



Logique & Analyse 209 (2010), 51–60

FREGE, THE COMPLEX NUMBERS, AND THE IDENTITY OF INDISCERNIBLES

CHRISTIAN HELMUT WENZEL

Abstract

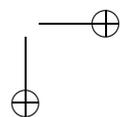
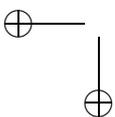
There are mathematical structures with elements that cannot be distinguished by the properties they have within that structure. For instance within the field of complex numbers the two square roots of -1 , i and $-i$, have the same algebraic properties in that field. So how do we distinguish between them? Imbedding the complex numbers in a bigger structure, the quaternions, allows us to algebraically tell them apart. But a similar problem appears for this larger structure. There seems to be always a background and a context that we rely upon. Thus mathematicians naturally make use of Kantian intuition and references fixed by names and denotations. I argue that such features cannot be avoided.

Part 1. Frege on Geometry and Arithmetic

In his *Foundations of Arithmetic*, Frege wrote:

One geometrical point, considered by itself, cannot be distinguished in any way from any other; the same applies to lines and planes. Only when several points, or lines, or planes, are included together in a single intuition [*Anschauung*], do we distinguish them. . . . But with numbers it is different; each number has its own peculiarities [*Eigentümlichkeit*]. [Frege 1884, § 13]

Brandom (2002) has pointed out that the last statement is not correct. The complex numbers i and $-i$ do not each have their own “peculiarities”, because there is an automorphism ϕ of the complex numbers, $\phi(a + bi) = (a - bi)$, carrying each complex number to its complex conjugate, in particular i to $-i$. An automorphism respects the algebraic operations and therefore preserves all algebraic properties. Performing a rational operation r on some



complex numbers and then applying ϕ gives you the same result as first applying ϕ and then r .¹

In defense of Frege two things should be mentioned here. First, in the paragraph immediately preceding the passage quoted above, Frege was concerned with the *real* numbers. He criticized Hankel for seeing the real numbers as based on “notiones communes” that still made use of Kantian “intuition” (*Anschauung*). Thus we can well read him as still referring to real numbers in this passage quoted above. Second, Frege was well aware of the fact that there are two square roots of -1 . Later on, in § 97, he writes:

Nothing prevents us from using the concept ‘square root of -1 ’; but we are not entitled to the definite article in front of it without more ado and take the expression ‘the square root of -1 ’ as having a sense. [Frege 1884, § 97]

Nevertheless, the problem still remains: how do we distinguish between these two complex numbers and what do we make of Kantian intuition?

Part 2. Complex Numbers, Quaternions, and Platonism

I will now phrase all this a little differently. In line with Frege’s views, let us imagine the complex numbers C as a cloud of points in a Platonic realm. We can then reason as follows: there is exactly one unit for multiplication. It must be one of the points in the cloud. Trying various multiplications, we might be lucky enough to find it (although the odds are against us, as there is only one such unit among the uncountably many complex numbers). Suppose we find it. Let us call it e . Next we “recall” that there must be two solutions to the equation $xx = -e$. Let us call them z and z' . They correspond to i and $-i$. Again, there is the problem of finding them. But we have an additional problem, the problem which concerns us here: there is no way to distinguish between z and z' in this Platonic cloud. The real numbers exist in the cloud, and any polynomial with real coefficients is zero for z if and only if it is zero for z' . If you perform a series of rational operations with real coefficients on z , you obtain a series of numbers; and if you perform the same operations on z' , you obtain a parallel series, and it turns out that the two series are conjugate to each other (via ϕ). Thus the two numbers z and z'

¹For a good account of the complex numbers, see chapter 1 of *Complex Analysis* by Lars V. Ahlfors. For the automorphism ϕ and the rational operation r , see page 7.

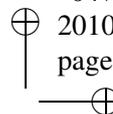
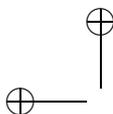
have the same algebraic properties and build up the complex plane in parallel and indistinguishable ways.

Let us put it another way: suppose there are two people and each of them picks a square-root of -1 from the Platonic cloud, without seeing what the other is doing, i.e., without seeing which of the two numbers (z and z') the other picks. Now suppose they perform a series of rational operations, starting with the number they have picked, and talk to each other. They will refer to the same point in the cloud whenever they mention a real number, and they know this (ϕ being constant on the real numbers). But they will never be sure whether they refer to the same number when they mention a complex number that is not a real number, and they know that, too. Thus, just by talking, they will never find out whether they started out with the same number or not. Although they know that the two numbers z and z' are distinct, one being the negative of the other, they can't tell which is which.

If you imagine yourself in front of the complex plane, with the real numbers forming a vertical line and the imaginary numbers a horizontal line, then you might imagine you can grasp the number i with your right hand and the number $-i$ with your left hand. You see the plane, and you use your body to distinguish between right and left. But just as there is no way of *telling* the difference between right and left in purely conceptual terms, there is no way of telling z and z' , or i and $-i$, *algebraically* apart (see Brandom, p. 288–289). Relying on our *body* or relying on the difference between notational *symbols* such as z and z' , or i and $-i$, are ways of importing *external* elements into the Platonic environment. Neither our body nor our signs or symbols are actually part of the Platonic realm.

But there is a way of enlarging the Platonic realm of the complex numbers so that one can *algebraically point* to z and z' *from within* that extended realm. The idea is to embed the complex numbers in a bigger structure, namely the *quaternions*.² This is not a field any more but merely a division ring. Let us call it K . It still has addition and multiplication with neutral elements and inverses, but multiplication is no longer commutative. There are different ways of embedding the complex numbers C in the quaternions K . One usually presents the quaternions in the form $a + bi + cj + dk$, with a, b, c, d real numbers, and introduces the relations $ii = jj = kk = -1$ and $ij = -ji = k, jk = -kj = i, ki = -ik = j$. If you set c and d equal to zero, you obtain C embedded in K .

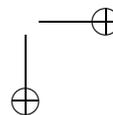
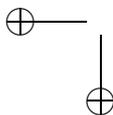
² See exercise 7 on page 493 in *Algebra*, by Serge Lang, Addison-Wesley 1984.



Let us turn to Frege again. The quaternions are numbers, too. So they must exist as some cloud in the Platonic realm. As there is an embedding of C in K , i.e., an injective homomorphism from C into K , so there must be one from the cloud of complex numbers to the cloud of quaternions. In the cloud of complex numbers we had the problem of distinguishing between z and z' . There was the problem of picking one of them, in so far as we did not algebraically see any difference between the two. But now, in K , we have an advantage. Once we have picked numbers v and w in the quaternion cloud corresponding to j and k , we can say: let z be the product vw , and let z' be the product wv . (This corresponds to $i = jk$ and $-i = kj$.) Now we have found a way of algebraically picking z and z' . But of course we have a new problem. How do we get those numbers corresponding to j and k ? The quaternions allow for automorphisms as well, automorphisms that are extensions of the complex conjugation, by sending j to $-j$ and k to k , or j to j and k to $-k$, or j to k and k to j , or j to $-k$ and k to $-j$. Just as i and $-i$ are indistinguishable, so, it appears, are j and k and their inverses. Thus things just got worse. But there is a point we should note. In K you cannot just conjugate the elements of C , in particular exchange i and $-i$, without *affecting the environment* of C in K . If you want to extend the automorphism ϕ of C to some automorphism ϕ' of K , you cannot leave j and k unchanged. Otherwise you would have $-i = \phi(i) = \phi'(i) = \phi'(jk) = \phi'(j)\phi'(k) = jk = i$. Thus the new environment does make a difference! With respect to multiplication, C is now *entangled* with this new environment. In particular i is *algebraically tied up* with the elements j and k from the third and fourth dimension. You can "penetrate" into C from the outside of C , but still from within K , namely by multiplying j and k .

The intuitive role of your right and left hand, with which you imagined you picked i and $-i$, appears to be taken over by j and k and multiplication. Perceptual intuition (what you can see, Kantian *Anschauung*) and the intuition you seem to have for your own body, seem to be "algebraicized". You can reach into C from K , and if you do something in C it has repercussions in K .

But Frege's statement, that "each number has its own peculiarities", has not been saved. Symmetries, automorphisms, and the resulting problems of distinguishing and picking objects remain. It seems that we, the intuiting and thinking subject, cannot be reduced to something purely objective and algebraic. Intuition (Kantian *Anschauung*) cannot be "algebraicized". But at least with respect to the initial structure C , we have found a way of replacing the additional (third) intuitive dimension (in which you find yourself with your body and your right and left hand facing C) by two additional algebraic dimensions (in which we find j and k that rationally generate C).



Part 3. The Current Discussion on the Identity of Indiscernibles

During the last few years there has been a lively discussion about the problem of how to distinguish between i and $-i$. Shapiro defends a structuralist view. Numbers are positions in structures (Shapiro 1997). Burgess (1999) and Keränen (2001) have challenged this view by pointing out that due to the automorphism of conjugation, there is no property that i has and that $-i$ does not have, and visa versa. Thus according to structuralism they should be identical, which they are not. Ladyman (2005) and Button (2006) have then pointed out that there is more to structures than just such properties. We must look at relations as well. To see that i and $-i$ are distinct, it is enough to notice that i is the additive inverse of $-i$ and not its own additive inverse. In other words, i satisfies the formula $x + (-i) = 0$ and $-i$ does not. Thus “ i ” and “ $-i$ ” cannot refer to the same number. The numbers i and $-i$ satisfy the irreflexive relation $x + y = 0$. The point here, it seems to me, is that we do not consider the two numbers in isolation but let them *interact* with each other. After all, the complex numbers are not just a collection of objects. They come with addition, multiplication, and inverses. Shapiro (2008) has emphasized that he meant this all along. There are places *and* relations. Places are like offices in organizations, and the two always come together.

If you ask a mathematician why it is that i and $-i$ are distinct, he might say that $i + (-i) = 0$ whereas $i + i = 2i \neq 0$ because 2 and i are not zero-divisors. A mathematician would have things to say about the complex numbers as a whole. MacBride (2006, p. 67) claims that talk of irreflexive relations (in order to establish that i and $-i$ are distinct) tacitly presupposes that i and $-i$ are distinct, and that Ladyman misses this point. But I don’t think this is correct. Mathematicians can introduce the two symbols “ i ” and “ $-i$ ” without knowing whether they denote the same number or not. In fields with characteristic 2 it turns out that numbers are equal to their additive inverses, whereas in fields with characteristic different from 2 it turns out that they are not (unless they are 0). Thus talk of irreflexive relations to show the distinctness i and $-i$ does make sense.

MacBride (2006) goes back to Russell to make the point that i and $-i$ must be “numerically diverse ‘before’ the [irreflexive] relation can obtain” (p. 67). Yes, and mathematicians indeed usually don’t introduce the complex numbers by invoking such irreflexive relations. They have other ways of doing it. Talk of such irreflexive relations is *post factum*. It is just a way of expressing the distinctness of i and $-i$.

A similar observation can be made about the points in Euclidean plane. In isolation from each other, the points are indistinguishable. But there is always a distance between two distinct points in the Euclidean plane. Thus one can say of two distinct points p and q that they are distinct, because p satisfies the formula $dist(x, q) > 0$, whereas q does not. There is a distance relation between the two points, as there is addition (and multiplication and complex conjugation) between complex numbers.

It has become customary to talk of “strong” and “weak” discernibility in this context. The points in the Euclidean plane, or a complex number and its complex conjugate, are indiscernible if you ask for strong discernibility, but they are discernible, if you are content with weak discernibility (they stand in some irreflexive relation to each other). Button (2006) speaks of “discernibility” and “distinguishability” (p. 218).

But what happens if two objects are not related in a structure? What happens if there is nothing like a distance function or an addition operation between two objects, or two places, in a structure? Shapiro has already mentioned “finite cardinality structures”, say a cardinal-four structure. There are four places and no relations at all. These places are structurally indiscernible, but they are distinct, because this is how it was meant and how it has been stipulated. There is nothing exotic or “metaphysically suspicious” (Button 2006, p. 220) about such structures. Ketland (2006) and Leitgeb and Ladyman (2007) have pointed out that there are many graphs that show such phenomena. The graph consisting of two nodes and one edge between the nodes allows for an irreflexive relation (so that we have a logical way for expressing the fact that the two nodes are distinct), but if we take the edge away, there is no such relation any more. So how should we express the fact that they are distinct from each other? Leitgeb and Ladyman have also pointed out that looking for instance at all possible (unlabelled) graphs with six nodes we can see that there are 11 that contain two isolated nodes. Thus in graph theory there is nothing exotic about isolated elements. They must be reckoned with. Whole subgraphs can show up several times in a bigger graph and be isolated. They are distinct objects, although there are no intra-structural relations that we could rely upon to express the fact that they are distinct. Hence there are plenty of graphs that have distinct but not even weakly discernible places, and graphs are decent mathematical objects.

Looking back at the case of i and $-i$, which are weakly discernible, there is still the problem that there is nothing within the structure that really distinguishes between them. We cannot tell which is which. Shapiro (2008) says: “Frankly, I am not sure what is being demanded.” But it seems to me that there still is a problem that causes some uneasiness and that has nothing to do

with weak or strong discernibility. This is a general problem of Platonism. The problem I have in mind is the problem of stipulating or picking objects and keeping them fixed in our "mind" while using them in operations. One might think that this is merely a problem of reference or epistemology, distinct from ontology. But I think there is more to this.

Let us return to our first quote from Frege:

Only when several points, or lines, or planes, are included together in a single intuition [*in einer Anschauung*], do we distinguish them. . . . But with numbers it is different; each number has its own peculiarities. [Frege 1884, § 13]

Now we see that the problem is actually not with the last, but with the first sentence! What is the role of "intuition" here? And what does it mean that these objects are "included" (actually: "grasped", *aufgefaßt*) in one intuition? And what does it mean that they are included "together" (actually: "simultaneously", *gleichzeitig*)? Shapiro, Leitgeb, Ladyman, and others agree that we should first look at mathematical practice and its implications, before embarking on metaphysical speculations. Shapiro (2008) says about the mathematical community that "they let i be one such square root, and *go on* from there", and he quotes Brandom saying: "Frege's practice . . . would seem to show that what matters for him is that we understand the *proper use* of the expression we *introduce*". But what exactly is required for this "going on", this "proper use", this "introducing", and 'including things in a single intuition'? Let us also have a look again at our second quote from Frege:

Nothing prevents us from using the concept 'square root of -1 '; but we are not entitled to put the definite article in front of it *without more ado* [*ohne weiteres*] and take the expression 'the square root of -1 ' as having a sense. [Frege 1884, § 97]

What is this "more ado"? When mathematicians say: "consider four points", or: "let i be a square root of -1 ", they imagine something and they are prepared to do something. Frege tried to dismiss this as a matter of psychology and as being irrelevant for mathematical objects themselves. But for Kant it was essential that not only such practice but also the objects considered involve "intuition". Something is given to us *in intuition*, and he meant "in intuition" not just adverbially. Intuition is part of what is given. Furthermore, he talks of "manifolds" of intuition (compare this with MacBride referring back to Russell talking of "bare particulars", p. 68). He also argues (in the Transcendental Aesthetic) that space and time are not concepts (universals).

Frege argued against Kant's notion of intuition, but in the end his own logicism failed and he turned to geometry. Now, when Shapiro says: "Frankly, I am not sure what is being demanded", what shall we tell him? Well, we could simply say that the identity relation for positions is primitive and be done with it. Ketland (2006) expresses such a view (p. 305).³ I would like to make three suggestions here.

1. Extension. If the structure is not rich enough to distinguish the objects in the way we would like to see them distinguished, we can enlarge the picture. For instance, we can imbed the complex numbers in the quaternions, as I have indicated in the previous part. Facing a graph with isolated nodes, we can think about the possibility of inserting edges. We can think of this graph as a *graph as such*, i.e. as an instance of the general concept of a graph which includes the idea of possible operations, such as adding edges. In such a context, the isolated nodes, or isolated subgraphs, do not appear so isolated any more. The world "as a whole" might turn out to be rigid, as Ketland (2006) invites us to imagine (p. 314). But with the quaternions this seems not to work, because there are still automorphisms, as we have seen. We, the subjects, seem to remain outside the picture, like an eye that cannot see itself.

2. Symbolism and Use. Mathematicians introduce symbols and operate with them, assuming that they refer uniquely and in a fixed way. An identity through time and communal usage is assumed. Kripke for instance, in his account of fixing the reference of "one meter" (Kripke 1980, pp. 54–57), assumes that objects in the world do not change chaotically and that there is communal agreement about the use of the newly introduced expression. But, although we can hold on to the meter-stick, we cannot actually hold on to the abstract meter itself. There still seems to be something solemn about the "ceremony" (p. 14) and the "baptizing" of the meter. Kripke argues that after the ceremony the statement that the stick is one meter long is contingently true and known *a priori*. But one can also argue that what is known *a priori* is actually a hypothetical statement and that this hypothetical statement relies on Kantian *a priori* intuition and the categories (Wenzel, 2003 and 2004).

³ Also Parsons asks: "Why should we require, for objects to be distinct, that there is *anything* that distinguishes them?" (Parsons 2004, p. 75). He gives a more conciliatory response to the objections raised by Burgess and Keränen though. He suggests distinguishing between basic and constructed structures in the case of the complex numbers, and he suggests reference to intuition, perception, mere quasi-concreteness plus idealization and abstraction in the case of the Euclidean plane (pp. 70–71). In general he concludes that "ideas from the metaphysical tradition can be misleading when imported into discussions of mathematical structuralism and perhaps into discussions of mathematical objects generally" (p. 74).

3. Intuition. Frege writes that we can "include" several points "in a single intuition" and thereby distinguish them (see the quote at the beginning of this article). Yes, we indeed can do that; we can also imagine that there are no structural relations between them, such as distances. Simply imagine one point and then another! But doing this involves time and space, and the notion of identity, continuity and persistence through time. We can think about and do something with these points, and for cases like these Kant argued for the *apriority* and necessity of certain elements, such as time, space, and the categories. These are transcendental elements. They are subjective in a special way: they make objectivity possible. If we could not distinguish between two points in this *a priori* way, experience would not be possible.

From an ontological or structuralist point of view, asking for distinguishing features of two points in a plane, or of i and $-i$, is asking for too much. But from an epistemological point of view, something is to be said for this demand. How exactly the ontological and the epistemological are related is an old and broad question that will not so easily be resolved.

National Taiwan University
Department of Philosophy
1 Roosevelt Road, Sec. 4
Taipei 10617
Taiwan

E-mail: wenzelchristian@yahoo.com

REFERENCES

- Ahlfors, Lars V. 1979: *Complex Analysis*. McGraw-Hill.
- Brandom, Robert 2002: 'The Significance of Complex Numbers for Frege's Philosophy of Mathematics', in *Tales of the Mighty Dead*, Harvard University Press, 277–297.
- Burgess, John P. 1999: Review of Stewart Shapiro 1997. *Notre Dame Journal of Formal Logic* 40, 283–291.
- Button, Tim 2006: 'Realistic structuralism's identity crisis: a hybrid solution', *Analysis* 66, 216–222.
- Frege, Gottlob 1884: *Die Grundlagen der Arithmetik*. Breslau: Koebner. *The Foundations of Arithmetic*. J. Austin, trans. 2nd ed. New York: Harper, 1960.
- Keränen, Jukka 2001: 'The identity problem for realist structuralism', *Philosophia Mathematica* 9, 308–330.
- Ketland, Jeffrey 2006: 'Structuralism and the identity of indiscernibles', *Analysis* 66, 303–315.

- Kripke, Saul 1980: *Naming and Necessity*. Harvard University Press.
- Ladyman, James 2005: 'Mathematical structuralism and the identity of indiscernibles', *Analysis* 65, 218–221.
- MacBride, Fraser 2006: 'What constitutes the numerical diversity of mathematical objects?', *Analysis* 66, 63–69.
- Parsons, Charles 2004: 'Structuralism and metaphysics', *Philosophical Quarterly* 54, 56–77.
- Shapiro, Stewart 1997: *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press.
- Shapiro, Stewart 2008: 'Identity, Indiscernibility, and *ante rem* Structuralism: The Tale of *i* and $-i$ ', *Philosophia Mathematica* 16, 285–309.
- Wenzel, Christian Helmut 2003: 'Knowledge, Belief, and the *A Priori*', *Papers of the 26th International Wittgenstein Symposium*, vol. XI, Kirchberg am Wechsel, Winfried Löffler and Paul Weingartner (eds.), pp. 369–370.
- Wenzel, Christian Helmut 2004: 'Kripke's Contingent A Priori: The Meter-Stick Example', in *Aufklärung durch Kritik: Festschrift für Manfred Baum*, Dieter Hüning, Karin Michel, Andreas Thomas (eds.), Berlin: Duncker & Humblot, 477–480.