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A CERTAIN CONSEQUENCE RELATION FOR SOLVING PARADOXES OF VAGUENESS

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1. *Introduction*

When people are presented with a series of colour patches, for instance from red to pink, and are asked to indicate with a number how confident they are as to a given colour if it is red or not red, their confidence throughout the region changes, and is indicated as the highest one only at the beginning and at the end of the series.[1] Experiments of this kind suggest that there is no sharp cut-off for the predicate 'red'. This fact may be explained either by deficiency of human discriminatory ability, or by deficiency of meaning of the predicate 'red'. However, there are some examples of vague predicates, like 'heap' or 'bald', for which the lack of cut-off strongly suggests the explanation by deficiency of meaning, and in particular deficiency of extension. No competent speaker of English can say that a certain number n of grains of sand forms a heap and an $(n + 1)$ number of grains does not form a heap, and no such a speaker can say that there is a number n of hairs on someone's head such that someone with that number of hairs is bald and with an $(n+1)$ number of hairs is not bald. Nevertheless, even in those cases no one is able to notice a difference between 100 and 101 grains of sand, or between 100 and 101 hairs on someone's head merely by looking, but not by counting. Each predicate for which the lack of sharp cut-off occurs has borderline cases, that is, those objects for which it is neither definitely true, nor definitely false that the predicate applies to them. Neither our better knowledge of the world, nor our better knowledge of language can reveal the hidden cut-off point for a vague predicate, since that cut-off point does not exist unless speakers make it arbitrarily in communicative acts. Our assumption is that the semantic aspect of vagueness is decisive for the phenomenon of vagueness, although it is connected with the epistemic one; in consequence the deficiency of meaning should be examined in connection with our discriminatory limitations. This dependence is observed in communicative acts, which have been taken as starting point of the present analysis of vagueness. The

[1] Cf. Changizi 1999, 42–43.

main question of this paper may be posed in the following way: To which extent is the classical logic adequate for a vague language? Only the supervaluational approach to vagueness, among the other familiar approaches, makes the assumption that the phenomenon of vagueness consists in a deficiency of extension, and obtains intuitively acceptable results. For that reason we shall examine supervaluational semantics more thoroughly than the other approaches considering it as our rival account of the phenomenon of vagueness. A new way of representing the logic of vagueness outlined in this paper is a modification of Belnap's four-valued bilattice-based logic. A preferential consequence relation defined in terms of the consequence relation of Belnap's logic may be regarded as an adequate theoretical model of inferences occurring in sorites paradoxes. Making use of this consequence relation we can show that the paradoxical reasonings are either invalid or unsound. This approach to vagueness will be compared with the supervaluational and subvaluational semantic, many-valued and fuzzy logic as rival approaches to vagueness.

2. *Supervaluationism and Vagueness*

Applied to vague predicates, the supervaluation theory makes use of the idea that the vague predicate allows for arbitrary precisifications.[2] Each precisification sets a sharp cut-off for the vague predicate. In consequence each atomic sentence containing a vague predicate is evaluated as true or false with respect to a given precisification. In this way each arbitrary precisification sets a classical valuation, but the classical truth-value of the same atomic vague sentence vary with precisification. However, there are compound sentences whose classical truth-value remains constant for all precisifications. In other words, there are sentences which are true with respect to all classical valuations, and there are sentences which are false with respect to all such valuations. They are called supertrue and superfalse, respectively.

Generally, the supervaluation semantics could be described in the following way.[3] If 'v' is a partial function of valuation which leaves the formula 'A' undefined when any its subformula 'B' is undefined, then there is a classical extension of 'v' such that it assigns either the value t or f whenever v assigns no value. A supervaluation 's' over 'v' is defined by the following conditions: (1) s(A) $= t$ (truth) if the value of 'A' on all classical extensions

[2] Cf. Fine 1975.

[3] The idea originates from van Fraassen 1966.

of 'v' is t; (2) s(A) $= f$ (falsehood) if the value of 'A' on all classical extensions of 'v' is f; and (3) s(A) is undefined otherwise. 'A' is valid if s(A) $= t$ for all supervaluations 's' over arbitrary 'v'.

Thus even though 'p' is an atomic vague sentence, which is undefined, 'p $\vee \neg$p' is supertrue and valid. One may say that this disjunction is true on each precisification, since on each precisification exactly one disjunct is true whereas the other is false. The supervaluation technique gives an answer to the question why 'This is red or pink' is true, but 'This is red and pink' is false if both atomic sentences 'This is red' and 'This is pink' are vague. Assuming that 'pink' and 'red' are contraries, each way of making them more precise makes the disjunction true, but the conjunction false. On each precisification only one of the two atomic sentences: 'This is red' and 'This is pink' is true, whereas the other is false, what is necessary and sufficient for the disjunction of these two sentences to be supertrue, and for the conjunction to be superfalse. Note that 'This is pink or small' and 'This is pink and small' remain undefined on this approach, if both atomic sentences are vague. There is a way of making the predicates 'pink' and 'small' more precise which makes the sentence 'This is pink' and 'This is small' false, and there is a way of making the predicates more precise which makes the two sentences true. Hence, they can be neither supertrue nor superfalse.

The supervaluation is not truth-functional in the sense that truth-value of each compound formula is not a function of the truth-values of its subformulae. As we have seen, for the same input <u, u>, where 'u' stands for "undefined", it gives the value t or 'u' as output in the case of disjunction, and f or 'u' in the case of conjunction. In this respect the supervaluation semantics differs from other approaches which consider a partial function of valuation. For example, the Kleene strong and weak valuation has uniformly 'u' as output for the input <u, u> in the case of disjunction and conjunction. The Kleene logic is then not able to provide an adequate account of the sentence 'This is red or pink', if the sentence attributes the property of being red or pink to a certain borderline case of the predicates 'red' and 'pink'.

2.1. *Supervaluationism and Penumbral Truths*

The question arises why the supervaluational semantics gives an adequate account of logical relations which hold between vague sentences, like 'p $\wedge$ r', 'p $\wedge \neg$p', 'p $\vee \neg$p', and 'p $\vee$ r', where 'p' and 'r' stand for 'This is pink' and 'This is red', respectively. Kit Fine argues that the supervaluational semantics respects what he calls penumbral truths, that is, truths about logical relations which hold between atomic sentences containing vague predicates having common borderline cases, like for instance 'pink' and 'red'. Fine assumes that such predicates are independent and exclusive upon their common borderline cases.[4] However, this assumption is evident for those who want vague predicates to behave like precise ones. And indeed each precisification makes extensions sharp for both predicates simultaneously. But unless a sharp cut-off is set, it is even difficult to say in which sense vague predicates are independent and exclusive upon their common penumbra. In natural language, there is no definition which could define the sharp extension of such predicates like 'red', 'pink', *etc*. While we can count as red many things, just because we recognise them as red, the question if an arbitrary object which cannot be recognised as definitely red belongs to the extension of the predicate 'red' is open, because there is no absolutely sharp extension for 'red'. But even if the sharp boundary had existed, it would not have been recognised as absolutely sharp, because of deficiency of our discriminatory ability.

Let us take the disjunction: 'p $\vee$ r'. Given an arbitrary object 'a', a certain obvious relation of exclusiveness, which holds between 'pink' and 'red' may be expressed by the following equivalence:

(E) 'a is pink' is true $\equiv \neg($'a is red' is true$)$
$\neg($'a is pink' is true$) \equiv$ 'a is red' is true.

The equivalence (E) is satisfied only if there is a cut-off for 'pink' and 'red'. The meaning of 'or' in the sentence 'a is pink or red' suggests that such a cut-off exists, since it needs only one of the two sentences 'a is pink' and 'a is red' to be true for 'a is pink or red' being true. Since in fact there is no sharp borderline between red and pink objects, the supervaluation semantics posits as many admissible cut-offs as it is possible without claiming which is the real or true one. Each is arbitrary and no better than the others, but each makes 'p $\vee$ r' true. This solution is a guarantee that disjunction which holds between sharp sentences, on each precisification has the same meaning for vague sentences containing predicates with common borderline cases. The idea which is close to the supervaluation technique is that 'a' could be defined as pink and 'a' could be defined as red, even if 'a' is not pink and 'a' is not red, but a borderline case of these two colours. Thus, the following equivalence holds:

(D) 'a' is defined as 'P'
$\equiv$ there is a precisification for which 'a is P' is true.

Now let us take the conjunction: 'p $\wedge$ r'. Since each cut-off for 'red' and 'pink' makes only one of the two sentences: 'a is pink' and 'a is red' true,

[4] Fine 1975, 270.

the conjunction is false for each precisification. In other words, if the conjunction retains the same meaning when occurring between sharp sentences, 'p ∧ r' will be false for each precisification. The same reasoning applies to 'p ∧ ¬p'. Each precisification makes only one of the two sentences: 'a is pink' and 'It is not the case that a is pink' ('a is not pink') true, and then makes 'p ∧ ¬p' false. In this way, the supervaluation semantics, making use of the idea of precisification, precisifies the meaning of 'and' for vague sentences and identifies it with the meaning of the classical conjunction. It also precisifies the meaning of 'or', which occurs between vague sentences, and identifies it with the meaning of the classical (exclusive) disjunction. For each precisification 'and' and 'or' behave like the respective classical connectives. The classical meaning of sentential connectives is indeed suggested by intuitively true conditionals containing predicates having common borderline cases, like: 'If 'a' is pink, then 'a' is not red', 'If 'a' is a tadpole, then 'a' is not a frog', and also by conditionals concerning different borderline cases of the same predicate, like 'If Tom is bald, then a man with fewer hairs on his head is bald'. Even though the atomic sentences occurring in these examples are vague, the conditionals are true on each precisification, and then supertrue.

On the Kleene valuations, the conditionals are undefined, if the respective atomic sentences are undefined.[5] On the other hand, in the Łukasiewicz 3-valued logic the implication is true if the antecedent and consequent have the value ½, but the intuitively true disjunction 'a is pink or red', where 'a' is a borderline case of 'pink' and 'red', has the value ½.[6] In the Bochvar logics, when the conditional is identified with the external implication, the conditionals of our example are true, if they have undefined antecedent and undefined consequent, but 'a is pink or red' is false, if 'or' is identified with the external disjunction.[7]

Taking into account the examples mentioned above, one cannot say that they provide an ultimate argument for the claim that sentential connectives occurring between vague sentences have the classical meaning. It seems rather that the connectives acquire classical meaning provided that the predicates involved are made sharp. 'If 'a' is pink, then 'a' is not red' is true only if it is not the case that 'a' is pink and red, and it is so if there is a sharp border between pink and red objects. 'If 'a' is a tadpole, then 'a' is not a frog' is true if there is a sharp borderline between tadpole stages and frog stages. 'If Tom is bald, then a man with fewer hairs on his head is bald' is true if

[5] Cf. Kleene 1952, 334.

[6] Cf. Łukasiewicz 1920.

[7] Cf. Bolc and Borowik 1992, 65–66.

Tom is definitely bald. The conditionals combining vague sentences either loose their classical meaning, because there is no hidden cut-off point for the predicates involved, or retain the classical meaning as elliptical statements. Thus, 'If 'a' is pink, then 'a' is not red' is elliptical for the sentence 'If 'a' is *defined* as pink, then a is not defined as red'. And similarly, 'If 'a' is a tadpole, then 'a' is not a frog' is elliptical for 'If 'a' is *defined* as a tadpole, then 'a' is not defined as a frog'. 'If Tom is bald, then a man with fewer hairs on his head is bald' is elliptical for 'If Tom is *defined* as bald, then a man with fewer hairs on his head is defined as bald'. The intuitively true disjunction 'a is pink or red', where the two atomic sentences: 'a is pink' and 'a is red' are vague, is elliptical for 'a is *defined* either as pink or as red'. The conjunction 'a is pink and red' is false on this reading, because it is elliptical for the false statement 'a is *defined* as pink and as red'.

This discussion leads to the conclusion that penumbral connections are not properly represented at the level of the object language, that is, at the level not involving semantic terminology. Penumbral truths are elliptical statements: Their full reading does not involve vague sentences, like 'a is pink' and 'a is red', but instead of these statements, their precise counterparts: 'a is defined as pink' and 'a is defined as red'. But on this reading, penumbral truths do not pose a special problem for logic of vagueness.

2.2. *Supervaluationism and Sorites Paradoxes*

Let us pass on to reasoning containing vague predicates and to their analysis in terms of the supervaluational semantics. Kit Fine claims that on the supervaluation theory the sentence (1):

(1) For all n: If a man with n hairs on his head is bald then a man with $(n+1)$ hairs on his head is bald

is false, because for all admissible precisifications it is the case that: 'A man with n hairs on his head is bald' is true and 'A man with $(n+1)$ hairs on his head is bald' is false.[8] However, in the classical logic if 'For all n: If a man with n hairs on his head is bald then a man with $(n+1)$ hairs on his head is bald' is false, then its negation:

(2) There is an n such that a man with n hairs on his head is bald and a man with $(n+1)$ hairs on his head is not bald

is true. On its literal meaning, the negation asserts that there is a sharp cut-off for 'bald' in that sense that a man with n hairs on his head is bald, but a

[8] Fine 1975, 285.

man with $(n+1)$ hairs on his head is not bald, for a certain number n. But if 'there is' has the meaning of the classical existential quantifier, the negation asserts the existence of a cut-off for at least one number n. On this liberal reading of 'there is' the sentence (2): 'There is a number n such that a man with n hairs on his head is bald and a man with $(n+1)$ hairs on his head is not bald' is supertrue, because it is true for all ways of making it completely precise. In other words, for any number n there is a sharp cut-off for 'bald', which makes the sentence (2) completely precise and in consequence makes it true. Thus the 'Bald Man' Paradox (I):

(I) A man with no hairs on his head is bald.
For all n: If a man with n hairs on his head is bald then a man with $(n+1)$ hairs on his head is bald.
$\therefore$ A man with 100.000 hairs on his head is bald.

is *valid*, but *unsound* on the supertruth theory, because its second premise is superfalse and the conclusion is superfalse. On the other hand, the following version of the 'Bald Man' Paradox (II):

(II) A man with no hairs on his head is bald.
A man with 100.000 hairs on his head is not bald.
$\therefore$ There is a number n such that a man with n hairs on his head is bald and a man with $(n+1)$ hairs on his head is not bald.

is *valid* and *sound* on the supertruth theory, because the premises and the conclusion are supertrue. These two arguments are also valid in the classical logic. There is no classical model which makes the premises of these arguments true and the conclusions false. Nevertheless, both conclusions intuitively are not true. The conclusion of this last paradox states that a man with a certain number n of hairs on his head is bald and a man with $(n+1)$ number of hairs on his head is not bald. This statement is not fully expressible on the supervaluational approach. A certain approximation of that statement provided by the supervaluational semantics looks like (3) below:

(3) 'A man with a certain number n of hairs on his head is bald and a man with $(n+1)$ hairs on his head is not bald' is true relative to an admissible precisification.

One may claim that the conclusion of the 'Bald Man' Paradox (II) is in the classical sense true and that in fact there is a number n being a cut-off point for 'bald', but we do not know which. However this view claims too much, as there is no any number n of hairs such that a man with n hairs on his head is bald and a man with $(n+1)$ hairs on his head is not bald. It would have been such a number n for 'bald' if the predicate 'bald' had been completely precise. Let us compare the paradox (II) with the following argument:

(A) 1 is an element of {1, 2, 3}.
8 is not an element of {1, 2, 3}.
$\therefore$ There is a number n such that n is an element of {1, 2, 3} and $(n+1)$ is not an element of {1, 2, 3}.

The number n is 3, because it belongs to {1, 2, 3}, but its successor does not belong to {1, 2, 3}. But the extension of the predicate 'bald' is not a set like {1, 2, 3}.

For a further discussion it would be good to have logical forms of both paradoxes. For this purpose we do not need more than the standard symbols for sentential connectives, numerals, one two-place predicate 'B' for 'is bald with' and an individual parameter 'a' for 'a man'. Thus the 'Bald Man'Paradox (I) may be regimented in the following way:

(I) B(a, 0)
[B(a, 0) ⇒ B(a, 1)] ∧ [B(a, 1) ⇒ B(a, 2)]
∧. . . ∧ [B(a, n) ⇒ B(a, n+1)]
∧. . . ∧ [B(a, 100.000 − 1) ⇒ B(a, 100.000)]
$\therefore$ B(a, 100.000)

On the supertruth theory, each precisification makes false one conjunct of the second premise, and hence makes the premise false. The premise being false on each precisification is superfalse. Hence the argument is unsound on the supervaluational semantics. Making use of the same formalism, we may give to the 'Bald Man' Paradox (II) the following logical form:

(II) B(a,0)
¬B(a, 100.000)
$\therefore$ [B(a, 0) ∧ ¬B(a, 1)] ∨ [B(a, 1) ∧ ¬B(a, 2)]
∨. . . ∨ [B(a, n) ∧ ¬B(a, n+1)]
∨. . . ∨ [B(a, 100.000 − 1) ∧ ¬B(a, 100.000)]

Each precisification makes true one disjunct of the conclusion, although in each case a different one, and hence makes true the conclusion. The conclusion being true on each precisification is supertrue. The supervaluational semantics does not conform then with our pre-theoretic intuitions concerning truth-value of this conclusion. In classical logic, where the Principle of Bivalence holds, both paradoxes are valid, since there are certain tacitly accepted assumptions, like:

(a) The predicate 'bald' has precise extension, and
(b) The predicate 'bald' has the same meaning in each occurrence in the arguments (I) and (II).

However, both assumptions may be questioned. Firstly, we take into account an obvious fact that the predicate 'bald' is vague in the natural language. Secondly, there must be a difference in meaning between 'bald' occurring in the premises of the 'Bald Man' Paradox (II) and 'bald' occurring in the conclusion of this paradox, since each premise has a definite truth-value, while for certain numbers n the conclusion is vague. If (a) and (b) do not hold for both paradoxes, then one may argue that the truth is not transmitted from the premises to the conclusion, and in particular the conclusion of the 'Bald Man' Paradox (II) is not true. But not being true the conclusion cannot be false, because if it is false, its negation is true in the classical logic. However, the negation takes the form of the second premise of the 'Bald Man' Paradox (I): 'For all n: if a man with n hairs on his head is bald then a man with $(n + 1)$ hairs on his head is bald', which is also contentious, since it leads to the intuitively false conclusion. This observation suggests that the Principle of Bivalence, which states that any sentence is either true or false, fails for vague sentences.

The supervaluation semantics rejects the Principle of Bivalence, but retains classical logic in this sense that all classical logical truths remain logical truths on supervaluational approach and all classically valid arguments are also validated by supervaluationism, where validity is defined as preservation of supertruth.[9] Kit Fine maintains that an adequate account of penumbral connections requires that the logic of vagueness be classical.[10] On our approach, penumbral connections are stated at the level of metalanguage, and the fact that they exist has no bearing on the decision as to whether the sentential connectives occurring between vague sentences of the object language have classical meaning or not.

Vague predicates play a double role: Each such predicate forms precise and vague sentences with respect to a given communicative act. If the Principle of Bivalence is rejected, vague atomic sentences do not admit a classical truth-value. But precise atomic sentences should be still evaluated as true or false. For compound sentences the situation is similar: Some of them are precise and the others are vague. To the first group belong (a) compound sentences whose atomic components are precise; and (b) compound sentences whose meaning depends solely on the meaning of classical connectives. The latter include classically valid sentences, like 'p $\vee$ $\neg$p', '(p $\wedge$

[9] See Williamson 1994, 148.

[10] Fine 1975, 286.

q) → p', *etc*., even though the atomic sentences which stand for 'p' and 'q' are vague. This view may be justified by the fact that the classical truths do not depend on the extensions of the predicates involved, and hold for predicates having extensions however precisified. This fact is accepted by the supervaluation semantics, where the idea of precisification is used to show that the classical logic may be saved even though the Principle of Bivalence is rejected. However, as we have seen, the idea of precisification fails when it comes to a satisfactory account of sorites paradoxes, like the 'Bald Man' Paradox (II), since it leads to unintuitive conclusion that the paradox is valid and sound.

3. *Many-Valued Logic and Vagueness*

Before we come to the details of our own approach to vagueness, we must say why we do not take up a familiar proposal provided by many-valued logics, in particular by a system of infinitely many valued logic. In such a system 0 represents falsehood and 1 represents truth, but all other real numbers of the closed interval from 0 to 1 represent intermediate truth-values. The logical constants are defined then in the following way:

$$V(\neg p) = 1 - V(p)$$
$$V(p \wedge q) = \min(V(p), V(q))$$
$$V(p \vee q) = \max(V(p), V(q)$$
$$V(p \rightarrow q) = \min(1, 1 - V(p)+V(q)).$$

This logic is known as the Łukasiewicz infinitely many valued logic or as fuzzy logic.[11] If we evaluate the conclusion of the 'Bald Man' Paradox (II), taking into account its logical form (II), according to the rules given above, it turns out that its value does not exceed 0.5. The conclusion is a disjunction of the following conjuncts: [B(a, n) ∧ ¬B(a, n+1)], where 0≤n≤100.000. We must assume that for each n the value of B(a, n) does not exceed the value of B(a, n+1) more than a certain constant number, for instance 0.001. Evaluated in that way, the 'Bald Man' Paradox (II) has true premises and the conclusion evaluated at approximately 0.5. Thus truth from the premises is not transmitted to the conclusion. This result is intuitively convincing, but applications of fuzzy logic to other vague sentences are rather contentious. Let us assume that a vague sentence p is evaluated at approximately 0.5. Then the negation of 'p' will have also the value 0.5, and the same value have 'p ∨ ¬p', 'p ∧ ¬p' and '¬(p ∧ ¬p)'. However, we want 'p ∨ ¬p'

[11] See Goguen 1968.

and '$\neg$(p $\wedge$ $\neg$p)' to be true independently of the fact that 'p' is vague and different with respect to truth-value from 'p $\wedge$ $\neg$p'. Moreover, fuzzy logic sets semantic distinctions where there is no perceptible evidence for making a difference whatever. Let us take the following example: It seems that there is no difference as to truth-value between the sentence 'A man with no hairs on his head is bald' and 'A man with one hair on his head is bald'. But in fuzzy logic, these two sentences differ as to truth-value. In our view, fuzzy logic as a theory of vagueness is not quite faithful to the facts of language use. But it is worth noting that fuzzy logic may be interpreted in a different way: Fuzzy logic provides an adequate account of facts connected with our ability of attributing the property of being truer to sentences.[12]

4. *Communicative Acts and Vagueness*

Vague predicates occurring in natural language are a source of inconsistent information about the world. To explain this fact we must turn attention to the role of vague predicates in communicative acts. Linguistic communicative acts take place in those cases when the speaker uses words in order to focus the addressee's attention on a certain object or event in the world, and when the addressee understands the communicative intentions of the speaker as directed towards his (her) attentional state with that intention.[13] Linguistic communicative acts are then a species of human intentional behaviour. Many experimental studies demonstrate that human intentional behaviour is developmentally prior to the ability of understanding communicative intentions.[14] Now, let us imagine that the speaker makes the statement 'a is pink' with the communicative intention to induce the other person to attend to a certain object 'a' in the world and that 'a' is on the border between pink and red. The communicative act performed by the speaker holds even though 'a' is a borderline case of pink objects: Uttering 'a is pink' in a given communicative situation, the speaker is able to draw attention of the listener to the object in question. One cannot imagine that 'a is pink' may be replaced by 'a is brown' in the same communicative situation, just because there are no brown 'a', and by uttering 'a is brown' the speaker cannot induce the listener to attend to 'a'. But if the speaker utters 'a is red' with the communicative intention directed towards the listener attentional state, the speaker is able

[12] Cf. Simons 1997.

[13] Cf. Tomasello 1998, 236.

[14] ibid., 238.

to draw the listener attention to the object 'a', and in consequence the communicative act holds. The speaker and listener share the same choices for linguistic representation of the intended referent, and the decision between 'a is pink' and 'a is red' is made for some pragmatic reasons, for instance with respect to the listener's interests.[15] This view confirms the fact that communicative acts are inherently joint acts.[16]

One can imagine two speakers 'A' and 'B' who act independently in similar communicative situations. Let 'A' makes the statement 'a is pink' and 'B' makes the statement 'a is red', and suppose that the listener is informed about 'a' by 'A' and 'B' in two different communicative acts. Now, let us imagine that the observer gathers the information about 'a' and encounters the two statements 'a is pink' and 'a is red'. Since the observer knows that if 'a' is defined as red then 'a' is not defined as pink and also if 'a' is defined as pink then 'a' is not defined as red, he (she) concludes that in fact two pairs of contradictory statements have been made: 'a is pink' and 'a is not pink', and 'a is red' and 'a is not red'. The classical concept of consequence applied to these two pairs of statements enables one to infer any statement, for instance 'a is brown'. The predicate 'pink' ('red') has been precisified in two different ways, one of which makes true the sentence 'a is pink' ('a is red'), and the other makes true the sentence 'a is not pink' ('a is not red'). There is a difference in this respect between these two pairs of sentences and the following pair of sentences: 'Frogs have four legs' and 'Frogs have six legs'. We know that only one of these two sentences is true and the other is false. But it would not be quite right to say the same about 'a is pink' and 'a is red' if 'a' is a borderline case of pink and red objects. This preliminary discussion leads to the following desiderata concerning an adequate theory for vagueness.

(a) The Principle of Bivalence is not valid for vague language.

(b) The sentence 'There is a number n such that a man with n hairs on his head is bald and a man with $(n+1)$ hairs on his head is not bald' is not true.

(c) Classically valid sentences remain valid even though they contain vague predicates.

(d) Two contradictory sentences containing a vague predicate applied to a borderline case do not imply any sentence.

(e) A framework for vagueness should be justified by facts of linguistic communication.

[15] Cf. Tomasello 1999, 517.

[16] See Clark 1996, 125.

5. *Four-Valued Bilattice-Based Logic as Theory of Vagueness*

There exist two different perspectives of looking at vagueness: the perspective of listener and the perspective of observer. The listener obtains statements made by the speaker which have a definite (classical) truth-value, or strings of words or sounds which are not intended as statements, and then without a definite (classical) truth-value. The observer obtains statements of many speakers (at least two); and it may happen that some of those statements contradict each other. Let us suppose now that the observer wants to define a valuation function for vague atomic sentences. The observer has to make a decision as to the truth-value of the sentence 'a is pink' taking into account the fact that the two statements have been made: 'a is pink' and 'a is not pink'. In other words, the information which has been obtained by the observer is that the sentence 'a is pink' is both true and false, since the sentence is classically false if and only if its negation is true. But the fact that 'a is pink' is both true and false means only that 'pink' has been precisified in two different ways, such that 'a is pink' is true for one way of making the predicate more precise and false for the other. There is a difference between being the sentence 'a is pink' both true and false and being it without a definite (classical) truth-value. In this last case this amounts to the fact that no information about 'a' has been obtained. There is also a difference between the logical status of being both true and false and the logical status of the classical truth-values, since being a sentence both true and false is understood as being known as true and known as false. To make a difference between sentences which do not have any classical truth-value, that is, those which are not known as true and are not known as false, and those sentences which are known as true and known as false, to the two classical truth-values will be added two epistemic ones: 'neither true nor false' designated as $\perp$ and 'both true and false' designated as 'T'. They may be called epistemic truth-values, because they reflect the observer's knowledge about truth-values of vague atomic sentences. Suppose that the observer knows that 'a is pink' is true and that 'a is pink' is false and that he also knows that 'a is red' is true and that 'a is red' is false. Now, suppose that the observer forms the disjunction: 'a is pink or red'. Does the disjunction change the information about 'a'? If each disjunct is known as true and false, also the disjunction is known as true and false. A similar observation concerns conjunction; if each conjunct is known as true and false, also conjunction is known as true and false.

This leads to Belnap's (1977) four-valued logic, which needs to be extended and modified if it is to be considered as a formal setting for vagueness. Belnap's logic, which is bilattice-based logic, is known as FOUR.[17]

[17] FOUR is the name of a bilattice and logic base on that bilattice. See Fitting 1989, 229.

The four truth-values: t (true), f (false), T (both true and false) and $\bot$ (neither true nor false) are arranged in two partial orderings denoted as: $\leq_{\mathrm{t}}$ and $\leq_{\mathrm{k}}$, where $\leq_{\mathrm{t}}$ represents increase in truth or measure of truth, and $\leq_{\mathrm{k}}$ represents an increase in knowledge or information. With respect to $\leq_{\mathrm{t}}$, t is the greatest element and f is the least one, but T and $\bot$ are intermediate values which are incomparable. With respect to $\leq_{\mathrm{k}}$, T is the greatest element and $\bot$ is the least one, but t and f are incomparable. Under both orderings we have a complete lattice, that is, $(\{\mathrm{t, f, T}, \bot\}, \leq_{\mathrm{t}})$ is a lattice with the meet and join denoted as $\wedge$ and $\vee$, respectively, and $(\{\mathrm{t, f, T}, \bot\}, \leq_{\mathrm{k}})$ is a lattice with the meet and join denoted as $\otimes$ and $\oplus$, respectively. Then these operations are defined in the standard way as:

Def. $\mathrm{a} \vee \mathrm{b} = \sup_{\leq_{\mathrm{t}}}\{\mathrm{a, b}\}$;
$\mathrm{a} \oplus \mathrm{b} = \sup_{\leq_{\mathrm{k}}}\{\mathrm{a, b}\}$;
$\mathrm{a} \wedge \mathrm{b} = \inf_{\leq_{\mathrm{t}}}\{\mathrm{a, b}\}$;
$\mathrm{a} \otimes \mathrm{b} = \inf_{\leq_{\mathrm{k}}}\{\mathrm{a, b}\}$.

The operation of negation, denoted as $\neg$, which behaves like the classical negation on the classical truth-values, has the following properties in FOUR:

1. If $\mathrm{a} \leq_{\mathrm{t}} \mathrm{b}$ then $\neg\mathrm{a} \geq_{\mathrm{t}} \neg\mathrm{b}$;
2. If $\mathrm{a} \leq_{\mathrm{k}} \mathrm{b}$ then $\neg\mathrm{a} \leq_{\mathrm{k}} \neg\mathrm{b}$;
3. $\neg\neg\mathrm{a} = \mathrm{a}$.

Thus, the FOUR negation is order preserving with respect to $\leq_{\mathrm{k}}$ and order reversing with respect to $\leq_{\mathrm{t}}$. The negation may be also defined by the following truth-table:

p	$\neg p$
t	f
T	T
$\bot$	$\bot$
f	t

Fig. 1. The truth-table of negation.

The four truth-values arranged in the two orderings together with the operation of negation form an algebraic structure called bilattice.[18]
The operators $\otimes$ and $\oplus$ (and also $\wedge$ and $\vee$) are monotone with respect to the $\leq_{\mathrm{t}}$-ordering and the operators $\wedge$, $\vee$ and $\neg$ (and also $\otimes$ and $\oplus$) are monotone with respect to the $\leq_{\mathrm{k}}$-ordering. It is easy to prove that the monotonicity

[18] This definition of bilattice comes from Ginsberg. See Ginsberg 1988, 270.

implies the following equations:[19]

1. $\bot \wedge \mathrm{T} = \mathrm{f}$.
2. $\bot \vee \mathrm{T} = \mathrm{t}$.
3. $\mathrm{f} \otimes \mathrm{t} = \bot$.
4. $\mathrm{f} \oplus \mathrm{t} = \mathrm{T}$.

Next we need a properly defined function of valuation for a certain formal language which will serve as a representation or model of natural language containing vague predicates. We shall assume that a propositional language with the operators defined above plays the role of that model. Intuitively adequate valuation should reflect connections between statements occurring in communicative acts and atomic sentences of that formal language. Let 'A', 'B' denote two different speakers, and 'X' denote any atomic sentence. We shall define mappings $\mathrm{l_A}$ and $\mathrm{l_B}$ corresponding to the speakers 'A' and 'B' in the following way:

Def. $\mathrm{l_A(X) = t}$ if A makes the statement X
$\mathrm{l_A(X) = f}$ if A makes the statement $\neg$X
$\mathrm{l_A(X)} = \bot$ otherwise
and similarly for the speaker 'B':
$\mathrm{l_B(X) = t}$ if B makes the statement X
$\mathrm{l_B(X) = f}$ if B makes the statement $\neg$X
$\mathrm{l_B(X)} = \bot$ otherwise.

The intuitive meaning of these two mappings is that the listener attributes truth, falsehood or no truth-value according to the respective statements made by the speaker in a given communicative situation. Next we can define the valuation function for atomic sentences as follows:

Def. $\mathrm{O(X) = l_A(X) \oplus l_B(X)}$.

Again, the intuitive meaning of this is that the atomic sentences are evaluated according to the observer's knowledge about the truth-values assigned by the listener to the corresponding statements made by the speakers 'A' and 'B'. The valuation O(X) takes into account the fact that in the case of disagreement between 'A' and 'B', the observer accepts both truth-values, and in consequence the atomic sentence 'X' has the truth-value 'T'.

But the function of valuation defined above represents only one possible approach to conflicting information. The other approach is when in the case of disagreement between speakers the observer assigns no truth-value to a

[19] Cf. Fitting 1989, 239.

given atomic sentence. In other words, in the case of conflicting information concerning truth-value of a vague atomic sentence, the sentence remains not known as true and not known as false, that is, it has the truth-value $\perp$. On this approach, the valuation for atomic sentences should be defined in the following way:

Def. $O(X) = l_A\ (X) \otimes l_B(X)$.

Now, let us say that the valuation v defined as usual as a function from the set of atomic sentences of the formal language into {t, f, T, $\perp$} is called intended if for each atomic sentence 'X' : v(X) = O(X). The valuation 'v' can be extended to compound sentences in the standard way, so that the connectives occurring in compound sentences correspond to the operations on {t, f, T, $\perp$}.

The language of FOUR should be augmented with a connective which would serve for representing implication. The first thought is to define the implication, as in the classical logic by '¬p ∨ q'. However, it is not good solution, because 'p → p' is not a tautology on this definition. This is a consequence of the fact that there are no tautologies in Belnap's logic, and in this case v(¬p ∨ p) = $\perp$ if v(p) = $\perp$, where $\perp$ is not a designated value. The implication connective, ⇒, will be defined then by the truth-table:

$p \Rightarrow q$	t	T	$\perp$	f
t	t	T	$\perp$	f
T	t	T	$\perp$	f
$\perp$	t	t	t	t
f	t	t	t	t

Fig. 2. The truth-table of implication.

The implication defined in this way makes *Modus Ponens* valid, and also the deduction theorem holds for ⇒.[20] Note that (1) ⇒ is a generalization of classical implication in this sense that on the classical truth-values ⇒ has the same truth-values as the classical implication, and (2) ⇒ is not monotone with respect to $\leq_k$. The function of valuation 'v' extends on the set of compound sentences of the language augmented with ⇒ in the standard way.

The logic FOUR has t and 'T' as designated values. We shall call a valuation which maps a given formula into an element of the set of designated values D = {t, T} a model of the formula. Having fixed the set of designated values, we shall define the basic consequence relation of the propositional

[20] Cf. Arieli and Avron 1996, 45.

language over the set of connectives $\{\neg, \wedge, \vee, \otimes, \oplus, \Rightarrow\}$, denoted by L, in the following way:

Def. We shall say that a formula α of the language L is a consequence of a set Γ of formulae of that language, what we shall symbolize as $\Gamma \models^4 \alpha$, if every model of Γ is a model of α.

This consequence relation of the logic FOUR is monotonic, compact, paraconsistent and it has sound and complete Gentzen-type proof system.[21] It is easy to show that the consequence relation defined in this way is paraconsistent in this sense that the pair of formulae α and $\neg\alpha$ does not imply an arbitrary formula. For instance, in the classical logic the set $\{p, \neg p\}$ has no model, because there is no valuation which maps each formulae of this set into t. But in four-valued bilattice-based logic FOUR $\{p, \neg p\}$ has a model for v(p) = T. Hence, in the classical logic an arbitrary formula 'q' follows from this set, but it is not the case in the logic FOUR. As we have seen, paraconsistency is a desirable property of reasoning containing vague predicates, because it blocks the inference of an arbitrary sentence from two contradictory sentences which contain a vague predicate applied to a borderline case. Note that the standard three-valued logics, like those of Kleene and Łukasiewicz, in which the non-classical value is not designated, are not paraconsistent. Another drawback of these logics from our point of view is that they invalidate classically valid formulae. For instance, 'p $\vee$ ¬p' is not valid in the Łukasiewicz 3-valued logic and also it is not valid in the Kleene 3-valued logic, where there are no tautologies at all. On the other hand, if one takes a three-valued logic with the set of values {t, f, T}, where t and 'T' are designated values, the logic is paraconsistent, but it still invalidates some classically valid formulae, like the Disjunctive Syllogism.[22] We have the same problem with the logic FOUR. While it is paraconsistent, it is strictly weaker than classical logic even for consistent languages, for example 'p $\vee$ ¬p' is not valid for this logic, since for v(p) = $\perp$, the formula has the value $\perp$. The Disjunctive Syllogism is also invalid with respect to the basic consequence relation of FOUR. A solution which we suggest concerns a certain modification of the consequence relation $\models^4$.

[21] See Arieli and Avron 2000: Proposition 3.2.

[22] Cf. Avron 1991, 285.

6. *Preferential Consequence Relation and Sorites Paradoxes*

The consequence relation $\models^4$ has been defined in the standard way, in this sense that all models of the set Γ have been taken into account. We shall define now a consequence relation with respect to a subset of all models of Γ.[23] The consequence relation which we are going to define may be called a preferential consequence relation, because it takes into account models chosen with respect to certain preference. In this case it will be the consequence relation preferring classical models whenever they are possible, but it allows for models with the non-classical truth-value 'T' only for those formulae which do not have classical models. The point is to define a consequence relation which preserves classical logical truths and classically valid arguments, but simultaneously is paraconsistent.

Let us denote the set of non-classical truth-values as J = {T, ⊥}. Let I(v, J) = {p: p is atomic and v(p) ∈ J} be the set of atomic formulae which have been assigned a non-classical truth-value, that is 'T' or ⊥. Let us say that a model 'm' is more consistent than a model 'n' with respect to J if I(m, J) ⊂ I(n, J), which means that 'm' has less atomic formulae evaluated as 'T' or ⊥ than 'n' does.

Def. We shall call 'm' a most consistent model of Γ with respect to J if there is no other model of Γ which is more consistent than m with respect to J.

These stipulations enable us to define the preferential consequence relation for the language L, which we shall denote as $\models^{4,J}$, in the following way:

Definition : $\Gamma \models^{4,J} \alpha$ if every most consistent model of Γ is a model of α.

Note that if Γ is classically consistent, then α classically follows from Γ if and only if $\Gamma \models^{4,J} \alpha$. An important observation concerning this preferential consequence relation is that it is nonmonotonic. In classical logic, if α is a logical consequence of a set of formulae Γ, and if Γ' is a set of formulae such that $\Gamma \subseteq \Gamma'$, then α is also a logical consequence of Γ'. In other words, the set of consequences increases monotonically with the set of premises. But it is not the case with $\models^{4,J}$. If we have the set of formulae Γ' such that $\Gamma \subseteq \Gamma'$, then for a certain formula α it is the case that $\Gamma \models^{4,J} \alpha$, but it is not the case that $\Gamma' \models^{4,J} \alpha$. The nonmonotonicity of this consequence relation is connected with the fact that Γ and Γ' represent information of different value. If we accept α as reasonable on the basis of Γ and a new formula in

[23] Cf. Arieli and Avron 1998, 116.

Γ' added to Γ contradicts the grounds which led us to accept α, or the new formula is regarded as representing information more reliable than the old reason to believe α, we shall not accept α any more as reasonable.[24] Let us take an example with vague sentences. Let Γ be the set of the following two sentences: 'a is red' and 'It is not the case that a is red and pink'. If r stands for 'a is red' and p stands for 'a is pink', then $\{r, \neg(r \wedge p)\} \models^{4,J} \neg p$, because the set of the premises Γ has only one most consistent model 'm': m(r) = t and m(p) = f, which is a model of the conclusion. Now, let us assume that the sentence 'a is pink' has been added to Γ which extends the set of premises to the set: $\Gamma' = \Gamma \cup \{p\}$. It is easy to show that it is not the case that $\{r, \neg(r \wedge p), p\} \models^{4,J} \neg p$, because there is the following most consistent model 'n': n(r) = T and n(p) = t of the premises Γ' which is not a model of the conclusion. The new sentence 'a is pink' added to the old premises may be considered as more reliable than the old reasons to believe that 'a is not pink'. This difference between 'a is pink' and 'a is red' is not perceptible in the case of the classical consequence relation where sentences are handled as if they were precise.

The preferential consequence relation makes possible a new approach to sorites paradoxes. Let us consider again the 'Bald Man' Paradox (II) which has been regimented in the following way:

(II) $B(a, 0)$
$\neg B(a, 100.000)$
$\therefore [B(a, 0) \wedge \neg B(a, 1)] \vee [B(a, 1) \wedge \neg B(a, 2)]$
$\vee \ldots \vee [B(a, n) \wedge \neg B(a, n+1)]$
$\vee \ldots \vee [B(a, 100.000 - 1) \wedge \neg B(a, 100.000)]$.

This argument is classically valid, since there is no classical model which makes the premises true and the conclusion false. It is also validated by the supervaluational semantics, because its premises and conclusion are supertrue, that is, true for all admissible precisifications. Now, the question arises if each disjunct of the conclusion of the 'Bald Man' Paradox (II) should have a classical truth-value. If the function of valuation is defined as the intended valuation $O(X) = l_A(X) \otimes l_B(X)$, the premises are true, but the conclusion does not have a classical truth-value. Let us notice that:

First Premise : $O(B(a, 0)) = l_A(B(a, 0)) \otimes l_B(B(a, 0)) = t \otimes t = t$.

Since $O(B(a, 100.000)) = l_A(B(a, 100.000)) \otimes l_B(B(a, 100.000)) = f \otimes f = f$, then

[24] Cf. Schlechta 1997, 3.

Second Premise : $O(\neg B(a, 100.000)) = t$.

The premises have the only most consistent model: O. However, the model O is not a model of the conclusion. The disjuncts which occur at the beginning and at the end of this disjunction are evaluated by the valuation O as false, but the others (at least one) are evaluated as $\perp$, and in consequence the conclusion takes on the value $\perp$. That is, for at least one n:

$O(B(a, n)) = l_A(B(a, n)) \otimes l_B(B(a, n) = t \otimes f = \perp$ and
$O(B(a, n{+}1)) = l_A(B(a, n{+}1)) \otimes l_B(B(a, n{+}1)) = t \otimes f = \perp$, and then
$O(\neg B(a, n{+}1)) = \perp$ and
$O[B(a, n) \wedge \neg B(a, n{+}1)] = \perp$.

Thus, the premises of the paradox (II) are evaluated as true, but its conclusion is evaluated as $\perp$, because it is a disjunction whose disjuncts have the value f or $\perp$, and $\sup\{f, \perp\} = \perp$. This shows that there is a most consistent model of premises which is not a model of the conclusion. The intended valuation O is a counter most consistent model for this argument. In this way one can show that the conclusion of the 'Bald Man' Paradox (II) does not follow from its premises, and then the argument is *invalid*.

Now, consider again the 'Bald Man' Paradox (I), which has been regimented as follows:

(I) $B(a, 0)$
$[B(a, 0) \Rightarrow B(a, 1)] \wedge [B(a, 1) \Rightarrow B(a, 2)]$
$\wedge \ldots \wedge [B(a, n) \Rightarrow B(a, n{+}1)]$
$\wedge \ldots \wedge [B(a, 100.000 - 1) \Rightarrow B(a, 100.000)]$
$\therefore B(a, 100.000)$.

The second premise evaluated by the intended valuation O defined by the operation $\otimes$ has true conjuncts at the beginning and at the end of the conjunction, but also (1) conjuncts with true antecedent and consequent evaluated as $\perp$, (2) conjuncts with antecedent and consequent evaluated as $\perp$, and (3) conjuncts with antecedent evaluated as $\perp$ and false consequent. That is, for at least one n:

(1) $O(B(a, n)) = t \otimes t = t$ and
$O(B(a, n{+}1)) = t \otimes f = \perp$, and then
$O[B(a, n) \Rightarrow B(a, n{+}1)] = t \Rightarrow \perp = \perp$;

(2) $O(B(a, n)) = t \otimes f = \perp$ and
$O(B(a, n{+}1)) = t \otimes f = \perp$, then
$O[B(a, n) \Rightarrow B(a, n{+}1)] = t$.

(3) O(B(a, n)) $= t \otimes f = \bot$ and
O(B(a, n+1)) $= f \otimes f = f$, and then
O[B(a, n) $\Rightarrow$ B(a, n+1)] $= \bot \Rightarrow f = t$.

Evaluated in this way the major premise of the 'Bald Man' Paradox (I) has the truth-value $\bot$, because it is a conjunction whose conjuncts have the value t or $\bot$, and $\inf\{\bot, t\} = \bot$. Having the conclusion evaluated by the intended valuation O as false and one premise evaluated by O as true, but the other as $\bot$, the argument is *unsound*. The argument remains unsound when the valuation O is defined with the help of the operation $\oplus$. It is easy to show that the second premise is evaluated then as false.

Note that on that analysis, when the valuation O is defined in terms of the operation $\otimes$, the conclusion of the 'Bald Man' Paradox (II) is the four-valued negation of the second premise of the 'Bald Man' Paradox (I). As we mentioned before, this conclusion is also the classical negation of the sentence 'For all n: If a man with n hairs on his head is bald then a man with $(n + 1)$ hairs on his head is bald', where 'for all n' and 'there is an n' are replaced respectively by the classical general and existential quantifiers and the implication is defined as '$\neg$p $\vee$ q'.

7. *Comparison with Subvaluational Semantics*

A paraconsistent approach to vagueness is characteristic of the so called subvaluational semantics.[25], but the idea that vagueness may lead to contradictory statements, like 'a is B' and 'a is not B', originates from Jaśkowski (1948). The subvaluational semantics makes use of the concept of admissible precisification, which occurs in the supervaluational semantics. However, borderline cases for a vague predicate like 'bald' are regarded as those to which the predicate both applies and does not apply, in this sense that if 'a' is a borderline case of 'bald' then 'a is bald' and 'a is not bald' are true, or 'a is bald' is true and 'a is bald' is false. On this approach, 'a is bald' is true (simpliciter) if 'a is bald' is true for a certain admissible precisification of 'bald'.[26] This is a claim opposed to that of supervaluational semantics, where a sentence is true (simpliciter) just in case it is true for all admissible precisifications. Kit Fine notes that on the supervaluational approach "truth is secured if it does not turn upon what one means".[27] But it is not the case

[25] See Hyde 1997.

[26] ibid., 647.

[27] Cf. Fine 1975, 278.

for the subvaluational semantics, where truth (simpliciter) depends on a particular way of making a given vague predicate more precise. The concept of truth in the subvaluational semantics contrasts with the conviction about objectivity of truth. In the four-valued bilattice-based logic, the sentence 'X' which is true for one admissible precisification and false for another is regarded as known as true and known as false, and it is evaluated in epistemic terms as 'T'. Subvaluationism attributes truth and falsehood to vague atomic sentences; but many of us could regard that assumption as unintuitive.

Another contentious feature of subvaluationism is that it does not preserve the classical consequence relation in those cases where it ought to be preserved, because it invalidates adjunction:

$$\mathrm{p}, \neg \mathrm{p} \text{ not} \vDash_{\mathrm{sb}} \mathrm{p} \wedge \neg \mathrm{p}.$$

If 'p' is a vague sentence, then 'p' may be true for a certain precisification and then true simpliciter and '¬p' may be true for a certain precisification and true simpliciter, but 'p ∧ ¬p' is not true for any precisification and then not true simpliciter. Note that

$$\mathrm{p}, \neg \mathrm{p} \vDash^{4,\mathrm{J}} \mathrm{p} \wedge \neg \mathrm{p},$$

since the only most consistent model of 'p' and '¬p' is m(p) = T, which is also the model of 'p ∧ ¬p'. This difference between the classical logic and subvaluational semantics points to the fact that the concept of subvaluational truth does not preserve classical meaning of conjunction. A similar problem appears in the supervaluational semantics where from the truth of a disjunction one cannot infer the truth of one of its disjuncts, in particular:

$$\mathrm{p} \vee \neg \mathrm{p} \text{ not} \vDash_{\mathrm{sp}} \mathrm{p}, \neg \mathrm{p}.$$

If 'p' is true for some precisification and false for another, it is not supertrue and its negation is not supertrue, but 'p ∨ ¬p' is supertrue, since it is true for any precisification. Thus, the concept of supertruth does not preserve the classical meaning of disjunction. Note that

$$\mathrm{p} \vee \neg \mathrm{p} \vDash^{4,\mathrm{J}} \mathrm{p}, \neg \mathrm{p},$$

since 'p ∨ ¬p' has two most consistent (and classical) models: m(p) = t and n(p) = f, and each is a model of some formula of the set {p, ¬p}.

8. *Higher-Order Vagueness*

An interesting fact concerning vagueness is that it is not possible to determine the boundaries of the borderline region, since borderline cases also have borderline cases. If a man has few fewer hairs on his head than Tom who is a borderline case of a bald man, the man is a borderline case of a borderline case of a bald man, and then a second order borderline case. It seems that borderline cases of a higher order vagueness than the second one are beyond our discriminatory ability, and they play no role in communicative acts. Anyway, it is beyond doubt that there is no sharp boundary between definitely positive and definitely negative instances of a given vague predicate, and there is no sharp boundary between definitely positive and borderline cases on the one hand, and definitely negative and borderline cases on the other. Let us consider how these facts could be accommodated to the present approach. The question arises then how truth-values of the four-valued logic FOUR are distributed among vague sentences. To see this let us take the second premise of the 'Bald Man' Paradox (I):

(P) If a man with n hairs on his head is bald then a man with $(n + 1)$ hairs on his head is bald.

If for each atomic sentence 'X', the function of valuation v(X) $=$ O(X) is defined with the help of the operation $\otimes$, the premise has the value $\bot$, because each intended valuation distributes truth-values among atomic constituents of this premise in such a way which gives the value $\bot$. For each O(X) there is a different number n, such that the following cases hold:

(1) "A man with n hairs on his head is bald" is true.
"A man with $(n + 1)$ hairs on his head is bald" is true.
(2) "A man with n hairs on his head is bald" is true.
"A man with $(n + 1)$ hairs on his head is bald" is $\bot$.
(3) "A man with n hairs on his head is bald" is $\bot$.
"A man with $(n + 1)$ hairs on his head is bald" is $\bot$.
(4) "A man with n hairs on his head is bald" is $\bot$.
"A man with $(n + 1)$ hairs on his head is bald" is false.
(5) "A man with n hairs on his head is bald" is false.
"A man with $(n + 1)$ hairs on his head is bald" is false.

Thus, for each valuation O(X) some conjuncts of the premise are true, but some other conjuncts have the value $\bot$. In a similar way is evaluated the conclusion of the 'Bald Man' Paradox (II) : For each valuation O(X) there is a number n, such that the following pair of sentences:

A man with n hairs on his head is bald
A man with $(n+1)$ hairs on his head is not bald

has values belonging to five groups: (1) t and f; (2) t and $\perp$; (3) $\perp$ and $\perp$; (4) $\perp$ and f; (5) f and t. Each valuation O(X) differs from the other as to the number n; but in each case the truth-values assigned to the pairs of sentences belong to one of the five groups, although for many numbers we are not able to point at exactly one such a pair. In this sense we can say that there is no sharp cut-off between definitely positive and negative cases of the predicate 'bald'; and in this sense there is no sharp cut-off between positive cases and borderline ones, that is, those cases which are evaluated as $\perp$; in the same sense, there is no sharp cut-off also between borderline cases of 'bald' and those cases which are definitely negative. There are no such sharp boundaries and we do not know about the existence of such boundaries: This idea is characteristic of our account of vagueness.

9. *Conclusions*

Everyone as a user of a natural language may be a speaker in one communicative act and a listener or observer in another. There is no trouble with vagueness for the speaker and listener. For communicative purposes, a vague predicate may be positively or negatively applied to its borderline case, and from the perspective of the speaker, each choice is good, since listener's understanding of speaker's statement does not depend on that choice. But the real problem emerges for the observer, because he has to do with contradictory statements. This paper shows how reasoning with vague language may be modeled from the perspective of observer. This has been done by using the four-valued bilattice-based logic FOUR, the intended valuation O, and the preferential consequence relation $\models^{4,\mathrm{J}}$, which is based on the consequence relation of FOUR. It has been shown that our formal setting for vagueness has the following desirable features:

1. It makes sorites paradoxes a kind of non-paradoxical reasoning, since it shows that they are either unsound or invalid.
2. It saves classical logical truths and classical consequence relation for classically consistent sets of sentences expressed in the language L.
3. It blocks inferences of an arbitrary sentence from two contradictory statements containing a vague predicate applied to a borderline case.
4. It may be adapted to higher-order vagueness.

5. It makes use of intuitively clear concepts, like the classical truth and the concept of being known as true and being known as false.
6. It accepts empirical facts concerning language communication.[28]

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