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STRUCTURALISM AND CATEGORY THEORY IN THE CONTEMPORARY PHILOSOPHY OF MATHEMATICS*

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Abstract

The set-theoretical (Bourbaki-style) and category-theoretical approach to structuralism in the philosophy of mathematics are compared. Advantages and disadvantages of those approaches are indicated. We come to the conclusion that category theory can be the language of mathematical structuralism but one should be careful to claim that it can serve as a foundation for the whole of mathematics.

1. One of the most popular trends of the contemporary philosophy of mathematics is structuralism usually connected with the slogan: mathematics is the science of structures. Mathematical structuralism can be characterized as a view that the subject of any branch of mathematics is a structure or structures. For example, we can define a natural number system to be a countably infinite collection of objects with one distinguished initial object and the successor relation that satisfy the principle of mathematical induction. Therefore the natural number is nothing more then a place in the structure of natural numbers. According to the structuralism, arithmetic is a science about the form or structure common to natural number systems.

Structuralism is consonant with current mathematical practice at least in two points: 1) objects considered by mathematicians are determined up to isomorphism, 2) at least some features of mathematical objects, some mathematical facts about them, depend solely on their structure. But what is a mathematical structure? What do we mean by "having the same structure"?

In the contemporary philosophy of mathematics various structuralistic conceptions have been formulated. They differ with respect to the way of defining structures and their existence. One can divide them into two groups corresponding to two main perspectives for the structural mathematics: foundational (set-theoretical) and categorical one.

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Let us begin with the foundational (set-theoretical) perspective in defining structures due to Bourbaki. Bourbaki structure is a domain of objects with some relations and functions defined on it. In this case we use such terms as set, function or relation, which are terms of set theory. Thus we can apply methods of model theory to investigate it.[1] Such description of mathematical objects leads us to a useful structural perspective but in many cases methods of model theory itself do not suffice to describe mathematical structures well enough.

Generally among set-theoretic structuralistic conceptions one can distinguish two main attitudes towards ontological problems:

(a) *in re* structuralism (called also eliminative structuralism) which states in particular that all statements about numbers are only generalizations. The *in re* structuralism claims that the natural number structure is nothing more than systems which are its instantiations. If such particular systems were destroyed then there would be also no structure of natural numbers.

(b) *ante rem* structuralism which claims that structures do exist apart from the existence of their particular examples. It is often said that *ante rem* structures have ontological priority with respect to their instantiations.

The main thesis of the eliminative structuralism is: statements about some kind of objects should be treated as universal statements about specific kind of structures. Thus number theory examines properties of all structures of order type. In case of arithmetic every sentence A expresses a property of all natural number systems, and can be understood as an implication:

For every system S, if S is an instance of natural number system, then $A(S)$.

This treatment of the natural numbers rests on two claims: the claim that simply infinite systems do exist and on the categoricity theorem. It is necessary to prove the existence of a natural number system, otherwise the above implication is always true, because every implication with a false predecessor is always true. So eliminative structuralism needs a basic ontology, a domain of considerations whose objects could take up places in structures *in re*. Such an ontology should be rich enough and we are not interested in the very nature of objects but rather in their quantity. The ontology of the *in re*

[1] By such an approach we change a statement that mathematics is a science about structures into a statement that mathematics is a science about sets, so we reduce mathematics not to the theory of structures but to set theory.

structuralism requires an infinite base. One of the methods proposed to eliminate this problem is to apply modalities. Hellman introduced, using second order modal logic, a theory containing the axiom stating the possibility of the existence of an infinite system.

In any case, in taking structures to be objects, we either run into the problem of having to assume a foundational background (eliminative structuralism) or of 'reification of structures' (*ante rem* structuralism) or we make mathematics dependent of the logic of possibility (modal structuralism).

2. The object of modern mathematical studies is rarely a specific set with relations or functions defined on it. As said above mathematicians investigate mostly objects determined up to isomorphism, relations between such objects bearing the same structure, relations between different kinds of structures on such objects and so on. So there is a need for a language and methods well suited to problems involving different kinds of structures. In response to this need category theory arose.

Category theory is an algebraic theory, which is a general mathematical theory of structures and of systems of structures. It is still evolving. At minimum, it is a powerful language, or conceptual framework, allowing us to see the common parts of a family of structures of a given kind as well as how structures of different kinds are interrelated.

The central role in this theory is played by the notion of category, which consists of objects A, B, C,... and morphism f, g, h,... such that: i) every f has a unique domain A and a unique codomain B, written f: A $\to$ B; ii) given any g: B $\to$ C there is a unique composite $g \circ f$: A $\to$ C, with composition being associative; iii) each B has an identity 1_B: B $\to$ B, which is a unit for composition, i.e., $1_B \circ f = f$ and $g \circ 1_B = g$ for any f and g as stated. A category is anything satisfying these axioms.

The objects need not to have elements; nor need the morphism be functions (for example category associated with any formal system of logic is a category, the objects of which are formulas and morphisms of which are deductions from premises).

Consider now whether a categorical perspective in structuralism is better, at least in some points, than the foundational (set-theoretical) one.

First of all the categorical notion of a structure is "syntax invariant", it does not depend on particular choice among the different possible set theoretic descriptions of a given kind of mathematical structures. For example spaces may be defined in several ways, the objects of the category Top (i.e., the category of all topological spaces) are described by various different Bourbaki structures.

The categorical notion of an isomorphism may serve as a definition of "having the same structure of a given type". Category theory provides a uniform notion of a structure: given any category, one automatically knows the right notion of having the same structure. Two objects may be said to bear the same structure if they are structurally indistinguishable, i.e., if any structural property enjoyed by one is also enjoyed by the other.

According to structuralism objects of mathematics (such as numbers, functions or points) are only places in structures, they do not have any properties which are not structural. Structuralists claim that mathematical objects have no important features outside the structure and all of their features have to and can be explained in terms of structural relations. For example the number 2 is nothing more then the successor of 1 and the predecessor of 3, so the essence of a natural number (for example 2) is determined by relations to other natural numbers (1 and 3). (Thus arithmetic is the science about relations between places of any system similar to the structure of natural numbers.) Category theory allows us to express structural properties of objects in a convenient way. Any mathematical property or construction given in terms of structure preserving mappings (in a given category) will necessarily respect isomorphism in that category and thus will be structural. Since all categorical properties are structural, the only properties which a given object in a given category may have, qua object in that category, are structural ones. As Awodey states in (Awodey, 1996):

> "Thus doing mathematics 'arrow-theoretically' automatically provides a structural approach, and this has proven quite effective in attacking certain kinds of mathematical problems having to do with mathematical structure" (pp. 214–215)

Furthermore many useful categories describe some structures which are not structures in the sense of Bourbaki. For example category whose objects are the open sets of a particular space and whose morphisms are inclusion maps between them is a kind of a mathematical structure on objects which is not a model of a Bourbaki structure in any conventional sense.

A further and very important advantage of the categorical approach to mathematical structure is that representing different kinds of structures as different categories provides a uniform notion of a structure. For example from a categorical point of view, a Cartesian product in set theory, a direct product of groups (Abelian or otherwise), a product of topological spaces, and a conjunction of propositions in a deductive system are all instances of a categorical product characterized by a universal property.

Formally, a *product* of two objects X and Y in a category C is an object Z of C *together* with two morphisms, called the projections, p: $Z \to X$ and q: $Z \to Y$ such that — and this is the universal property — for all objects W with morphisms f: $W \to X$ and g: $W \to Y$, there is a unique morphism h: $W \to Z$ such that $p \circ h = f$ and $q \circ h = g$.

It is totally consonant with mathematical structuralism. Note that we have defined *a* product for X and Y and not *the* product for X and Y. Indeed, products and other objects with a universal property are defined only up to a (unique) isomorphism.

In category theory, the nature of elements constituting a certain construction is irrelevant. What matters is the way in which an object is related to other objects of the category, that is, the morphisms going in and the morphisms going out, or, in other words, how certain structures can be mapped into a given object and how a given object can map its structure into other structures of the same kind.

Category theory reveals how different kinds of structures are related to one another (it is not so easy in the case of set-theoretic approach to structuralism). For instance, in algebraic topology, topological spaces are related to groups (and modules, rings, etc.) in various ways (such as homology, cohomology, homotopy, K-theory). Groups with group homomorphisms constitute a category. Eilenberg and Mac Lane invented category theory precisely in order to clarify and compare these connections. What matters are the morphisms between categories, given by functors. Homology, cohomology, homotopy, K-theory are all examples of functors. Informally, functors are structure-preserving maps between categories. Given two categories C and D, a functor F from C to D sends objects of C to objects of D, and morphisms of C to morphisms of D, in such a way that the composition of morphisms in C is preserved, i.e., $F(g \circ f) = F(g) \circ F(f)$, and identity morphisms are preserved, i.e., $F(id_X) = id_{FX}$. It immediately follows that a functor preserves commutativity of diagrams between categories.

Following Awodey (1996) we can characterize categorical structuralism in the following way:

> "The structural perspective on mathematics codified by categorical methods might be summarized in the slogan: The subject matter of pure mathematics is invariant form, not a universe of mathematical objects consisting of logical atoms." (p. 235)

3. Some philosophers claim that category theory is an alternative to set theory as a foundation for mathematics and that methods of category theory will suffice for many present-day mathematical purposes. But there are some problems and objections connected with this claim.

The first one is a problem of the autonomy of category theory from set theory. Is category theory really independent from set theory? If we agree that category theory uses set-theoretic notions such as domain, codomain and function, then structuralism framed by category theory falls under set-theoretic variety of structuralism. Moreover category theory cannot be treated as an alternative for set theory in any reasonable sense of 'alternative'.

Another important problem announced by Hellman (in Hellman 2003) is a problem of mathematical existence. "This problem as it confronts category theory can be put very simply: the question just does not seem to be addressed! (We might dub this the *problem of the 'home address'* : *where do categories come from and where do they live?*)" (p. 136)

Axioms defining categories include existence claims, but if we want to read this axioms 'structurally' (à la algebra), they are only defining conditions, not absolute assertions of truths based on established meanings of primitive terms (the axioms of set theory, as usually read, are not 'structural' in this sense).

To sum up, Hellman (2003) claims that category theory is defective as a framework for structuralism in at least two major interrelated ways: 1) it is not independent from set theory and 2) it lacks substantive axioms of mathematical existence.

As Awodey noticed in (2004) the questions asked about mathematical existence such as: "Where do categories come from and where do they exist" are reasonable only from the foundational perspective. He proposed to use category theory to avoid the whole business of 'foundations'. The idea of 'doing mathematics categorically' involves a point of view different from the foundational one, which is based on the idea of specifying for a given theorem or theory only the required or relevant degree of information or structure, the important features of a given situation, without assuming some knowledge or specification of the 'objects' involved. He writes in (2004): "The laws, rules and axioms involved in a particular piece of reasoning, or a field of mathematics, may vary from one to the next, or even from one mathematician or epoch to another." (p. 56)

Mathematical theorems are schematic, they do not involve the specific nature of structures or their components in an absolute sense. It does not matter what structures are supposed to be or to 'consists of'. In mathematical statements particular nature of the entities involved plays no role. Rather their relations, operations, etc. are important and crucial. In this sense mathematical statements (theorems, proofs, even definitions) are about connections, operations, relations, properties of connections, operations on relations, connections between those operations and so on.

Thus according to this view there is no absolute universe of all mathematical objects, there is no unique context that provides us with conditions for the actual or possible existence of structure or structured systems. In a categorical framework the context, systematized by the category-theoretic axioms, varies, so mathematical concepts has to be thought of in a context that can be varied in a systematic fashion. Categorical framework provides us with the conditions a context has to satisfy in order to talk about or to do mathematics. So we cannot say what the natural numbers are, but in which contexts we can talk about them.

Category theory describes conditions under which we can talk about the same type of systems. Category should be treated not as a system of statements about objects (i.e., neither about "structures" nor about possible types of systems possessing a structure), but rather as a context describing conditions, which have to be fulfilled to talk about particular type of objects. Axioms of a given category provide context in which one can talk about the common structure of systems in terms of morphism between them, without necessity of appealing to the theory of sets, theory of structures or modal logic.

Supporter of such structuralism does not have to determine what is a structure or what is a category, in ontological or modal sense of the word "is". Everything what has to be done is to provide a proper context, in which one can talk about a common structure of systems.

An advantage of such an approach to structuralism is that it does not provide "constructive basis" for mathematics, but rather provides "descriptive basis" for the structuralistic claim that mathematics is a science of structures (it is interpreted as a claim that mathematics is a science of systems possessing structures).

So what is the difference between set-theoretical structuralism and the categorical one? Hale named categorical structuralism the *pure* structuralism and described it as algebraic structuralism *in re* from the *top-down* perspective.

Now the natural question appears: what is the difference between *top-down* structuralism and the *bottom-up* (set-theoretical) one? Structuralism from the *bottom-up* perspective should have, as said above, a basic ontology. The notion of structure is built from the objects of this ontology in the process of abstraction. "The direction" of this abstraction is clear: from details to the whole, so *bottom-up*. For all versions of set-theoretical structuralism the same conditions, actual or modal, for the existence or possible existence of systems possessing structure, have to be assumed.

In the case of *top-down* structuralism this demand can be omitted by introducing a basic theory in Hilbert's sense. Instead of asking what structures are, there appears the question: what does it mean that two systems have the same structure. *Top-down* structuralism is called pure because axioms of a category provide a framework for talking about particular structural systems without considering what those systems are built from.

In the *top-down* perspective one starts from the concept of an abstract system, in the algebraic sense, understood as a language for description of the common structure of systems: it allows to talk about systems possessing the same structure as examples of the same type of structure without the necessity of considering from what those systems are built. So in this perspective instead of asking what the structure is one asks what does it mean "to have the same structure".

Therefore category theory provides a framework for *top-down*, *in-re* interpretations of mathematical structuralism, because category provides context, in which one can talk about "common structure" of systems, regardless of what this systems are built from. Such *top-down* algebraic structuralism, expressed in the language of the category theory, does not require neither treating structures as "objects" (actual or possible) nor understanding axioms as truths or assertions. In contrast to Shapiro, categorical structuralist does not have to claim that categories exist as objects independently of abstract systems, which are examples of them: he/she does not even claim that categories exist in the sense of "objects" in some system. Categorical structuralism can be summarized by words of Awodey (1996):

> "The subject matter of pure mathematics is an invariant form and not a universe of mathematical objects consisting of logical atoms." (p. 235)

To sum up, we must distinguish the claim that category theory can be the language of mathematical structuralism from the claim that it can be an alternative for set theory as a basic theory for mathematics. Category theory is a more convenient tool for exploring mathematical structuralism than set theory but one should be careful to claim that category theory can serve as a foundation for the whole of mathematics. Indeed, it is not clear if category theory is really independent from set theory, moreover we do not know enough about the ontological and epistemological status of categories. Category theory is the useful language for talking about mathematical structuralism but it is not a tool for "doing" mathematics structurally.

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