discourse and B is true in another plausible discourse.
$A \rightarrow_d B$ is true iff if A is true in certain plausible discourse then B is true in a discourse.

The interpretations of $\vee_d$ and $\leftrightarrow_d$ can be obtained by definition. The important point here is that the primitive discursive connective is $\neg_d$.

In our approach, two kinds of negations are used and it is necessary to compare them. $\sim$ is a constructive negation which can express constructive falsity of the proposition, whereas $\neg_d$ is a discursive negation of the proposition with modal flavor.

They can express the possibility operator needed in discursive logic as $\sim \neg_d$. Here, $\sim$ behaves classical-like negation and $\neg_d$ modal-like negation. We know in classical modal logic that $\Diamond A \equiv -\Box - A$ holds. Here, $-$ is classical negation and $\equiv$ is classical equivalence. It is therefore natural to consider two negations in classical-like and modal-like way.

From the above discussion, $CDLSN$ is shown to be a constructive discursive logic which is compatible with Jaskowski's original ideas. It means that a constructivist can formally perform discursive reasoning.

5. *Concluding Remarks*

We proposed a constructive version of discursive logic $CDLSN$ with an axiomatization and semantics. We set up it as a natural modification of Almukdad and Nelson's N^- [1] with $\neg_d$. We gave some formal properties of $CDLSN$ including completeness.

Alternatively, $CDLSN$ can be interpreted as the system which weakens intuitionistic negation $\neg$ in N^-. However, the alternative formulation does not affect the results in this paper. We believe that this system seems to be new in the literature.

There are two advantages of the proposed system. First, it is constructively intuitive because we have a Kripke semantics. In view of the incompleteness of discourse, constructive approach seems attractive for discursive logic.

Second, it dispenses with modal operators to define discursive connectives. In other words, the possibility operator used in standard discursive logic can be replaced by the combination of two negations. However, it may be possible to introduce other types of discursive connectives as primitives.

Although this paper focuses on theoretical aspects of constructive discursive logic, the logic has many possible applications. For example, it may be worth studying *non-monotonic reasoning* and *multi-agent* in constructive discursive logic. We hope to report interesting applications of the proposed logic in future papers.

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