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CANTOR-VON NEUMANN SET-THEORY

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Abstract

In this elementary paper we establish a few novel results in set theory; their interest is wholly foundational-philosophical in motivation. We show that in *Cantor-Von Neumann Set-Theory (CVN)*, which is a reformulation of Von Neumann's original theory of *functions and things* that does not introduce 'classes' (let alone 'proper classes'), developed in the 1920ies, both the Pairing Axiom and 'half' the Axiom of Limitation are redundant — the last result is novel. Further we show, in contrast to how things are usually done, that some theorems, notably the Pairing Axiom, can be proved *without* invoking the Replacement Schema (F) and the Power-Set Axiom. Also the Axiom of Choice is redundant in CVN, because it a *theorem* of CVN. The philosophical interest of Cantor-Von Neumann Set-Theory, which is very succinctly indicated, lies in the fact that it is far better suited than Zermelo-Fraenkel Set-Theory as an axiomatisation of what Hilbert famously called *Cantor's Paradise*. From Cantor one needs to jump to Von Neumann, over the heads of Zermelo and Fraenkel, and then reformulate.

0. *Introduction*

In 1928, Von Neumann published his grand axiomatisation of Cantorian Set-Theory [1925; 1928]. Although Von Neumann's motivation was thoroughly Cantorian, he did not take the concept of a set and the membership-relation as primitive notions, but the concepts of a *thing* and a *function* — for reasons we do not go into here. This, and Von Neumann's cumbersome notation and terminology (II-things, II.I-things) are the main reasons why initially his theory remained comparatively obscure. Then came Paul Bernays [1937–1953; 1957]. He dressed up Von Neumann's theory in logicist *haute couture*, notably with *classes* (extensions of predicates), and cut out its Cantorian heart, the Axiom of Limitation (see below). And then, in 1938, came Gödel. He took this theory of *sets and classes* as the framework for proving his famous consistency results of the Axiom of Choice and the Generalised Continuum Hypothesis. Gödel also added the notion of a 'proper class' — as if extensions of predicates (i.e. classes) suddenly stop being extensions and become 'improper' when they happen to be sets too. The resulting theory of *sets and classes*, which is usually called 'Von Neumann-Bernays', 'Von Neumann-Bernays-Gödel' or even 'Gödel-Bernays' Set-Theory,[1] thus became known and was used more and more, as time passed, by logicians and set-theoreticians; it has however remained little known among working mathematicians other than set-theoreticians or logicians. The standard axiomatisation still is *Zermelo-Fraenkel Set-Theory* (ZFC), glossing over possible qualms concerning the Axiom of Choice.[2]

Elsewhere we have argued that not ZFC, but what we propose to call *Cantor-Von Neumann Set-Theory* (CVN) is the best available axiomatisation of what Hilbert famously baptised "Cantor's Paradise".[3] The theory CVN results when Von Neumann's theory of functions and things is reformulated in the standard, 1st-order language of pure set-theory (denoted by $\mathcal{L}_\in$) extended with a single primitive set V, and certain redundant axioms are deleted.

The purpose of the present note is to write down Cantor-Von Neumann Set-Theory formally (Section 2), to prove that 'one-and-a-half' axiom is redundant, and to prove some axioms of ZFC in CVN in a manner that differs from the usual deductions (Section 3). The fact that half of the axiom of Limitation is redundant has gone unnoticed for about eighty years — as far as this author is aware of. But first, in order to have some idea what the *conceptual* watershed between ZFC and CVN consists in, and *a fortiori* to have a solid motivation for considering the theory CVN at all, we begin by providing a *very* succinct overview of this watershed (Section 1). We emphasise that *the subject of the present paper is not this conceptual watershed, but a few rigourous results that are the spin-off of a philosophical-foundational inquiry into Cantorian Set-Theory* (cf. Muller [2010]).

[1] See Stoll [1963: 318], Fraenkel *et al.* [1973: 128], Jech [1978: 76], Mostowski in Müller [1976: 325], Enderton [1977: 10] and Kunen [1980: 35]. To add to the confusion, Fraenkel *et al.* [1973: 137] call what is almost our Cantor-Von Neumann Set-Theory '$G\wedge(*)$', where $(*)$ stands for the Axiom of Limitation (the language $\mathcal{L}_\in$ is then extended with 'class-variables' to language $\mathcal{L}_\in^*$). But this is *Von Neumann's* theory, not *Gödel's*!

[2] Cf. Fraenkel *et al.*'s overview [1973], Ch. II. Ironically, the name 'Zermelo-Fraenkel' is due to Von Neumann [1961: 321, 348], who also provided the (correct formulation of the) Axiom of Replacement, who added the Axiom of Regularity, and who created the canonical theory of ordinal and cardinal numbers; all of this is standardly transplanted to Zermelo's [1908] axiomatisation in order to obtain ZFC.

[3] Muller [2010]. This work gratefully builds on Hallett's seminal monograph [1984] on the philosophy and history of Cantorian Set-Theory.

1. *Zermelo-Fraenkel versus Cantor-Von Neumann*

First comes a string of definitions (which we do not spell out formally[4]): a set is *potential-infinite* iff it sustains a linear ordering that has no top in the set; a set is *finite* iff it can be bijected to $\{1, 2, 3, \ldots, n\}$ for some $n \in \omega$, where ω is the first limit Von Neumann-ordinal; a set is *actual-infinite* iff a proper subset of it can be surjected onto it; a set is *ultimate* (Quine) iff no set has it as a member; a set is *Cantoresque* iff it is not ultimate; set is *absolute-infinite* iff it is equinumerous to the set of all Cantoresque sets (Von Neumann); a set is *increasable* iff it can be surjected onto some more inclusive set; a set is *transfinite* iff it is actual-infinite and increasable. A set is a *Cantor-set* (Cantor's concept of a *Menge*) iff it is Cantoresque, increasable, well-founded, well-orderable, has a unique cardinal number, and has a unique ordinal number as soon as it is ordered well; and, finally, a set is *combinatorially inept* iff it does not arise in the cumulative hierarchy.

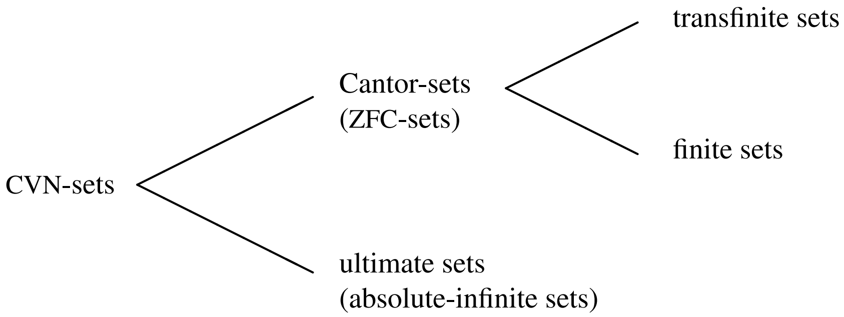
Standard *Zermelo-Fraenkel Set-Theory* (ZFC) is defined as the 1st-order deductive closure of the Axioms of Extensionality, Union, Power, Infinity, Replacement, Regularity and Choice; Replacement entails Separation, and Replacement and Power together entail Pairing.[5] *Cantor-Von Neumann Set-Theory* (CVN) is formulated in $\mathcal{L}_\in$ extended with a single 'logical constant' V, a primitive *set* (call this language $\mathcal{L}_\in^{\mathsf{V}}$); we define CVN as the 1st-order deductive closure of the Axioms of Universe (V *includes* but does not *contain* every set), Extensionality, Power, Infinity, Set-Existence (every predicate with only bounded quantifiers, possibly with set-parameters, has a set-extension of Cantoresque members), Regularity and Weak Limitation (all ultimate sets are absolute-infinite). Von Neumann's Axiom of Limitation is the conjunction of Weak Limitation and its converse: exactly ultimate sets are absolute-infinite (cf. Section 2 for details).

One proves that in ZFC all sets are Cantor-sets and that neither absolute-infinite nor ultimate sets exist. In CVN one proves that a set is potential-infinite iff it is actual-infinite; that a set is ultimate iff it is absolute-infinite (Limitation); that a set is ultimate iff it is not Cantoresque; that every set is either absolute-infinite or transfinite or finite; that every set is either ultimate or a Cantor-set; and that every set is either increasable or ultimate.

A.H. Kruse essentially proved that CVN is a conservative, hence an equiconsistent deductive extension of ZFC: every theorem of CVN in which only

[4] See Muller [2010] for the formal definitions and the easy proofs of the theorems we are about to report.

[5] See Fraenkel *et al.* [1973: 22, 52], Lévy [1979: 23–24], Suppes [1960: 237], Stoll [1963: 304].



Cantor-sets occur already is a theorem in ZFC.[6] Thus in CVN one remarkably demonstrates, rather than postulates (as in ZFC), that *every Cantor-set has a union-set and a choice-set* from axioms that do not smack of choice.[7] Further, CVN is finitely axiomatisable, in contradistinction to ZFC. Model-theoretically, the whole in every model of CVN that contains exactly the Cantor-sets is a model of ZFC; and every model of ZFC can be extended to become a model of CVN, such that the whole of all sets (in the model of ZFC) becomes the model of V of CVN in the extended model of CVN.

Now, various assertions of Cantor are *proved* in CVN, whereas they are *disproved* in ZFC, e.g. that the "actual-infinite has to be subdivided into the *increasable actual-infinite* and the *unincreasable actual-infinite*"[8] (which Cantor called the "transfinite" and the "absolute-infinite", respectively[9]); that the whole of all Cantor-sets is a "perfectly well-defined" absolute-infinite set;[10] that "every potential infinity presupposes an actual-infinity";[11] that an absolute-infinite set is "mathematically indeterminable"[12] (when interpreted as combinatorial ineptitude, admittedly with a dosis of wisdom with

[6] Cf. Fraenkel *et al.* [1973: 136–137]. Kruse proved this for VN^* (see Table); it then follows that it also holds for CVN.

[7] See Fraenkel *et al.* [1973: 137].

[8] Cantor [1932: 375].

[9] Cantor [1932: 405].

[10] Cantor [1932: 448].

[11] Cantor [1932: 410–411].

[12] Cantor [1932: 375].

hindsight) and "cannot be conceived of as a member of another set"[13] (they are ultimate); that absolute-infinite sets "have to be admitted and acknowledged"[14] (they exist); and more. All in all, CVN provides a rigourous *legalisation* for a host of informal claims of Cantor, whereas ZFC *outlaws* them. For this reason, CVN is far better suited to be the axiomatisation of Cantor's paradise than ZFC. In a single sentence: to obtain a rigorous legislation of Cantor's Paradise, jump straight to Von Neumann, over the heads of Zermelo and Fraenkel, and then reformulate.

So much for a brief comparison between ZFC and CVN. We refer to Muller [2010] for an elaborate inquiry into CVN, its heuristics and motivation, and into how it compares with ZFC conceptually.

2. *Axiomatics*

To rehearse, the 1st-order formal language $\mathcal{L}_\in$ of pure Set-Theory has only *set-variables* $(A, B, C, D, F, \ldots\ X, Y, Z$; occasionally we use m, n, p as finite-ordinal-variables), and the membership-relation $(\in)$ as its only dyadic predicate-constant. The background logic, which is classical 1st-order predicate logic with identity $(=)$. Then $\ulcorner X \in Y \urcorner$ and $\ulcorner X = Y \urcorner$ are the only types of atomic sentence. In the language of CVN, denoted by $\mathcal{L}_\in^{\mathsf{V}}$, we have in addition to $\mathcal{L}_\in$ one primitive *set*, V. We use $\ulcorner \equiv \urcorner$ for term-definition, and $\ulcorner$iff$\urcorner$ for sentence- and predicate-definition.

Throughout we assume that all the usual definitions are in force (power-set $\wp X$ of set X, union-set $\bigcup X$ of set X, the empty set $\varnothing$, etc.; see Fraenkel *et al.* [1973], Chapter II). We emphasise that $\mathcal{L}_\in^{\mathsf{V}}$ does *not* contain Bernaysian 'classes', Quinean 'virtual sets' or Gödelian 'proper classes'. We use boldfaced capitals for ultimate set-names (Kunen-convention). We next spell out the axioms of CVN formally.

The *Universe Axiom* says that V includes every set:

$$\text{(Univ)} \quad \forall X : \ X \subseteq \mathsf{V} \ . \tag{1}$$

We introduce the concepts of ultimacy and Cantoresqueness formally, which are each other's negations:

$$\text{Ultim}(X) \ \text{ iff } \ \forall Y : \ X \notin Y \ ; \qquad \text{Cantq}(X) \ \text{ iff } \ \exists Y : \ X \in Y \ . \tag{2}$$

[13] Grattan-Guinness [1971: 119].

[14] Cantor [1932: 205].

Lemma 1: (Ultimacy Lemma) *A set is ultimate iff* V *does not contain it, or in other words:* V *is the set of all and only Cantoresque sets:* $\forall X$: $\text{Ultim}(X) \longleftrightarrow X \in \mathsf{V}$.

Proof (Univ). Let X be ultimate. Then by definition (2), X is not the member of every set, V included. Hence $X \notin \mathsf{V}$.

Let $X \notin \mathsf{V}$. By Univ (1), $Y \subset \mathsf{V}$ for every Y, which means that $Z \in Y \longrightarrow Z \in \mathsf{V}$ for every Z. Contraposing yields that $Z \notin \mathsf{V} \longrightarrow Z \notin Y$ for *every* Z, hence also for X, which permits us to deduce that $X \notin Y$. Since Y is arbitrary, X is ultimate. $\Box$

The Axiom of Set-Existence (SetEx) asserts that for any predicate $\varphi(\cdot, Y)$, which may have any number of arbitrary set-parameters $Y_1, Y_2, \ldots, Y_n$ (abbreviated by Y), and which does not contain unbounded quantifiers but may contain quantifiers running over *all* Cantoresque sets (collected in V), there exists a set S of all Cantoresque sets for which the predicate holds:[15]

$$\text{(SetEx)} \quad \forall Y,\ \exists S \subseteq \mathsf{V},\ \forall X \in \mathsf{V} \big(X \in S \ \longleftrightarrow\ \varphi(X, Y) \big)\ . \qquad (3)$$

Notice that in general the set-extension S of SetEx (3) can be *ultimate*, but all its members, the sets that fall under the predicate, are Cantoresque (they cannot be ultimate because they are *members*, of S); whether set S actually is ultimate or not is something we have to prove on the basis of the other axioms. Further, the fact that the variable X in (3) is bounded to V makes it possible to reduce this list of denumerably many axioms to eight axioms, which means that CVN is finitely axiomatisable.[16] The restriction to bounded quantifiers in Set-Existence betrays that it has a whiff of predicativity in it — but certainly not more than a whiff, because quantification over *all* Cantorian sets in V still is light-years removed from Russellian typification and Quinean stratification.

For convenience, we define $\mathsf{V}_{\varnothing}$ as the set of all Cantoresque, non-empty sets (Set-Existence):

$$\mathsf{V}_{\varnothing} \ \equiv\ \{X \in \mathsf{V} \mid \exists Y \in \mathsf{V} : Y \in X\}\ . \qquad (4)$$

That *some* set exists is a theorem of logic; this set may be V and V may be empty; to prove that $\mathsf{V} \neq \varnothing$, another axiom besides SetEx is needed,

[15] Von Neumann did not proceed from SetEx *ab ovo*; but remember we announced to erect a reformulation of Von Neumann's original theory of things and functions.

[16] Fraenkel *et al.* [1973: 129–130] for how this is achieved; we are not going to repeat it *hic et nunc*.

such as Infinity (which asserts the existence of ω); the existence of $\varnothing$ as a Cantoresque set follows from Infinity, because $\varnothing \in \omega$. Next come the familiar axioms of Extensionality, Pairing, Union, Power, Infinity, Separation and Regularity (there is implicit universal quantification over all variables that occur free).

$$
\begin{array}{ll}
\text{(Ext)} & (X \subseteq Y \wedge Y \subseteq X) \;\longrightarrow\; X = Y \;. \\
\text{(Pair)} & X, Y \in \mathsf{V} \;\longrightarrow\; \{X, Y\} \in \mathsf{V} \;. \\
\text{(Union)} & X \in \mathsf{V} \;\longrightarrow\; \bigcup X \in \mathsf{V} \;. \\
\text{(Power)} & X \in \mathsf{V} \;\longrightarrow\; \wp X \in \mathsf{V} \;. \\
\text{(Inf)} & \omega \in \mathsf{V} \;. \\
\text{(Sep)} & Z \in \mathsf{V},\; Y \in \mathsf{V},\; \exists\, A \in \mathsf{V}, \\
 & \forall\, X \big(X \in A \;\longleftrightarrow\; X \in Z \wedge \varphi(X, Y)\big) \;. \\
\text{(Reg)} & \forall\, X \in \mathsf{V},\; \exists\, Y \in \mathsf{V}\,(Y \in X \wedge Y \cap X = \varnothing) \;.
\end{array}
\tag{5}
$$

SetEx (3) provides us already with pair-sets, union-sets and power-sets, and even with ω, but nothing can be said as to whether these sets are ultimate or not; the Axioms of Pairing, Union, Power and Infinity decide this by asserting that these sets are *not* ultimate.

We employ the usual definitions of a *function* F from its *domain* D to co-domain C, denoted by $F : D \to C$:

$$F : D \to C \;\text{ iff }\; \forall\, X \in D,\; \exists! Y \in C :\; \langle X, Y \rangle \in F \;. \tag{6}$$

So $F \subseteq D \times C$. The *range of* F is the set of everything reached by F from D:

$$R_F \;\equiv\; \{Y \in C \mid \exists\, X \in D :\; \langle X, Y \rangle \in F\} \;. \tag{7}$$

Then $R_F \subseteq C$. The Axiom of Replacement then reads that for every function F from domain D to co-domain C it holds that if its domain is Cantoresque, then so is its range:

$$\text{(F)} \quad (F : D \to C) \;\longrightarrow\; (D \in \mathsf{V} \;\longrightarrow\; R_F \in \mathsf{V}) \;. \tag{8}$$

The Axiom of Global Choice reads that there is some function $F \subset \mathsf{V}$ (also called a 'choice-function') that sends every non-empty set to a member of it:

$$\text{(GChoice)} \;\exists\, F \subset \mathsf{V}\, \big(F : \mathsf{V}_\varnothing \to \mathsf{V}_\varnothing \wedge \forall\, X \in \mathsf{V}_\varnothing : F(X) \in X\big). \tag{9}$$

GChoice (9) implies Choice as we know it from ZFC: restrict F in (9) to Cantoresque subsets of $\mathsf{V}_\varnothing$ (replace $\ulcorner F \subset \mathsf{V}\urcorner$ with $\ulcorner F \in \mathsf{V}\urcorner$).

We now arrive at the Cantorian heart of CVN. Von Neumann essentially proposed two precise renditions of Cantor's idea of an 'absolute-infinite' set: first, as a set that *cannot be collected further into any other set*, which is of course ultimacy (2); and secondly, as being *absolute-infinite*, defined as being equinumerous to the set V of *all* Cantoresque sets:

$$\text{AbsInf}(X) \;\text{ iff }\; X \sim \mathsf{V} \,, \tag{10}$$

where $\sim$ is the 'equinumerosity-relation'. Definition: set X is *equinumerous* to set Y iff there is 'bijection' between them:

$$X \sim Y \;\text{ iff }\; \exists\, F \subset \mathsf{V} : X \rightarrowtail\!\!\!\!\rightarrow Y \,, \tag{11}$$

where a *bijection* F *from* X *to* Y, denoted by $F : X \rightarrowtail\!\!\!\!\rightarrow Y$, is defined as a function whose range Y is such that every member of Y comes from exactly one domain-member:

$$\begin{array}{l} F : X \rightarrowtail\!\!\!\!\rightarrow Y \\ \quad\text{iff } \big(F : X \to Y \wedge \forall\, B \in Y,\ \exists! A \in X : \langle A, B\rangle \in F \big) \,. \end{array} \tag{12}$$

Thus for Von Neumann, 'too big' means 'most encompassing': the only way for a set to become absolute-infinitely big is to be as big as what encompasses everything. Since Von Neumann intended ultimacy as a new way of looking at absolute-infinity, whence the following axiom

Axiom 2: (Limitation) *All and only the ultimate sets are absolute-infinite:* $\forall\, X$*:* Ultim$(X) \;\longleftrightarrow\;$ AbsInf(X).

The Weak Axiom of Limitation asserts one conjunct of Limitation (Axiom 2).

Axiom 3: (Weak Limitation: Absolute-Inifinity of the Ultimate) *Every ultimate set is absolute-infinite:* $\forall\, X$: Ultim$(X) \;\longrightarrow\;$ AbsInf(X).

For the sake of reference and overview, we define the following theories, as the 1st-order deductive closures of the axioms in the language mentioned:

Theory	Language	Axioms	Theorems
ZFC	$\mathcal{L}_\in$	Ext, Inf, Reg, Pow, Un, F, Choice	Sep, Pair
CVN_0	$\mathcal{L}_\in^V$	SetEx, Ext, Inf, Reg, Univ	
CVN	$\mathcal{L}_\in^V$	SetEx, Ext, Inf, Reg, Univ, Pow, WkLim	GChoice, F, Sep, Un, Pair, Lim
VN^*	$\mathcal{L}_\in^*$	SetEx, Ext, Inf, Reg, Pow, Un, Pair, Lim	GChoice, F, Sep (Un, Pair)

VN^* comes closest to Von Neumann's original theory (VN), when reformulated in $\mathcal{L}_\in$ enriched with 'class-variables' in Bernaysian fashion ($\mathcal{L}_\in^*$);[17] Un and Pair between brackets in the 'Theorems'-column indicate they need not be taken as axioms, as Von Neumann originally did, because they can be proved on the basis of the other axioms. For reasons indicated in Section 1, we baptise the theory in the third row of the Table above *Cantor-Von Neumann Set-Theory* (CVN).

Axiom 2 of Limitation provokes the question whether it is not some philosophical ornament, solely put forward by Von Neumann to propitiate the Cantorian spirit. The answer is a resounding denial, for Limitation is, in the presence of CVN_0 plus Power and Pairing, equivalent to the conjunction of Global Choice, Separation, Replacement (this was essentially proved by Von Neumann [1928]) and Union (proved by Lévy [1968])! This is one excellent reason why Von Neumann adopted the theory VN (with Union), because then Global Choice, Separation and Replacement become *theorems*; consequently one then finds ZFC among its deductive offspring so that VN deductively extends ZFC. Thus calling the Axiom of Limitation, perhaps pejoratively, a 'Cantorian ornament' does not even begin to do justice to it. Besides its deductive strength, Von Neumann motivated the Axiom of Limitation on two independent grounds: (i) it captures Cantor's notion of an "unincreasable, actual-infinite set" and "recognises and admits their existence" (all Cantor's words); and (ii) it blocks the deduction of the well-known antinomies (Russell, Burali-Forti) and simultaneously, seemingly *per impossibile*, it almost saves the Peano-Frege principle of full comprehension according to which *every* predicate has an extension (by binding variables mildly to V). If all this is 'ornamental', then we better reconstrue mathematics as the

[17] Fraenkel *et al.* [1973: 128] call VN^* *without* Limitation 'Von Neumann-Bernays' (VNB) and VN^* with Choice 'VNBC'. Of course they employ a two-sorted language, and moreover add redundant axioms: Union, Pairing, Replacement, Separation

rigorous inquiry into ornaments.

From the axioms we turn to the theorems.

3. *Weak Limitation and its Consequences*

By means of a series of theorems, we shall work our way to the central result: CVN entails VN, which is to say that Pairing and the converse of Weak Limitation can be proved in CVN:

$$\text{CVN} \vdash \text{WkLim, Union} . \qquad (13)$$

Of course, as soon as we have Limitation, we have Union, Separation, Replacement and (Global) Choice, due to Von Neumann's result mentioned earlier, but it is interesting to know, we believe, that several of these theorems can be proved *without* invoking Limitation. We commence with

Lemma 4: *Set* V *contains every set that is included or contained in some Cantoresque set:*

$$\exists Y \in \mathsf{V}(X \in Y \ \vee \ X \subseteq Y) \ \longrightarrow \ X \in \mathsf{V} .$$

Proof. It is sufficient to prove that both disjuncts separately imply that $X \in \mathsf{V}$. (i) If $X \in Y \in \mathsf{V}$, then X is contained in some set and therefore not ultimate (2), which is the same as being a member of V according to Lemma 1. (Thus V is a transitive set.)

(ii) If $X \subset Y \in \mathsf{V}$, then $X \in \wp Y \in \mathsf{V}$ (Power) and then the transitivity of V (i) yields $X \in \mathsf{V}$. $\Box$

Theorem 5: (Ultimacy of V) *Set* V *is ultimate.*

Proof (CVN_0, Power; Infinity of CVN_0 will not be employed.) We prove it by *reductio ad absurdum*. Assume that V is Cantoresque (Reductio Assumption; henceforth: RA).

To steer at a contradiction, we remark that RA entails that V is self-membered because V contains by Lemma 1 *all* Cantoresque sets:

(i) $\mathsf{V} \in \mathsf{V}$.

Pow guarantees that the singleton-set $\{X\}$ of every Cantoresque set $X \in \mathsf{V}$ (whose existence follows from SetEx) is Cantoresque, because $\{X\} \in \wp\wp X \subseteq \mathsf{V}$. Then by (RA) $\{\mathsf{V}\} \in \mathsf{V}$, because of Lemma 1. We obviously also have by definition of the singleton-set of V:

(ii) $\mathsf{V} \in \{\mathsf{V}\}$.

Regularity, when applied to Cantoresque set $\{\mathsf{V}\}$, now says there is some $Y \in \{\mathsf{V}\}$ such that $\{\mathsf{V}\}$ and Y have no members in common. This Y must be V, because V is the only member of $\{\mathsf{V}\}$. So V and $\{\mathsf{V}\}$ have no members in common. But according to (i) and (ii), they do have a member in common: V. Contradiction. $\Box$

In ZFC, Pairing (5) is proved on the basis of Power and Replacement; we prove it here without using any of these axioms. We prove it on the basis of CVN_0 enriched with WkLim (see Table).

Theorem 6: (Paring) *For every pair of Cantoresque sets, their pair-set exists and is Cantoresque.*

Proof (CVN_0, WkLim). Let $X, Y \in \mathsf{V}$. By Set-Existence and Extensionality, the pair-set $\{X, Y\} \subseteq \mathsf{V}$ (Univ) exists uniquely. To prove that it is a member of V, assume, for *reductio*, that $\{X, Y\}$ is not a member of V, i.e. that it is ultimate (RA): $\{X, Y\} \notin \mathsf{V}$.

Then $\{X, Y\}$ is equinumerous to V, due to Weak Limitation (*Axiom-3*). Let f be the bijection from V onto $\{X, Y\}$, the existence of which is by definition (11) logically equivalent to the equinumerosity of $\{X, Y\}$ and V. Well, Infinity gives us $\omega \in \mathsf{V}$ (Univ). Then $\omega \subseteq \mathsf{V}$. Let $m, n \in \omega$ be distinct ($n \neq m$). If $f(m)$ or $f(n)$ is not equal to X or Y, then f is not even a function with range $\{X, Y\}$ and we already have a contradiction. To avoid this contradiction, we must have that $f(m) = X$ and $f(n) = Y$, or with X and Y interchanged. Obviously, there is a $p \in \omega$ that is distinct from both n and m. Hence there *must* be a member of $\{X, Y\}$, say Z, distinct from X and Y, such that $f(p) = Z$. But $\{X, Y\}$ has no other members besides X and Y. Contradiction. So if $X, Y \in \mathsf{V}$, then $\{X, Y\} \in \mathsf{V}$. $\Box$

Now that we have Pairing in CVN_0 plus WkLim, we can make pairs, ordered pairs and sets of ordered pairs, etc. All functions, relations and operations are now available. Notice that neither Replacement nor Power was needed to prove Pairing, whereas in ZFC precisely these, and only these, axioms are used to prove Pairing — these proofs do not carry over to CVN because we do not have Replacement (yet) and our proof of Replacement (as well as Von Neumann's proof) requires the presence of Pairing.[18] Instead of

[18] Von Neumann had Pairing as an axiom [1928; 1961: 344]. To prove Pairing from Replacement and Power, consider $\varnothing$, which exists according to Separation (a consequence

Inf, Power can be used in the proof above to produce more than 2 members of V via $\wp A$, $\wp B$, $\wp\wp A$, $\wp\wp B$ etc. so as to get at a contradiction.

The absolute-infinity of V is an immediate consequence of *Theorem 5* as soon as we have Weak Limitation. But it can be proved *without* explicitly appealing to Weak Limitation, on the basis of Universe, Extensionality, Set-Existence and Pairing.

Theorem 7: (Absolute-Infinity of V) *Set* V *is absolute-infinite.*

Proof (Univ, Ext, SetEx, Pairing). We prove something stronger: *every set is equinumerous to itself*, hence V included. First we define for an arbitrary set $X \subseteq \mathsf{V}$, the identity $\Delta_X \subseteq \mathsf{V}$ as the 'diagonal set' of ordered pairs $\langle A, A\rangle$, for all $A \in X$. The unique existence of Δ_X is guaranteed by SetEx, Ext and Pairing. According to definition (11), X is equinumerous to itself iff there is a bijection between set X and itself, i.e. there is a set of ordered pairs $\langle A, B\rangle$, where $A, B \in X$, such that every $A \in X$ and every $B \in X$ occur exactly once. Clearly the diagonal set Δ_X qualifies as such a set.

The equinumerosity of V with itself yields its absolute-infinite character by definition (10). $\Box$

As a corollary of *Theorems 5* and *7*, we have an instance of the Axiom of Limitation for V (theorem of logic: $\psi \wedge \varphi$ entails $\psi \longleftrightarrow \varphi$):

$$\text{CVN}_0,\ \text{WkLim} \ \vdash\ \text{Ultim}(\mathsf{V}) \longleftrightarrow \text{AbsInf}(\mathsf{V})\ . \tag{14}$$

Now we prove Zermelo's Axiom Schema of Separation.

Theorem 8: (Separation Schema) *Every predicate has a Cantoresque set-extension of members in a given Cantoresque set:*

$$\text{CVN}_0,\ \text{Pow} \ \vdash\ Z \in \mathsf{V},\ Y \in \mathsf{V},\ \exists\, A \in \mathsf{V},\ \forall\, X \left(X \in A \ \longleftrightarrow\ X \in Z \wedge \varphi(X, Y)\right).$$

Proof (CVN_0, Pow; of CVN_0 neither Regularity nor Infinity will be used). Let Z be an arbitrary Cantoresque set, $Z \in \mathsf{V}$, and $\varphi(\cdot, Y)$ some sentence with one free variable and n set-parameters, abbreviated by Y. Set-Existence gives us the set-extension A of predicate $\ulcorner\varphi(X, Y) \wedge X \in Z\urcorner$, which has Z

of Replacement); make $\wp\wp\varnothing = \{\{\varnothing\}, \varnothing\}$, which is a Cantoresque set (Pow); biject it to the set $\{A, B\}$, where A and B are distinct but further arbitrary sets; and finally invoke Replacement to conclude that $\{A, B\}$ also is Cantoresque.

as an additional Cantoresque set-parameter. Is A a Cantoresque set? Yes, by application of Lemma 4. $\square$

The Schema of Separation can be replaced with a single sentence (notably this *cannot* be done in ZFC, where V is not available): a subset of a Cantoresque set is Cantoresque; formally,

$$X \in \mathsf{V} \longrightarrow (X \cap \mathsf{V}) \in \mathsf{V} . \tag{15}$$

Since it is a theorem of logic that $(X \cap \mathsf{V}) \subseteq X$, Lemma 4 does the rest.

Definitions: X is *minumerous or equinumerous to* Y, denoted by $X \preccurlyeq Y$, iff X can be bijected to a subset of Y; and X is *minumerous to* Y, or synonymously, Y is *amplinumerous to* X, denoted by $X \prec Y$, iff X is minumerous or equinumerous to Y and Y is not minumerous or equinumerous to X:

$$\begin{aligned} X \preccurlyeq Y \quad &\text{iff} \quad \exists\, Z \subseteq Y,\ \exists\, F \subseteq \mathsf{V} : X \rightarrowtail\!\!\!\!\twoheadrightarrow Z . \\ X \prec Y \quad &\text{iff} \quad X \preccurlyeq Y \wedge \neg(Y \preccurlyeq X) . \end{aligned} \tag{16}$$

We report two theorems: Cantor's Power Theorem, according to which every set is minumerous to its power-set, and the Cantor-Dedekind-Bernstein 'Minumerosity Theorem', which asserts the a-symmetry of the relation $\preccurlyeq$:

$$\begin{aligned} &\text{CVN}_0,\ \text{Pow} \vdash\ X \prec \wp X \\ &\text{CVN}_0,\ \text{Pow} \vdash\ (X \preccurlyeq Y \wedge Y \preccurlyeq X) \longleftrightarrow\ X \sim Y . \end{aligned} \tag{17}$$

The following theorem directly follows from the definition of minumerosity (16) and Theorem (17):

$$\begin{aligned} \text{CVN} \vdash\quad X \prec Y \quad &\longleftrightarrow\ \big(\neg(Y \preccurlyeq X) \wedge X \not\sim Y\big) \\ &\longleftrightarrow\ \big(X \preccurlyeq Y \wedge X \not\sim Y\big) . \end{aligned} \tag{18}$$

Cantor's Power Theorem (17) can be proved on the basis of SetEx (Separation suffices), Ext, Power and Pairing, which together yield that (a) $X \preccurlyeq \wp X$ (easy: $A \mapsto \{A\}$ bijects X onto a subset of $\wp X$); and (b) $X \not\sim \wp X$ (by means of a *reductio* argument); in the final step of the proof, the Minumerosity Theorem is invoked, via version (18), to deduce from (a) and (b) that $X \prec \wp X$.[19] The Minumerosity Theorem can be proved from Sep, Ext

[19] See the proofs of Cantor's Power Theorem and the Minumerosity Theorem in, for instance, Stoll [1963: 81–82, 86], Lévy [1979: 85, 87].

and Pair, hence also in CVN, which has Sep (*Thm 8*) and Pairing (*Thm 6*) as theorems (in ZFC), and Ext as an axiom.

Now we are in a position to prove the converse of Weak Limitation in CVN.

Theorem 9: (Ultimacy of the Absolute-Infinite) *Every absolute-infinite set is ultimate and therefore not Cantoresque.*

Proof (CVN). Let X be an absolute-infinite set: $X \sim \mathsf{V}$ (Assumption). We have to prove that X is ultimate. The *reductio* assumption is that X is *not* ultimate: $X \in Y$ for some Y (RA).

When X is not ultimate, then by *Lemma 1*, $X \in \mathsf{V}$, and then $\wp X \in \mathsf{V}$ (Pow). Every set $Y \in \mathsf{V}$ can be bijected to a subset of V, namely to itself by means of the identity function; hence $Y \preccurlyeq \mathsf{V}$. In combination with $Y \prec \wp Y$ (17), we then deduce that $X \prec \mathsf{V}$. By means of (18), we conclude that $\mathsf{V} \not\sim X$, which contradicts the Assumption. $\Box$

Every axiom of CVN is invoked to prove that absolute-infinite sets are ultimate (*Thm 9*), all via Pairing (*Thm 6*) and the Minumerosity Thm (17). We then arrive at:

Theorem 10: (Limitation Theorem) *In* CVN*, the absolute-infinite sets are exactly the ultimate sets.*

From the Limitation Theorem (Thm 10) and Pairing (Thm 6), our main result follows: CVN entails VN (13).

The Axiom of Replacement, which in ZFC is an axiom schema, is reduced in CVN to a single sentence of $\mathcal{L}_\in$:

Theorem 11: (Replacement) *If the domain of a function is Cantoresque, then so is its range* (8)*.*

Proof (CVN). We prove it on the basis of CVN; we submit the proof as a simpler one than Von Neumann's proof [928; 1961: 365]; only WkLim will be involved.

Let F be a function whose domain D_F is Cantoresque: $D_F \in \mathsf{V}$ (Assumption). Define the set of members of D_F which F sends to a given member $Y \in R_F$ (SetEx, Ext):

$$[Y]_F \equiv \{X \in D_F \mid F(X) = Y\} . \tag{19}$$

Since $[Y]_F \subseteq D_F$, and thus $[Y]_F \in \wp D_F$ (Pow), we may conclude that $[Y]_F$ is Cantoresque (for every $Y \in R_F$). We next collect them in a set

(SetEx, Ext):

$$Z_F \equiv \left\{[Y]_F \in \wp D_F \mid Y \in R_F\right\} .$$

Then $Z_F \subseteq \wp D_F$, which implies that

(i) $Z_F \preccurlyeq \wp D_F$.

The range R_F is equinumerous to Z_F, because $Y \mapsto [Y]_F$ is bijective from R_F to Z_F. When we combine $R_F \sim Z_F$ with (i), we obtain that $R_F \preccurlyeq \wp D_F$. From this and $\wp D_F \prec \wp\wp D_F$ (17) and $\wp\wp D_F \preccurlyeq \mathsf{V}$ (because $\wp\wp D_F \subseteq \mathsf{V}$), we then have that $R_F \prec \mathsf{V}$. By virtue of (18), we then have

(ii) $\neg(\mathsf{V} \preccurlyeq R_F)$.

If R_F were equinumerous to V, then we would trivially have that $\mathsf{V} \preccurlyeq R_F$, in contradiction to (ii); hence V is not equinumerous to R_F. But then, by WkLim (3), R_F is not ultimate either. Hence if $D_F \in \mathsf{V}$, then $R_F \in \mathsf{V}$. □

We finally consider our last theorem.

Theorem 12: *Cantor-Von Neumann set-theory has Union and Global Choice as theorems.*

Proof Sketch (CVN). Lévy [1968] surprisingly proved Union, which had been considered as an unprovable axiom for more than forty years. The proof crucially employs Replacement and further WkLim; it carries over to CVN. We sketch it.

Set Ω is the largest *ordinal* number, which means that Ω is a set well-ordered by $\in$. Since Ω is ultimate (Burali-Forti), it is absolute-infinite (WkLim) and therefore equinumerous to V. Then there is a one-one correspondence $F : \Omega \rightarrowtail\!\!\!\!\twoheadrightarrow \mathsf{V}$, $\alpha \mapsto F(\alpha)$. Call its inverse $\beta : \mathsf{V} \rightarrowtail\!\!\!\!\twoheadrightarrow \Omega$, $X \mapsto \beta(X)$; let $\beta(X)$ be the least ordinal such that $X = F(\beta(X))$.

First it is proved, using Replacement, that every set of ordinals is bounded by some ordinal (this is essentially a result from ordinal arithmetic: the ordinal-sum of an arbitrary number of ordinal numbers exist); then the set of ordinals is a subset of this bound, because for every $\alpha, \beta \in \Omega$: if $\alpha \leqslant \beta$ ($\alpha \in \beta$ or $\alpha = \beta$), then $\alpha \subseteq \beta$. SetEx yields the existence of set $\bigcup X$ for every Cantoresque set X. The challenge is now to establish that $\bigcup X$ is Cantoresque.

Replacement gives us the Cantoresque set of all Cantoresque sets $\beta[Y]$ for sets $Y \in \bigcup X$. The set of all $\sup \beta[Y] \in \Omega$, i.e. the least upperbound for the set of ordinals $\beta[Y]$, for $Y \in X$, is bounded by some Cantoresque ordinal, α say. Then Cantoresque set $F[\alpha] \supseteq \bigcup X$. Replacement yields that set $\bigcup X$, then, is also Cantoresque. So $\bigcup X \in \mathsf{V}$ whenever $X \in \mathsf{V}$. So much for Union; next Global Choice.

The bijection $F : \Omega \rightarrowtail\!\!\!\!\rightarrow \mathsf{V}$, considered above, can be exploited to show that V can be well-ordered.[20] Loosely speaking, since every ordinal $\alpha \in \Omega$ is by definition well-ordered by $\in$, so that we have a woset $\langle \alpha, \in_\alpha \rangle$, we can replace every member $\langle \beta, \gamma \rangle$ of the membership-relation $\in_\alpha$ on α, that is, of

$$\in_\alpha \equiv \big\{ \langle \beta, \gamma \rangle \in \alpha \times \alpha \mid \beta \in \gamma \big\} , \tag{20}$$

with the pair $\langle F(\beta),\, F(\gamma) \rangle$; this replacement yields the set:

$$\widetilde{\in}_\alpha \;\equiv\; \big\{ \langle F(\beta),\, F(\gamma) \rangle \in \mathsf{V} \times \mathsf{V} \mid \langle \beta, \gamma \rangle \;\in \in_\alpha \;\big\} . \tag{21}$$

The set $\widetilde{\in}_\alpha$ is Cantoresque (due to Replacement) and is a well-ordering on the set $F(\alpha)$: $\langle F(\alpha),\, \widetilde{\in}_\alpha \rangle$ is a woset. The images of the unique $\widetilde{\in}_\alpha$-bottoms of every woset $F(\alpha)$ can be collected in a set equinumerous to V (because F is bijective), which results in a choice-set of V. $\square$

The main conclusions of this paper are, besides that Pairing is redundant axiom in Cantor-Von Neumann Set-Theory (CVN), that, first, 'half' of Von Neumann's Axiom of Limitation, which is the Cantorian heart of CVN, is redundant; and secondly, that the proofs of the important theorems in CVN (Pairing, Separation, Replacement, Global Choice, Union) reveal that this 'half' of Limitation (which we call Weak Limitation) is the part of Limitation that performs all the deductive labour. The strength of Weak Limitation is as Herculean as Von Neumann's intellectual powers were.

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[20] Neumann [1928: 720–721].

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