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POSITIVE ABSTRACTION AND EXTENSIONALITY REVISITED

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Abstract
We here exhibit Scott-style models for positive abstraction with extensionality, which have recently revealed connections between that positive set theory and an illative system of lambda calculus. These Scott-style models also bring out a very characteristic feature of that theory, some consequences of which, such as a quantifier elimination result, are presented here.

1. *Introduction*

This paper is the continuation of [4] & [6] which treat of Positive Set Theory. It is to be seen as a complementary paper of both [6] and [8], which latter is concerned with an extensional system of illative lambda calculus. We shall recall some results of those papers to motivate our investigations.

Let us consider the following rules of formation of terms and formulas in first-order predicate calculus with equality:

(1) Any variable x is a term;

(2) $\perp$ and $\top$ are atomic formulas;

(3) If τ and σ are terms, then $\tau \in \sigma$ is an atomic formula;

(4) If τ and σ are terms, then $\tau = \sigma$ is an atomic formula;

(5) If φ and ψ are formulas, so are $\varphi \wedge \psi$ and $\varphi \vee \psi$;

(6) If φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas;

(7) If φ is a formula, so is $\neg\varphi$;

(8) If φ is a formula and x is a variable, then $\{x \mid \varphi\}$ is a term.

We denote by $\mathscr{L}$ the language obtained from (1)–(7), and for any fragment Σ of $\mathscr{L}$, we let $\mathrm{Comp}[\Sigma]$ stand for the scheme of formulas

$$\forall \bar{z} \exists y \forall x (x \in y \leftrightarrow \varphi(x, \bar{z}))$$

where $\varphi(x, \bar{z})$ is in Σ. The language extended by rule (8) is designated by $\mathscr{L}_\tau$, and then, given a fragment Σ of $\mathscr{L}_\tau$, we let $\mathrm{Abst}[\Sigma]$ stand for the scheme of formulas

$$\forall \bar{z} \forall x (x \in \{x \mid \varphi\} \leftrightarrow \varphi(x, \bar{z}))$$

with $\varphi(x, \bar{z})$ in Σ. The *positive* fragments of $\mathscr{L}$ and $\mathscr{L}_\tau$ are defined by proscribing the use of (7), which we indicate by adding the superscript '$+$'. Adding '$*$' as subscript means that we do not consider the use of (4) either. At last, given a set-theoretic structure $\mathcal{U} \equiv \langle U; \in_{\mathcal{U}} \rangle$, any of these languages may conveniently be extended by constants naming the elements of U, and we shall indicate this by juxtaposing '(U)' to the right, e.g. $\mathscr{L}_{\tau *}^{+}(U)$.

Positive Set Theory originated in Skolem's papers [10, 11] where he showed that $\mathrm{Comp}[\mathscr{L}_*^+]$ is consistent with Ext, the axiom of extensionality, i.e.

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

The consistency of $\mathrm{Comp}[\mathscr{L}^+] + \mathrm{Ext}$ came out much later of [4] in which was proved the existence of κ-topological models for any *weakly compact* cardinal κ. By a κ-topological model is meant here an extensional model $\mathcal{U} \equiv \langle U; \in_{\mathcal{U}} \rangle$ in which the *coded* subsets of U — i.e. those of the form $\{u \in U \mid u \in_{\mathcal{U}} v\}$ for some (unique) $v \in U$ — are exactly the *closed* sets of some κ-topology on U; and by a κ-topology is meant a topology in which any union of strictly less than κ closed sets is still closed.

On the other hand, $\mathrm{Abst}[\mathscr{L}_\tau{}^+]$ is easily proved to be consistent by a term model construction, but it is known to be inconsistent together with Ext. Nevertheless, it was shown in [6] that there exist extensional *term* models of $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}]$, and the proof of this is quite subtle. The consistency of $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}] + \mathrm{Ext}$ will fall out more easily of this paper as we are going to show the existence of κ-topological models for any *regular* cardinal κ.

As shown in [9], the interest of topological models for set theory can be related to the consistency of the following principle:

$$(\dagger) \qquad \forall x \forall y (x \mathrel{\dot{\leqslant}} y \rightarrow x \mathrel{\underset{\cdot}{\leqslant}} y)$$

where $x \mathrel{\underset{\cdot}{\leqslant}} y$ stands for $\forall z (z \in x \rightarrow z \in y)$ and $x \mathrel{\dot{\leqslant}} y$ for $\forall z (x \in z \rightarrow y \in z)$. It is easy to see that $(\dagger)$ holds in any set theory in which '$\{x \mid y \in x\}$' exists for all y, or more obviously in any one in which '$\{x \mid y = x\}$' exists for all y. Thus, in particular, $(\dagger)$ follows from $\mathrm{Comp}[\mathscr{L}^+]$ as well as from $\mathrm{Abst}[\mathscr{L}_{\tau*}^+]$. Now it is proved in [9] that if $(\dagger)$ holds then $\bigcap`\{x \mid \psi(x,\bar{z})\}'$ and $\bigcup`\{x \mid \psi(x,\bar{z})\}'$ cannot both exist for all $\psi(x,\bar{z})$. At least, in a κ-topological model $\mathcal{U}$ of $(\dagger)$, $\bigcap`\{x \mid \psi(x,\bar{z})\}'$ exists for all $\psi(x,\bar{z})$ while at the same time $\bigcup`\{x \mid \psi(x,\bar{z})\}'$ can exist for each $\psi(x,\bar{z})$ such that $|\{u \in U \mid \mathcal{U} \models \psi(u,\bar{w})\}| < \kappa$ for all $\bar{w}$ in U. Thus, it is the existence of κ-topological models of $\mathrm{Comp}[\mathscr{L}^+] + \mathrm{Ext}$ for some $\kappa > \aleph_0$ that led to showing in [4] that a natural extension of that theory is compatible with a relevant axiom of infinity, so the resulting system is strong enough to interpret ZF and much more (see [3]). In another context, the existence of κ-topological models for $\mathrm{Abst}[\mathscr{L}_{\tau*}^+] + \mathrm{Ext}$ will here reveal that this theory satisfies the converse of $(\dagger)$, and this will result in rather unexpected features such as quantifiers elimination.

At last, the discovery of topological models for $\mathrm{Abst}[\mathscr{L}_{\tau*}^+] + \mathrm{Ext}$ — and particularly their resemblance with topological models of lambda calculus — inspired the author to inquire into possible connections between that positive set theory and some extensional systems of illative lambda calculus. This has proved successful and such a system is presented in [8].

2. *Duality*

We begin by a simple observation that will clarify later considerations.

Define the $(\cdot)^*$-operator on formulas and terms of $\mathscr{L}_{\tau*}$ as follows:

$$(\tau \in \sigma)^* \text{ is } \tau^* \in \sigma^*,$$

$$\bot^* \text{ is } \top, \qquad \top^* \text{ is } \bot,$$

$$(\varphi \wedge \psi)^* \text{ is } \varphi^* \vee \psi^*, \qquad (\varphi \vee \psi)^* \text{ is } \varphi^* \wedge \psi^*,$$

$$(\forall x\varphi)^* \text{ is } \exists x\varphi^*, \qquad (\exists x\varphi)^* \text{ is } \forall x\varphi^*,$$

$$(\neg\varphi)^* \text{ is } \neg\varphi^*,$$

$$\{x \mid \varphi\}^* \text{ is } \{x \mid \varphi^*\},$$

$$\text{and } x^* \text{ is } x \text{ for any variable } x.$$

Clearly φ^{**} is φ for any φ in $\mathscr{L}_{\tau*}$, and we notice that if φ is in $\mathscr{L}_{\tau*}^+$, so is φ^*.

Now, given a set-theoretic structure $\mathcal{U} \equiv \langle U; \in_{\mathcal{U}} \rangle$, let $\mathcal{U}^*$ stand for $\langle U; \notin_{\mathcal{U}} \rangle$, in which the interpretation of the abstractor for $\mathscr{L}_{\tau*}^+$-formulas — provided such an interpretation is given in $\mathcal{U}$ — is defined by $\{x \mid \varphi\}^{\mathcal{U}^*}(\bar{v}) :=$

$\{x \mid \varphi^*\}^{\mathcal{U}}(\bar{v})$ for every $\varphi(x, \bar{y})$ in $\mathscr{L}_{\tau*}^+$ and $\bar{y} := \bar{v}$ in U. Clearly $\mathcal{U}^{**} = \mathcal{U}$, and one can show by induction on the complexity of a formula that

Fact 2.1: $\mathcal{U}^* \models \varphi(\bar{v})$ iff $\mathcal{U} \models \neg\varphi^*(\bar{v})$, for any $\mathscr{L}_{\tau*}$-formula $\varphi(\bar{y})$ whose atomic formulas are in $\mathscr{L}_{\tau*}^+$, and for any $\bar{y} := \bar{v}$ in U.

It is therefrom easy to see that

Fact 2.2: $\mathcal{U}^*$ fulfils $\mathrm{Abst}[\mathscr{L}_{\tau*}^+]$ if and only if $\mathcal{U}$ does.

We have not yet specified the interpretation of = in $\mathcal{U}^*$. Recall that $x = y$ is definable by $\Psi(x, y) :\equiv \forall z(z \in x \leftrightarrow z \in y)$ under Ext. Now we notice that $\neg\Psi^*(x, y)$ is just equivalent to $\Psi(x, y)$. Therefore, any interpretation of = making $\mathcal{U}^*$ extensional whenever $\mathcal{U}$ is must coincide with the one in $\mathcal{U}$ — and in all the structures we consider this is understood to be the *identity*, i.e. $\mathcal{U} \models u = v$ iff u and v are the same element in U.

Accordingly, if we define $(\tau = \sigma)^*$ and $(\tau \neq \sigma)^*$ to be respectively $\tau^* \neq \sigma^*$ and $\tau^* = \sigma^*$, then one can extend Fact 2.1 to the case where φ contains atomic formulas $\tau = \sigma$ and $\tau \neq \sigma$ with τ, σ in $\mathscr{L}_{\tau*}^+$. This, together with Fact 2.2, yields the following *duality principle*:

Fact 2.3: Let Σ stand for $\mathrm{Abst}[\mathscr{L}_{\tau*}^+]$ or for $\mathrm{Abst}[\mathscr{L}_{\tau*}^+] + \mathrm{Ext}$, and let φ be an $\mathscr{L}_{\tau}$-formula whose atomic formulas are in $\mathscr{L}_{\tau*}^+$, or of the form $\tau = \sigma$ and $\tau \neq \sigma$ with τ, σ in $\mathscr{L}_{\tau*}^+$. Then $\Sigma \vdash \varphi$ if and only if $\Sigma \vdash \neg\varphi^*$.

A simple consequence we mention here is the following. Let us say that an $\mathscr{L}_{\tau*}^+$-term τ is *autodual* if τ^* is τ. Then, for all pairs of autodual *closed* terms τ, σ, we have $\Sigma \vdash \tau \in \sigma$ iff $\Sigma \vdash \neg(\tau \in \sigma)^*$ iff $\Sigma \vdash \tau \notin \sigma$, and it follows that $\tau \in \sigma$ is not decidable from Σ — for we recall that Σ is consistent. The simplest example of an autodual term is $\{x \mid x \in x\}$, which we denote by W. Note that this also shows that $\mathcal{U}$ and $\mathcal{U}^*$ can never be isomorphic as set-theoretic structures, otherwise we would have $\mathcal{U} \models \mathrm{W} \notin \mathrm{W}$ iff $\mathcal{U} \models \mathrm{W} \in \mathrm{W}$.

Remark 2.1: Such a duality principle as Fact 2.3 is not provable for $\mathrm{Comp}[\mathscr{L}^+] + \mathrm{Ext}$. Indeed, if it was to hold, any model of that theory should satisfy $\forall z \exists y \forall x(x \in y \leftrightarrow x \neq z)$, but this is false in a topological model. We do not know however if this can be true in some model of $\mathrm{Comp}[\mathscr{L}^+] + \mathrm{Ext}$.

3. *Topological models*

On the one hand, the topological models used in [4] to prove the consistency of $\mathrm{Comp}[\mathscr{L}^+] + \mathrm{Ext}$ are T_2 spaces. These appear as κ-compact 0-dimensional κ-uniform spaces that are uniformly isomorphic to their space of closed subsets with the induced κ-uniformity (see e.g. [5]). Note that any topological model of $\mathrm{Comp}[\mathscr{L}^+]$ should at least be T_1 because of the existence of '$\{x \mid y = x\}$' for all y, but we do not know however whether there can be non-T_2 ones.

On the other hand, the topological models of $\mathrm{Abst}[\mathscr{L}_{\tau^*}^{+}]$ we present in this paper are T_0 spaces; anyhow, there cannot be T_1 ones here because $\mathrm{Abst}[\mathscr{L}_{\tau^*}^{+}] + \mathrm{Ext}$ proves the non-existence of '$\{x \mid y = x\}$' for some y — this is related to the inconsistency of $\mathrm{Abst}[\mathscr{L}_{\tau}{}^{+}] + \mathrm{Ext}$, of course. What we are going to show is that any κ-dcpo which is Scott isomorphic to its set of closed subsets ordered by reverse inclusion, or dually to its set of open ones ordered by inclusion, gives rise to a model of $\mathrm{Abst}[\mathscr{L}_{\tau^*}^{+}] + \mathrm{Ext}$. So we shall next introduce what is strictly needed to understand and prove that statement.

Let κ be an infinite regular cardinal. Given a partially ordered set $\langle U; \leqslant\rangle$, we say that a subset D of U is *κ-directed* if every subset of D of cardinality strictly less than κ has an upper bound in D (so $D \neq \emptyset$). Then U is said to be a *κ-dcpo* if each κ-directed subset D of U has a least upper bound $\bigvee D$.

Any κ-dcpo U inherits of a κ-topology, called the *Scott κ-topology*, whose closed subsets are the lower sets S of U satisfying the following closure property: if D is a κ-directed subset of S, then $\bigvee D \in S$. It is easy to see that a map $f : U \longrightarrow V$ between κ-dcpo's is continuous w.r.t. the Scott κ-topologies if and only if f preserves κ-directed suprema, i.e. $f(\bigvee D) = \bigvee\{f(d) \mid d \in D\}$ for each κ-directed subset D of U. We denote the set of all Scott continuous functions from U to V, ordered pointwise, by $[U \to V]$. This is a κ-dcpo as well, and even a complete lattice provided V is. Examples of interest to us are $[U \to 2]$ and $[U \to 2']$, where 2 is $\{0, 1\}$ with $0 < 1$ and $2'$ is the opposite ordered set. These are respectively isomorphic to $\mathcal{P}_{\mathrm{op}}(U)$, the complete lattice of Scott open sets ordered by inclusion, and to $\mathcal{P}_{\mathrm{cl}}(U)$, the one of Scott closed sets ordered by reverse inclusion — which are obviously isomorphic to each other, for any given U, as $2 \cong 2'$.

We designate the category of κ-dcpo's with Scott continuous maps as morphisms by DCPO_κ. It can be seen that this is a cartesian closed category, the exponential of which being given by $[\cdot \to \cdot]$ — the proof of this is just a routine generalization of the original case $\kappa = \aleph_0$, as treated in [1] for instance. This observation is mainly all we need in order to show that any solution to $U \cong [U \to 2]$ in DCPO_κ yields a model for positive abstraction, in a similar way that it was originally proved by Scott that any solution to $U \cong [U \to U]$

in $\mathsf{DCPO}_{\aleph_0}$ gives a model for the untyped λ-calculus. Note that by a solution to a reflexive equation $U \cong \mathcal{F}(U)$ in DCPO_κ we always mean a κ-dcpo U *together with* a Scott homeomorphism $f : U \longrightarrow \mathcal{F}(U)$.

Theorem 3.1: *Suppose* $\langle U; f\rangle$ *is a solution to* $U \cong [U \to 2]$ *in* DCPO_κ. *Then the set-theoretic structure* $\mathcal{U} \equiv \langle U; \in_{\mathcal{U}}\rangle$ *defined by* $u \in_{\mathcal{U}} v$ *iff* $f(v)(u) = 1$ *is an extensional model of* $\mathrm{Abst}[\mathscr{L}^+_{\tau*}]$.

Proof. For any *closed* formula φ of $\mathscr{L}_\tau(U)$, let $|\varphi|_{\mathcal{U}}$ stand for the truth value of φ interpreted in $\mathcal{U}$. In particular, for all $u, v \in U$, we have $|u \in v|_{\mathcal{U}} = \epsilon_{\mathcal{U}}(u, v)$, where $\epsilon_{\mathcal{U}} : U \times U \longrightarrow 2 : (u, v) \longmapsto f(v)(u)$, which is Scott continuous in both of its arguments. We are going to show by induction on the complexity that each term $\tau(\bar{y})$ of $\mathscr{L}^+_{\tau*}(U)$, given with a list of variables $\bar{y} = y_1, \ldots, y_n$ to which its free variables belong, has a 'suitable' Scott continuous interpretation $\tau^{\mathcal{U}} : U^n \longrightarrow U : (\bar{v}) \longmapsto \tau^{\mathcal{U}}(\bar{v})$.

First, if $\tau(\bar{y})$ is just a variable, say y_k in $\bar{y}$, then we take $\tau^{\mathcal{U}} : (\bar{v}) \longmapsto v_k$, which is clearly Scott continuous; and if $\tau(\bar{y})$ is any fixed $u \in U$, then we take $\tau^{\mathcal{U}} : (\bar{v}) \longmapsto u$, which is also Scott continuous.

Now we turn to the case where $\tau(\bar{y})$ is a set abstract $\{x \mid \varphi\}$ for a $\mathscr{L}^+_{\tau*}(U)$-formula $\varphi(x, \bar{y})$. Here, that the interpretation is 'suitable' means, of course, that $|u \in \tau^{\mathcal{U}}(\bar{v})|_{\mathcal{U}} = |\varphi(u, \bar{v})|_{\mathcal{U}}$ for any $u, \bar{v}$ in U, from which incidentally results the uniqueness of such a suitable interpretation. The proof goes by induction on the complexity of $\varphi(x, \bar{y})$:

- φ is $\bot$ / $\top$. Let $\curlywedge$ / $\curlyvee$ stand for $f^{-1}(u \mapsto 0)$ / $f^{-1}(u \mapsto 1)$.
  Then $\tau^{\mathcal{U}} : (\bar{v}) \longmapsto \curlywedge$ / $\curlyvee$ is the suitable interpretation of $\{x \mid \varphi\}$.
- φ is $\sigma(x, \bar{y}) \in \rho(x, \bar{y})$, where σ, ρ are $\mathscr{L}_\tau(U)$-terms.
  Then $\tau^{\mathcal{U}} : (\bar{v}) \longmapsto f^{-1}(u \mapsto \epsilon_{\mathcal{U}}(\sigma^{\mathcal{U}}(u, \bar{v}), \rho^{\mathcal{U}}(u, \bar{v})))$ is Scott continuous, and this is the suitable interpretation of $\{x \mid \varphi\}$.
- φ is $\psi(x, \bar{y}) \vee \chi(x, \bar{y})$. Let $\sigma(\bar{y})$ stand for $\{x \mid \psi\}$ and $\rho(\bar{y})$ for $\{x \mid \chi\}$.
  Then $\tau^{\mathcal{U}} : (\bar{v}) \longmapsto f^{-1}(u \mapsto \underline{\vee}(\epsilon_{\mathcal{U}}(u, \sigma^{\mathcal{U}}(\bar{v}))\,,\, \epsilon_{\mathcal{U}}(u, \rho^{\mathcal{U}}(\bar{v})))$ is the suitable interpretation of $\{x \mid \varphi\}$; it is Scott continuous for so is $\underline{\vee} : 2 \times 2 \to 2 : (a, b) \mapsto \max\{a, b\}$.
- φ is $\psi(x, \bar{y}) \wedge \chi(x, \bar{y})$.
  As above, but with $\underline{\wedge} : 2 \times 2 \to 2 : (a, b) \mapsto \min\{a, b\}$ instead of $\underline{\vee}$.

- φ is $\exists z\psi(x,\bar{y},z)$. Let $\sigma(\bar{y},z)$ stand for $\{x \mid \psi\}$.
  Then, for all $u,\bar{v}$ in U, we have

$$\begin{aligned}|\exists z\psi(u,\bar{v},z)|_{\mathcal{U}} &= \max_{w\in U} |\psi(u,\bar{v},w)|_{\mathcal{U}} \\ &= \max_{w\in U} |u \in \sigma^{\mathcal{U}}(\bar{v},w)|_{\mathcal{U}} \\ &= |u \in \sigma^{\mathcal{U}}(\bar{v},\curlyvee)|_{\mathcal{U}}.\end{aligned}$$

  It follows therefrom that $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto f^{-1}(v \mapsto \epsilon_{\mathcal{U}}(u, \sigma^{\mathcal{U}}(\bar{v},\curlyvee)))$ is the suitable Scott continuous interpretation of $\{x \mid \varphi\}$.

- φ is $\forall z\psi(x,\bar{y},z)$.
  As above, but with 'min' instead of '$\max$' and then $\curlywedge$ instead of $\curlyvee$.

To have a clear conscience, one would make sure that the interpretation of $\{\cdot \mid -\}$ so defined satisfies the following natural substitutivity property: for any $\mathscr{L}_{\tau*}^{+}$-formula $\varphi(x,\bar{y})$ and list of $\mathscr{L}_{\tau*}^{+}$-terms $\bar{\tau}(\bar{z})$ of the same length as $\bar{y}$, we have $\{x \mid \varphi\}^{\mathcal{U}}(\bar{\tau}^{\mathcal{U}}(\bar{u})) = \{x \mid \psi\}^{\mathcal{U}}(\bar{u})$ for all $\bar{u}$ in U, where ψ is the formula $\varphi(x,\bar{\tau}(\bar{z}))$. This actually follows from the fact that

$$\mathcal{U} \models \mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}.$$

$\dashv$

As $2 \cong 2'$, any solution to $U \cong [U \to 2']$ will likewise give rise to a model of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}]$. In fact, every solution $\langle U; f\rangle$ to $U \cong [U \to 2]$ can be turned into a solution $\langle U; g\rangle$ to $U \cong [U \to 2']$, and vice versa, by setting $g(v)(u) = \underline{\neg}(f(v)(u))$ for all $u,v \in U$, where $\underline{\neg} : 2 \longrightarrow 2' : 0/1 \mapsto 1/0$. Clearly, if $\mathcal{U}$ is the model of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}]$ associated with $\langle U; f\rangle$, then the one corresponding to $\langle U; g\rangle$ is just $\mathcal{U}^*$, the dual of $\mathcal{U}$, as defined in Section 2. Although they are based upon the same underlying complete lattice U, it is worth recalling that $\mathcal{U}$ and $\mathcal{U}^*$ are not isomorphic as set-theoretic structures. On the contrary, U and the opposite ordered set U' must be isomorphic as κ-dcpo's, for we have $U' \cong [U \to 2]' = [U \to 2'] \cong U$.

Remark 3.1: We mention that natural Scott-style models of the system of illative λ-calculus considered in [8] can be obtained as solutions to a reflexive equation of the form $U \cong [U \to U] + 2$ in $\mathsf{DCPO}_{\aleph_0}$. Such solutions naturally contains $[U \to 2]$, which models the pure set-theoretic part of that system. This would show how $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ is related — on the semantic side at least — to the system called 'Positive Frege' in [8].

We shall now exhibit one particular solution to $U \cong [U \to 2]$ [resp. $U \cong [U \to 2']$] in DCPO_κ for each regular cardinal κ.

4. *Canonical solutions*

Roughly, following [1], the *canonical* solution to $U \cong [U \to 2]$ [resp. $U \cong [U \to 2']$] in DCPO_κ can be obtained as inverse limit by iterating the functor $[\cdot \to 2]$ [resp. $[\cdot \to 2']$] from the initial object $\{\curlywedge\}$, where $\curlywedge$ is a bottom element. Note that ω iterations will not suffice when $\kappa > \aleph_0$, and then limit steps in the iteration process will introduce some complications that fortunately we need not discuss here. The simplicity of the solution we are about to describe will convince ourselves that this must be the canonical one.

For any regular cardinal κ, let S_κ be a linearly ordered set of type $\kappa+1+\kappa'$, where κ' is the reverse of κ, say $S_\kappa := \{a_\alpha \mid \alpha \in \kappa\} \cup \{c\} \cup \{b_\alpha \mid \alpha \in \kappa\}$ with $a_\alpha < c < b_\beta$ for all $\alpha, \beta \in \kappa$, and $a_\alpha < a_\beta$ iff $b_\beta < b_\alpha$ iff $\alpha \in \beta$. Clearly S_κ is a complete linear order, and it may then be looked at as a κ-dcpo. Its lower sets are $A_\beta := \{a_\alpha \mid \alpha \in \beta\}$ and $B_\beta := S_\kappa \setminus \{b_\alpha \mid \alpha \in \beta\}$, for all $\beta \in \kappa + 1$. Notice that each B_β has a maximum, so it is Scott closed. Also is A_β for all $\beta \in \kappa$, because then $|A_\beta| < \kappa$. But A_κ is not Scott closed. Indeed, A_κ is a κ-directed subset of itself (because κ is regular) but $\bigvee A_\kappa = c \notin A_\kappa$. It follows that $\mathcal{P}_{\mathrm{cl}}(S_\kappa)$, ordered by reverse inclusion $\supseteq$, is also of order type $\kappa + 1 + \kappa'$, the order isomorphism $g : S_\kappa \longrightarrow \mathcal{P}_{\mathrm{cl}}(S_\kappa)$ being defined by $g(a_\beta) := B_\beta$, $g(b_\beta) := A_\beta$, for all $\beta \in \kappa$, and $g(c) := B_\kappa$. Moreover, it is easy to see that g preserves all suprema, i.e. $g(\bigvee D) = \bigcap\{g(d) \mid d \in D\}$ for all $D \subseteq S_\kappa$ — the key observation here is that $\bigcap\{B_\beta \mid \beta \in \kappa\} = B_\kappa$ — so that g is in particular a Scott homeomorphism. Dually, one defines a Scott homeomorphism $f : S_\kappa \longrightarrow \mathcal{P}_{\mathrm{op}}(S_\kappa)$ by $f(a_\beta) := S_\kappa \setminus B_\beta$, $f(b_\beta) := S_\kappa \setminus A_\beta$, for all $\beta \in \kappa$, and $f(c) := S_\kappa \setminus B_\kappa$. The pairs $\langle S_\kappa; f\rangle$ and $\langle S_\kappa; g\rangle$ give respectively the canonical solutions to $U \cong [U \to 2]$ and to $U \cong [U \to 2']$ in DCPO_κ. According to Theorem 3.1, we now have, for each regular cardinal κ, two canonical (and non-isomorphic) models of $\mathrm{Abst}[\mathscr{L}_{\tau^*}^{+}]$ denoted by $\mathcal{S}_\kappa$ and $\mathcal{S}^*_\kappa$.

5. *Monotonicity*

In any model $\mathcal{U}$ associated with a solution $\langle U; f\rangle$ to $U \cong [U \to 2]$, it is easy to see that the underlying order $\leqslant$ of U coincides with $\dot{\leqslant}_{\mathcal{U}}$.[1] In fact, it does

[1] We recall that $[U \to 2]$ is naturally isomorphic to $\mathcal{P}_{\mathrm{op}}(U)$ ordered by *inclusion*, whereas $[U \to 2']$ is isomorphic to $\mathcal{P}_{\mathrm{cl}}(U)$ ordered by *reverse inclusion*.

coincide with $\dot{\leqslant}_{\mathcal{U}}$ too. We already know that

$$\mathcal{U} \models \forall x \forall y (x \mathrel{\dot{\leqslant}} y \rightarrow x \mathrel{\underset{\cdot}{\leqslant}} y).$$

And on the other hand, we have $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v$ iff $f(w)(u) \leqslant f(w)(v)$ for all $w \in U$, which will then be true if we assume $u \mathrel{\underset{\cdot}{\leqslant}} v$, because each $f(w)$ is *monotone*. Therefore $\mathcal{U} \models \forall x \forall y (x \mathrel{\underset{\cdot}{\leqslant}} y \rightarrow x \mathrel{\dot{\leqslant}} y)$ as well, and thus $\underset{\cdot}{\leqslant}$ and $\dot{\leqslant}$ coincide in a Scott-style model. This actually holds in any model of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$:

Theorem 5.1: $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} \vdash \forall x \forall y (x \mathrel{\underset{\cdot}{\leqslant}} y \rightarrow x \mathrel{\dot{\leqslant}} y)$.

Proof. Assume $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}]$ and let a, b be such that $a \mathrel{\underset{\cdot}{\leqslant}} b$ but $a \mathrel{\dot{\nleqslant}} b$. Then define $\sigma(x) = \{z \mid z \in a \vee (x \in x \wedge z \in b)\}$. It easily follows from $a \mathrel{\underset{\cdot}{\leqslant}} b$ that $\forall x((x \in x \rightarrow \sigma(x) = b) \wedge (x \notin x \rightarrow \sigma(x) = a))$ (note that Ext is needed here). Now, as $a \mathrel{\dot{\nleqslant}} b$, take c such that $a \in c, b \notin c$, and define $\rho = \{x \mid \sigma(x) \in c\}$. Thus we have $\forall x(x \in \rho \leftrightarrow x \notin x)$, and therefore $\rho \in \rho \leftrightarrow \rho \notin \rho$. ⊣

Remark 5.1: We stress that the use of Ext is essential, e.g. it is shown in [7] that there are term models of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}]$ not fulfilling

$$\forall x \forall y (x \mathrel{\underset{\cdot}{\leqslant}} y \rightarrow x \mathrel{\dot{\leqslant}} y).$$

A more explicit formulation of Theorem 5.1 is as follows:

$$\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} \vdash \forall x \forall y \forall z (x \mathrel{\underset{\cdot}{\leqslant}} y \wedge x \in z \rightarrow y \in z)\,,$$

which clearly says that in any model $\mathcal{U}$ of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ whatsoever, $\{u \in U \mid u \in_{\mathcal{U}} v\}$ is an *upper set* of $\langle U; \underset{\cdot}{\leqslant}_{\mathcal{U}} \rangle$ for each $v \in U$; in other words, its characteristic function $f(v)$, defined by $f(v)(u) := 1$ if $u \in_{\mathcal{U}} v$ and $:= 0$ otherwise, is *monotone* from $\langle U; \underset{\cdot}{\leqslant}_{\mathcal{U}} \rangle$ to $\langle 2; \leqslant \rangle$.

Now, another equivalent formulation and an important consequence of Theorem 5.1 are the following.

Corollary 5.2: *For any formula* $\varphi(x, \bar{s})$ *of* $\mathscr{L}_{\tau*}^{+}$,

$$\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} \vdash \forall x \forall y \forall \bar{s} (x \mathrel{\underset{\cdot}{\leqslant}} y \wedge \varphi(x, \bar{s}) \rightarrow \varphi(y, \bar{s})).$$

Proof. Take $z := \{x \mid \varphi\}$ in the previous formulation. ⊣

Corollary 5.3: *For every formula* φ *of* $\mathscr{L}_{\tau*}^{+}$*, there exists a quantifier-free formula* φ_0 *of* $\mathscr{L}_{\tau*}^{+}$ *such that* $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} \vdash \varphi \leftrightarrow \varphi_0$.

Proof. Let Λ and V respectively denote $\{x \mid \bot\}$ and $\{x \mid \top\}$. Using Corollary 5.2, we have $\exists x\varphi(x, \bar{s}) \leftrightarrow \varphi(\mathrm{V}, \bar{s})$ and $\forall x\varphi(x, \bar{s}) \leftrightarrow \varphi(\Lambda, \bar{s})$ for every formula $\varphi(x, \bar{s})$ of $\mathscr{L}_{\tau*}^{+}$. We can accordingly eliminate the quantifiers of each $\mathscr{L}_{\tau*}^{+}$-formula written in prenex form. Note that by saying that φ_0 is quantifier-free it is meant here that there are no quantifiers at all, even in the formulas defining the set abstracts occurring in φ_0, and so forth. $\dashv$

By duality, a version of Corollary 5.3 holds as well for negations of $\mathscr{L}_{\tau*}^{+}$-formulas. But monotonicity is no longer a property of $\mathscr{L}_{\tau}{}^{+}$-formulas, seeing for instance that $\exists y(y = \{x \mid y \in x\})$ is not equivalent to $\Lambda = \{x \mid \Lambda \in x\}$, as this latter is false whereas the former can be true as we shall see in Section 7.

6. *Numerals*

Let $\mathscr{B}(y) := \{x \mid y \in x\}$, $\Lambda := \{x \mid \bot\}$ and $\mathrm{V} := \{x \mid \top\}$. Then define inductively Λ_n and V_n for each natural number n as follows:

$$\begin{cases} \Lambda_0 := \Lambda & \mathrm{V}_0 := \mathrm{V} \\ \Lambda_{n+1} := \mathscr{B}(\Lambda_n) & \mathrm{V}_{n+1} := \mathscr{B}(\mathrm{V}_n). \end{cases}$$

Notice that Λ_n^* is V_n for all $n \in \mathbb{N}$. We are going to show, as a consequence of Theorem 5.1, that $\Lambda_n =$ '$\{\mathrm{V}_k \mid k < n\}$' and so, by duality, that $\mathrm{V}_n = \mathrm{V}\backslash$'$\{\Lambda_k \mid k < n\}$'. More precisely, for each naturel number n, we define the following pair of dual sentences:

$$\begin{cases} \Psi_n :\equiv \forall x(x \in \Lambda_n \leftrightarrow \bigvee_{k<n} x = \mathrm{V}_k) \\ \Psi'_n :\equiv \forall x(x \in \mathrm{V}_n \leftrightarrow \bigwedge_{k<n} x \neq \Lambda_k) \end{cases}$$

and we show that

Proposition 6.1: $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} \vdash \Psi_n \wedge \Psi'_n$*, for all* $n \in \mathbb{N}$.

Proof. Of course, by Fact 2.3, we may concentrate on inferring Ψ_n for all n. We proceed by induction. For $n = 0$ this is immediate. Suppose that Ψ_k, and so Ψ'_k, holds for all $k \leqslant n$, and let $x \in \Lambda_{n+1}$. Then $\Lambda_n \in x$ and it follows from Theorem 5.1 that if $\Lambda_n \mathrel{\dot{\leqslant}} y$ then $y \in x$. Now, by Ψ_n, we have $\Lambda_n \mathrel{\dot{\leqslant}} y$ iff $\mathrm{V}_k \in y$ for all $k \leqslant n-1$, which is equivalent to $y \in \mathrm{V}_{k+1}$ for all

$k \leqslant n-1$, and this reduces to $y \in \mathrm{V}_n$ because $\mathrm{V}_n \underset{\cdot}{\leqslant} \mathrm{V}_{n-1} \leqslant \cdots \underset{\cdot}{\leqslant} \mathrm{V}_0$ by Ψ'_k for $k \leqslant n$. Hence $\mathrm{V}_n \underset{\cdot}{\leqslant} x$. If $\mathrm{V}_n \neq x$, then, by Ψ'_n, there exists $l < n$ such that $\Lambda_l \in x$, that is $x \in \Lambda_{l+1}$, and then, by Ψ_{l+1}, we have $x = \mathrm{V}_k$ for some $k < l+1 \leqslant n$. Thus Ψ_{n+1} is true. ⊣

One can even say more about the structure of the Λ_n's and the V_n's from Proposition 6.1. Let us define the following pair of dual sentences:

$$\begin{cases} \Upsilon_n :\equiv \forall x(\Lambda_n \underset{\cdot}{\leqslant} x \wedge \Lambda_n \neq x \rightarrow \Lambda_{n+1} \underset{\cdot}{\leqslant} x) \\ \Upsilon'_n :\equiv \forall x(x \underset{\cdot}{\leqslant} \mathrm{V}_n \wedge x \neq \mathrm{V}_n \rightarrow x \underset{\cdot}{\leqslant} \mathrm{V}_{n+1}). \end{cases}$$

Proposition 6.2: $\mathrm{Abst}[\mathscr{L}^+_{\tau*}] + \mathrm{Ext} \vdash \Upsilon_n \wedge \Upsilon'_n$, *for all* $n \in \mathbb{N}$.

Proof. Again, by Fact 2.3, we may concentrate on inferring Υ_n for each n. Assume $\Lambda_n \underset{\cdot}{\leqslant} x$ and $\Lambda_n \neq x$. To prove that $\Lambda_{n+1} \underset{\cdot}{\leqslant} x$, it suffices to show that $\mathrm{V}_n \in x$, that is, $x \in \mathrm{V}_{n+1}$. But this is true since $x \neq \Lambda_k$ for all $k \leqslant n$. ⊣

Accordingly, the universe U of any model $\mathcal{U}$ of $\mathrm{Abst}[\mathscr{L}^+_{\tau*}] + \mathrm{Ext}$, ordered by $\underset{\cdot}{\leqslant}_{\mathcal{U}}$, has the following form:

$$\underset{\Lambda_0}{\bullet}\ \underset{\Lambda_1}{\bullet}\ \underset{\Lambda_2}{\bullet}\ \cdots\ \langle \mathrm{Ker}\,\mathcal{U} \rangle\ \cdots\ \underset{\mathrm{V}_2}{\bullet}\ \underset{\mathrm{V}_1}{\bullet}\ \underset{\mathrm{V}_0}{\bullet}$$

where $\mathrm{Ker}\,\mathcal{U}$, the *kernel* of $\mathcal{U}$, is

$$\{u \in U \mid u \neq \Lambda_n^{\mathcal{U}} \text{ and } u \neq \mathrm{V}_n^{\mathcal{U}} \text{ for all } n \in \mathbb{N}\}.$$

Notice that neither '$\{\Lambda_n \mid n \in \mathbb{N}\}$' nor '$\{\mathrm{V}_n \mid n \in \mathbb{N}\}$' can be proved to exist as a set from $\mathrm{Abst}[\mathscr{L}^+_{\tau*}] + \mathrm{Ext}$, since the former is not even a $\underset{\cdot}{\leqslant}$-upper set and the latter is not a set in $\mathcal{S}^*_\kappa$ (but it is in $\mathcal{S}_\kappa$). In spite of this, and in view of the characterization above, any of these two classes might legitimately be taken as system of *numerals*. Then, to know to which degree this representation is faithful, one could for instance have a look at those functions $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ that are *representable*, that is, for which there exists a term $\tau(y_1, \ldots, y_k)$ of $\mathscr{L}^+_{\tau*}$ such that

$$\mathrm{Abst}[\mathscr{L}^+_{\tau*}] + \mathrm{Ext} \vdash \tau(\Lambda_{n_1}, \ldots, \Lambda_{n_k}) = \Lambda_{f(n_1,\ldots,n_k)}$$

— or equally

$$\mathrm{Abst}[\mathscr{L}^+_{\tau*}] + \mathrm{Ext} \vdash \tau^*(\mathrm{V}_{n_1}, \ldots, \mathrm{V}_{n_k}) = \mathrm{V}_{f(n_1,\ldots,n_k)}$$

by duality — for all $n_1, \ldots, n_k \in \mathbb{N}$. We shall only content ourselves here with pointing out a couple of examples.

Obviously, the successor function is represented by $\mathscr{B}(y)$. The predecessor function is also representable by taking $\mathscr{B}'(y) := \{x \mid \mathscr{B}(x) \in y\}$. Indeed, we have $\mathscr{B}'(\mathscr{B}(y)) = \{x \mid \mathscr{B}(x) \in \mathscr{B}(y)\} = \{x \mid x \in y\} = y$, so $\mathscr{B}'(\Lambda_{n+1}) = \Lambda_n$ for all $n \in \mathbb{N}$, and clearly $\mathscr{B}'(\Lambda_0) = \Lambda_0$.

Other functions that are representable are 'max' and 'min', seeing that $\Lambda_{n_1} \cup \Lambda_{n_2} = \Lambda_{\max(n_1,n_2)}$ and $\Lambda_{n_1} \cap \Lambda_{n_2} = \Lambda_{\min(n_1,n_2)}$, where the terms $y_1 \cup y_2$ and $y_1 \cap y_2$ are defined as usual by $\{x \mid x \in y_1 \vee x \in y_2\}$ and $\{x \mid x \in y_1 \wedge x \in y_2\}$. More generally, the existence of these terms shows that in any model $\mathcal{U}$ of $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}] + \mathrm{Ext}$, $\langle U; \dot{\leqslant}_{\mathcal{U}} \rangle$ is a lattice, as we have $\bigvee\{a,b\} = (a \cup b)^{\mathcal{U}}$ and $\bigwedge\{a,b\} = (a \cap b)^{\mathcal{U}}$ for all $a, b \in U$. It is very unlikely that more can be said about the ordered set $\langle \mathrm{Ker}\,\mathcal{U}; \dot{\leqslant}_{\mathcal{U}} \rangle$ from $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}] + \mathrm{Ext}$ only, though the question we examine in the next section is somewhat related to this.

7. *Wanted: the fixed-point property*

Given a model $\mathcal{U}$ of $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}] + \mathrm{Ext}$, the interpretation of each term $\sigma(y)$ of $\mathscr{L}_{\tau *}^{+}(U)$ is a monotone function on $\langle U; \dot{\leqslant}_{\mathcal{U}} \rangle$. Now this latter is a complete lattice when $\mathcal{U}$ is a Scott-style model, so we do have $\mathcal{U} \models \exists y(y = \sigma(y))$ in that case, for any monotone function on a complete lattice has a fixed point. The question as to know whether that fixed-point property is derivable from $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}] + \mathrm{Ext}$ is tackled here.

At least, we can give a positive answer in a very particular case, which may be viewed as a sort of set-theoretic remains of the fixed-point theorem of the lambda calculus:[2]

Theorem 7.1: $\mathrm{Abst}[\mathscr{L}_{\tau *}^{+}] + \mathrm{Ext} \vdash \forall \bar{z} \exists y(y = \sigma(y, \bar{z}))$, *for any* $\mathscr{L}_{\tau *}^{+}$*-term* $\sigma(y, \bar{z})$ *of the form* $\{x \mid \varphi(x; \psi(y, \bar{z}); \bar{z})\}$, *where this notation means that we can distinguish a subformula* ψ *of* φ *having* y *and possibly* $\bar{z}$, *but not* x, *as free variables.*

Proof. Let

$$\Omega := \{t \mid \psi(\{x \mid \varphi(x; t \in t; \bar{z})\}, \bar{z})\}$$

and

$$\tau := \{x \mid \varphi(x; \Omega \in \Omega; \bar{z})\}\,.$$

[2] I am grateful to Marcel Crabbé for this observation.

Thus we have $\Omega \in \Omega$ iff $\psi(\tau, \bar{z})$, and then

$$\tau = \{x \mid \varphi(x; \psi(\tau, \bar{z}); \bar{z})\} = \sigma(\tau, \bar{z}).$$

$\dashv$

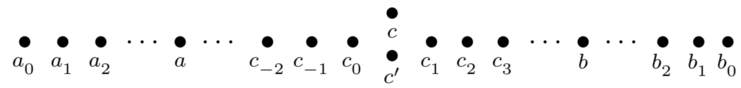
The condition here on the variable y to appear within a subformula is very restrictive as it excludes such simple terms as $\mathscr{B}(y)$ and $\mathscr{B}'(y)$ of Section 6.

For the latter a fixed point is easily found, namely $\mathrm{W} = \{x \mid x \in x\}$, as we have

$$\begin{aligned}\mathscr{B}'(\mathrm{W}) &= \{x \mid \mathscr{B}(x) \in \mathrm{W}\}\\ &= \{x \mid \mathscr{B}(x) \in \mathscr{B}(x)\}\\ &= \{x \mid x \in x\}\\ &= \mathrm{W}.\end{aligned}$$

But for the former the situation is not as simple. In the canonical models of Section 4 it is always the case that W — which is interpreted by c in those models — is a fixed point of $\mathscr{B}(y)$ too. This, however, does not follow from $\mathrm{Abst}[\mathscr{L}^+_{\tau *}]+\mathrm{Ext}$ as we shall now see by exhibiting another Scott-style model.

Let U be the complete lattice depicted as follows, with the ordering $\leqslant$ from left to right:



Given $y \in U$, we let $)y]$ stand for $\{x \in U \mid x \leqslant y\}$ and $)y[$ for $\{x \in U \mid x < y\}$. The lower sets of U, ordered by reverse inclusion, are $)b_n], n \in \omega$, $)b]$, $)b[$, $)c_n], n \in \omega \setminus \{0\}$, $)c_1[$, $)c]$, $)c']$, $)c_{-n}], n \in \omega$, $)a]$, $)a[$, $)a_n[, n \in \omega$. Among these, only $)a[$ and $)b[$ are not Scott closed, so we have an order isomorphism g from $\langle U, \leqslant\rangle$ onto $\langle \mathcal{P}_{\mathrm{cl}}(U), \supseteq\rangle$ defined as follows:

$$g(a_n) :=)b_n], \text{ for all } n\ ; \qquad g(a) :=)b]\ ;$$

$$g(c_{-n}) :=)c_n], \text{ for all } n \geqslant 1, \text{ and } g(c_0) :=)c_1[\ ;$$

$$g(c) :=)c]\ ; \qquad g(c') :=)c']\ ;$$

$$g(c_n) :=)c_{-n+1}], \text{ for all } n \geqslant 1\ ;$$

$$g(b) :=)a]\ ; \qquad g(b_n) :=)a_n[, \text{ for all } n.$$

Clearly g preserves all suprema, so $\langle U; g\rangle$ yields another example of a solution to $U \cong [U \to 2']$ in $\mathsf{DCPO}_{\aleph_0}$. Incidentally, it can be seen that this corresponds to the one obtained as inverse limit by iterating $[\cdot \to 2']$ from the following ordered set:

$$\curlywedge\bullet < \begin{matrix}\bullet\, c\\ \bullet\, c'\end{matrix} \qquad \text{(where } \curlywedge \text{ is a bottom element).}$$

We thus have another model $\mathcal{U}$ of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ with $u \in_{\mathcal{U}} v$ iff $u \in g(v)$. Now we notice that $u \in_{\mathcal{U}} u$ iff $u < c_1$; hence $\mathrm{W}^{\mathcal{U}} = c_0$. Then we observe that $c_0 \in_{\mathcal{U}} u$ iff $u \leqslant c_1$; so $\mathscr{B}(\mathrm{W})^{\mathcal{U}} = c_{-1}$, and it follows that $\mathcal{U} \not\models \mathrm{W} = \mathscr{B}(\mathrm{W})$. We also mention that the only fixed points of $\mathscr{B}(y)$ on U are a, b, c, c', and that each of these does not seem to be definable in $\mathcal{U}$ by a closed $\mathscr{L}_{\tau*}^{+}$-term.

That said, we still don't know whether $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} \vdash \exists y(y = \mathscr{B}(y))$. We shall only notice that finding a fixed point of $\mathscr{B}(y)$ within a *term* model $\mathcal{M}$ — as the one given in [6] — amounts to finding a quantifier-free formula $\varphi(x)$ of $\mathscr{L}_{\tau*}^{+}$ such that $\mathcal{M} \models \varphi(\{x \mid \psi\}) \leftrightarrow \psi(\{x \mid \varphi\})$ for *all* quantifier-free formula $\psi(x)$ of $\mathscr{L}_{\tau*}^{+}$, which is rather puzzling at first sight.

8. *Reflexive abstraction*

Whether or not the fixed-point property holds, one may consider the use of a syntactical device to name in a uniform way one potential solution to any reflexive equation $y = \sigma(y, \bar{z})$.

This can be done by replacing the formation rule (8) of Section 1 by the following:

(8)′ If φ is a formula and x, y are distinct variables, then $\{x \mid_y \varphi\}$ is a term.

It is understood here that both x and y are bound in $\{x \mid_y \varphi\}$; we will then let $\{x \mid \varphi\}$ stand for $\{x \mid_y \varphi\}$ whenever y does not occur free in φ.

We use the same notation $\mathscr{L}_\tau$ for the language obtained from (1)–(8)′ and now, given a fragment Σ of $\mathscr{L}_\tau$, we let $\mathrm{Abst}_{\circlearrowleft}[\Sigma]$ stand for the scheme of formulas '$\forall x(x \in \{x \mid_y \varphi\} \leftrightarrow \varphi(x, \{x \mid_y \varphi\}, \bar{z}))$' with $\varphi(x, y, \bar{z})$ in Σ. Thus, given a formula $\varphi(x, y, \bar{z})$, $\{x \mid_y \varphi\}(\bar{z})$ is meant to represent one particular solution to $y = \{x \mid \varphi\}(y, \bar{z})$.

As expected, provided φ is in $\mathscr{L}_{\tau*}^{+}$, this is consistent:

Proposition 8.1: *Any Scott-style model* $\mathcal{U}$ *of* $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ *can be turned into a model of* $\mathrm{Abst}_{\circlearrowleft}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$.

The key to the proof of Proposition 8.1 is that fixed points can be chosen in a continuous way on a Scott-style model. This is well known for fixed points of Scott continuous maps on a pointed $\aleph_0$-dcpo (see [1] for instance). But caveat: a Scott continuous function on a pointed κ-dcpo need not even have a fixed point if $\kappa > \aleph_0$ (take for instance $U = \omega$, seen as a $\aleph_1$-dcpo, with $f : \omega \to \omega : n \mapsto n + 1$). This will hold, however, as far as we are concerned with complete lattices, which is the case here.[3]

Fact 8.1: Let U be a complete lattice seen as a κ-dcpo and let $f \in [U \to U]$. Then f has a least fixed point $\mu(f)$; moreover, the application $f \longmapsto \mu(f)$ is Scott continuous from $[U \to U]$ to U.

Proof. Let $f \in [U \to U]$ and define $f_\alpha(\curlywedge) \in U$ for each ordinal α inductively as follows: $f_{\beta+1}(\curlywedge) := f(f_\beta(\curlywedge))$ and $f_\lambda(\curlywedge) := \bigvee \{f_\beta(\curlywedge) \mid \beta < \lambda\}$ for λ limit (and so $f_0(\curlywedge) = \curlywedge$). Because U is a complete lattice this is well defined. Clearly $\alpha < \beta$ implies $f_\alpha(\curlywedge) \leqslant f_\beta(\curlywedge)$, so that $\{f_\beta(\curlywedge) \mid \beta < \kappa\}$ is κ-directed. It follows therefrom that $f(f_\kappa(\curlywedge)) = f_\kappa(\curlywedge)$, and this is the least fixed point of f for it is easily seen that $f_\alpha(\curlywedge) \leqslant w$ for all α and all w such that $f(w) = w$. Now it is routine to show by induction that the application $f \longmapsto f_\alpha(\curlywedge)$ is Scott continuous for each α, and so in particular is $f \longmapsto \mu(f) := f_\kappa(\curlywedge)$. ⊣

We are now ready to prove Proposition 8.1.

Proof of Proposition 8.1. Let $\mathcal{U}$ be the set-theoretic structure associated with a given solution to $U \cong [U \to 2]$ in DCPO_κ.

As in the proof of Theorem 3.1, one can show by induction on the complexity that each term $\{x \mid \varphi\}(\bar{z})$, where φ is in $\mathscr{L}^+_{\tau*}(U)$ and its free variables other than x are among $\bar{z} = z_1, \ldots, z_n$, has a (unique) suitable Scott continuous interpretation $\tau^{\mathcal{U}} : U^n \longrightarrow U : (\bar{w}) \longmapsto \{x \mid \varphi\}^{\mathcal{U}}(\bar{w})$. So it remains to consider the case of a reflexive set abstract $\{x \mid_y \varphi\}(\bar{z})$ with $\varphi(x, y, \bar{z})$ in $\mathscr{L}^+_{\tau*}(U)$. Let $\tau^{\mathcal{U}}$ be the suitable Scott continuous interpretation of $\{x \mid \varphi\}(y, \bar{z})$ and then, given $\bar{w}$ in U, let $f_{\bar{w}} : U \longrightarrow U : v \longmapsto \tau^{\mathcal{U}}(v, \bar{w})$. Clearly $f_{\bar{w}}$ is Scott continuous on U, so let $\mu(f_{\bar{w}})$ be its least fixed point. As the application $(\bar{w}) \longmapsto f_{\bar{w}}$ is Scott continuous, so is $(\bar{w}) \longmapsto \mu(f_{\bar{w}})$ according to Fact 8.1. And as we have $|u \in \mu(f_{\bar{w}})|_{\mathcal{U}} = |u \in \tau^{\mathcal{U}}(\mu(f_{\bar{w}}), \bar{w})|_{\mathcal{U}} = |\varphi(u, \mu(f_{\bar{w}}), \bar{w})|_{\mathcal{U}}$, it follows that $\sigma^U : (\bar{w}) \longmapsto \mu(f_{\bar{w}})$ is a suitable Scott continuous interpretation of $\{x \mid_y \varphi\}(\bar{z})$. Note that such a suitable interpretation is not necessarily unique here. ⊣

[3] The reader familiar with domain theory will have noticed that all the κ-dcpo's we consider are in fact *κ-continuous lattices* (again see [1] for the definition when $\kappa = \aleph_0$).

The consideration of the case $\kappa > \aleph_0$ was of course dispensable to prove the consistency of $\mathrm{Abst}_{\circlearrowleft}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ alone. Besides, it should be remarked that this could also be established by a term model construction as in [6].

9. *Topological extensions*

Finally, we shall briefly indicate that topological extensions of $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ or $\mathrm{Abst}_{\circlearrowleft}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ are conceivable too.

Let $(\Box)$ and $(\Diamond)$ be the topological axiom schemes defined as follows:

$(\Box)$: For every formula φ,
$$\exists y(\forall x(x \in y \to \varphi) \wedge \forall z(\forall x(x \in z \to \varphi) \to z \leqslant y))$$

$(\Diamond)$: For every formula φ,
$$\exists y(\forall x(\varphi \to x \in y) \wedge \forall z(\forall x(\varphi \to x \in z) \to y \leqslant z))$$

As shown in [9], $(\Box)$ [resp. $(\Diamond)$] is equivalent to asserting the existence of $\bigcup$'$\{x \mid \psi(x, \bar{z})\}$' [resp. $\bigcap$'$\{x \mid \psi(x, \bar{z})\}$'] for any formula $\psi(x, \bar{z})$; so $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext} + (\Box) + (\Diamond)$ is inconsistent (see the remark about (†) in Section 1). Nevertheless, each of the axiom schemes $(\Diamond)$ and $(\Box)$ taken individually can consistently be added here, by duality. For $(\Box)$ [resp. $(\Diamond)$], this naturally results from the existence of models associated with solutions to $U \cong [U \to 2]$ [resp. $U \cong [U \to 2']$] in DCPO_κ. Again, this contrasts with the positive set theory $\mathrm{Comp}[\mathscr{L}^{+}] + \mathrm{Ext}$, which admits κ-topological models (κ weakly compact), but is easily proved to be incompatible with $(\Box)$ (see [9]).

10. *Related works*

Positive Set Theory has been studied by several authors from different perspectives. A retrospective view of related systems, such as positive abstraction, can be found in [7], where most results of this paper first appeared. A proof theoretic analysis of positive abstraction can also be found in [2], where it is shown that the sequent calculus corresponding to $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}]$ has cut-elimination, whereas the sequent calculus for $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$ has not — which is in fact another consequence of the monotonicity property of Theorem 5.1. So the proof-theoretic strength of the system $\mathrm{Abst}[\mathscr{L}_{\tau*}^{+}] + \mathrm{Ext}$, or some of its extensions, might be worth investigating, especially in the light of its connection with the system of illative lambda calculus presented in

[8]. At least, further investigations on this latter, inspired by the results of this paper, are now being considered.

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