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RELEVANCE LOGICS, PARADOXES OF CONSISTENCY AND THE K RULE*

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1. *Introduction*

The aim of this paper is to study the effect of adding the K rule to relevance logics in the presence of a constructive negation and in respect of the paradoxes of consistency.

In the literature on relevance logics, paradoxes of implication have customarily been classified into paradoxes of relevance and paradoxes of consistency (see, e.g., [2], p. 349). A characteristic exemplar of the former is the K axiom

(a). $\vdash A \to (B \to A)$

or the K rule

(b). $\vdash A \Rightarrow \ \vdash B \to A$

and a representative member of the latter is the ECQ axiom ("e contradictione quodlibet" axiom)

(c). $\vdash (A \wedge \neg A) \to B$

and related theses such as

(d). $\vdash A \to (\neg A \to B)$

and

(e). $\vdash \neg A \to (A \to B)$

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In passing, it should be noted that Lewis (in so many ways, a precursor of relevance logics) was not unaware of the distinction as it is readily deducible from the following remark on the paradoxes of strict implication ([3], p. 513).

> "It remains to suggest why these paradoxes of strict implication are paradoxical. Let us observe that they concern two questions: what is to be taken as consequence of an assumption which, being self-contradictory, could not possibly be the case; and what is to be taken as sufficient premise for that which being analytic and self-certifying, could not possibly fail to be the case".

But let us return to the literature on relevance logics. Relevance logicians have always been interested in exploring the frontiers between relevance and non-relevance logics. A notorious example of this fact is the considerable attention paid to the paradoxical logic R-Mingle in *Entailment I* (see [1]), or the work of Routley, Meyer and others on the logics KR, CR and CE (see [4], [6], [7] and [11]). (The logic KR is the result of adding the axiom ECQ (c) to the Logic of Relevance R and on the other hand, the logic CR and the logic CE are obtained by adding a boolean negation to R and to the Logic of Entailment E, respectively).

Now, it is to be noted that these investigations we are remarking are developed in the context of the standard negation in relevance logics, i.e, De Morgan negation. Well, what happens if the context is one of a constructive negation? The aim of this paper is to answer this question. Our results (some of them surprising, we think) can be summarized as follows. By B_+, we refer to Routley and Meyer's well-known basic positive logic (see [11]). Then, B_{K+} is the result of adding the K rule to B_+ and $B_{K'+}$ is an S4-type extension of B_{K+}.

Next, the logics B_{Kcr} and $B_{K'cr}$ are the extensions of B_{K+} and $B_{K'+}$ with the weak contraposition axioms (A8, A9; §6), constructive double negation as a rule (T10; §6) and constructive reductio as a rule (T8, T9; §6).

Of course, these logics are subsystems of minimal intuitionistic logic, but let us stress that (from B_{Kcr} up) they are not included in Lewis's modal Logic S5 (so, they are not included in Lewis's S4 or in the Logic of Entailment E either): A8 is not valid in S5 (supposing, of course, that the arrow is read as S5 strict implication).

Let us remark, on the other hand, that the logics we introduce in this paper are not included in the logics KR and CR (so neither are they in CE) mentioned above consequently providing a different perspective (from that considered until now) on the borderlines between relevance and non-relevance logics. Finally, we note that none of the logics we define has the K axiom or any of the versions (c), (d) and (e) of ECQ.

Lewis notes ([3], p. 511):

> "In material implication, the key paradoxes, implicating all the others are: A false proposition implies any proposition; a true proposition is implied by any; any two false propositions are equivalent; any two true propositions are equivalent. Correspondingly, the key paradoxes of strict implication are: A contradictory (self inconsistent) proposition implies any proposition; an analytic proposition is implied by any; any two contradictory propositions are equivalent; any two analytic propositions are equivalent".

The logics we develop here have, of course, paradoxes of relevance: an analytic proposition is implied by any (K rule); any two analytic propositions are equivalent (K rule). What about paradoxes of consistency? In general, they do not have paradoxes of consistency: not any proposition is implied by a contradictory proposition, but, certainly, two contradictory propositions are equivalent (cfr. T19). So, interestingly, we think, our logics cut across Lewis's classification of paradoxes of implication.

The structure of the paper is as follows. In sections 2–6 the logics B_+, B_{K+} and B_{Km} are described. In sections 7, 8, the logics B_{Kcr} and $B_{K'cr}$ are introduced, respectively. We discuss the reductio axioms in the context of the present paper in section 9. In sections 10, 11, we show how to strengthen the logics previously defined. Finally, we include two appendices: the first one presents a list of prominent theorems of B_+, B_{K+} and $B_{K'+}$ and the second provides simple matrix proofs of some interesting facts claimed throughout the paper.

2. *The positive logic B_{K+}*

B_{K+} is axiomatized with

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B)$ / $B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of derivation are

$$\text{Modus ponens (MP): } (\vdash A \ \& \ \vdash A \to B) \Rightarrow \ \vdash B$$
$$\text{Adjunction (Adj.): } (\vdash A \ \& \ \vdash B) \Rightarrow \ \vdash A \wedge B$$
$$\text{Suffixing (Suf.): } \vdash A \to B \Rightarrow \ \vdash (B \to C) \to (A \to C)$$
$$\text{Prefixing (Pref.): } \vdash A \to B \Rightarrow \ \vdash (C \to A) \to (C \to B)$$
$$\text{K: } \vdash A \Rightarrow \ \vdash B \to A$$

Therefore, B_{K+} is B_{+} with the addition of the K rule.

3. *Semantics for* B_{K+}

A B_{K+} *model* is a triple $\langle K, R, \vDash\rangle$where K is a non-empty set, and R is a ternary relation on K subject to the following definitions and postulates for all a, b, c, $d \in K$ with quantifiers ranging over K:

d1. $a \leq b =_{df} \exists x Rxab$

d2. $R^2abcd =_{df} \exists x(Rabx \ \& \ Rxcd)$

P1. $a \leq a$

P2. $(a \leq b \ \& \ Rbcd) \Rightarrow Racd$

P3. $(b \leq d \ \& \ Radc) \Rightarrow Rabc$

Finally, $\vDash$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositional variables p, wff A, B and $a \in K$:

(i). $(a \leq b \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii). $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv). $a \vDash A \to B$ iff for all b, $c \in K$, $(Rabc \ \& \ b \vDash A) \Rightarrow c \vDash B$

A formula A is B_{K+} *valid* ($\vDash_{B_{k+}} A$) iff $a \vDash A$ for all $a \in K$ in all models. Note that the postulates

P4. $Rabc \Rightarrow b \leq c$

P5. $(a \leq b \ \& \ b \leq c) \Rightarrow a \leq c$

and

P6. $R^2abcd \Rightarrow Rbcd$

are immediate in all B_{K+} models.

Regarding semantic consistency (soundness), the proof that all theorems of B_{K+} are valid is left to the reader (see, for example, [2] or [5] for a general strategy).

A final note. As it is known, there is a set of "designated points" in the standard semantics for relevance logics (see the two items just quoted above). It is in respect of this set that d1 is introduced and wff are evaluated. The absence of this set in B_{K+} semantics (and the corresponding changes in d1 and the definition of validity) are the only (but crucial) differences between B_{+} models and B_{K+} models.

4. *Completeness of* B_{K+}

We begin by recalling some definitions:

A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory a is *regular* iff all the theorems of B_{K+} belong to a. Finally, a is *null* iff no wff belong to a.

Now, we define the B_{K+} canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and $a, b, c \in K^T$, $R^T abc$ iff if $A \to B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime non-null theories and R^C be the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the B_{K+} canonical model is the triple $\langle K^C, R^C, \vDash^C \rangle$.

Next, we sketch a proof of the completeness theorem.

Lemma 1: *If a is a non-null theory, then a is regular.*

Proof. Let $A \in a$ and B be a theorem. By the K rule, $A \to B$ is a theorem. So, $B \in a$. □

Lemmas 2–6 below are an easy adaptation of the corresponding B_{+} lemmas (see, e.g., [5]) to the case of non-null theories (as it is known, theories are not necessarily non-null in the B_{+} canonical model and, in fact, in the canonical model of any standard relevance logic).

Lemma 2: *Let A be any wff, a, a non-null element in K^T and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 3: *Let a be a non-null element in K^T, $b \in K^T$ and c a prime member in K^T such that $R^T abc$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 4: *Let $a \in K^T$, b a non-null element in K^T and c a prime member in K^T such that $R^T abc$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Now, we set

Definition 1: *Let a, $b \in K^T$. Then, $a \leq^T b$ iff $R^T xab$ and $x \in K^C$.*

We have

Lemma 5: *Let $a \in K^T$ and b be a prime element in K^T. Then, $a \leq^T b$ iff $a \subseteq b$.*

And consequently,

Lemma 6: *$a \leq^C b$ iff $a \subseteq b$.*

Note that b and c in lemma 3 and a and c in lemma 4 need not be non-null. On the other hand, lemma 7 below follows immediately from lemma 2.

Lemma 7: *If $\nvdash_{B_{K+}} A$, then there is some $x \in K^C$ such that $A \notin x$.*

Lemma 8: *Let a, b be non-null theories. The set $x = \{B \mid \exists A[A \to B \in a$ and $A \in b]\}$ is a non-null theory such that $R^T abx$.*

Proof. It is easy to prove that x is a theory such that $R^T abx$. We prove that x is non-null. Let $A \in b$. By lemma 1, $A \to A \in a$. So, $A \in x$ by $R^T abx$. $\square$

The following three lemmas are proved similarly as in the standard semantics (use lemma 8 in the proof of the canonical adequacy of clause (iv)).

Lemma 9: *The canonical postulates hold in the B_{K+} canonical model.*

Lemma 10: *$\vDash^C$ is a valuation relation satisfying conditions (i)–(iv) above.*

Lemma 11: *The canonical model B_{K+} is in fact a model.*

By lemmas 7 and 11, we have

Theorem 1: (Completeness of B_{K+}) *If* $\vDash_{B_{K+}} A$, *then* $\vdash_{B_{K+}} A$.

5. *The logic* $B_{K'+}$

The logic $\mathrm{B}_{K'+}$ is the result of adding the axiom

A7. $(A \rightarrow B) \rightarrow [C \rightarrow (A \rightarrow B)]$

to B_{K+} (we note that B_{K+} and $\mathrm{B}_{K'+}$ are different logics. See Appendix B). A $\mathrm{B}_{K'+}$ model is defined similarly as a B_{K+} model save for the addition of the postulate

P7. $R^2abcd \Rightarrow Racd$

In order to prove semantic consistency, it remains to prove that A7 is valid (use P7). On the other hand, to prove completeness, it remains to prove that P7 is canonically valid. So, suppose R^2abcd, i.e., R^Cabx and R^Cxcd for some $x \in K^C$. Further, suppose $A \rightarrow B \in a$, $A \in c$ for some wff A, B. We have to prove $B \in d$. Now, let $C \in b$. By A7, $C \rightarrow (A \rightarrow B) \in a$. So, $A \rightarrow B \in x$ (R^Cabx, $C \in b$). Therefore, $B \in d$ (R^Cxcd, $A \in c$).

6. B_{K+} *with minimal negation: the logic* B_{Km}

The logic B_{Km} is an extension of the language of B_{K+} with the propositional falsity constant F. We add the constant F to the positive language and define

$\neg A =_{df} A \rightarrow F$

No new axioms, however, are added. The following theses are some characteristic theorems of B_{Km} (a sketch of the proof for each one is to their right; cfr. Appendix A on the theorems employed).

T1. $\vdash A \rightarrow B \Rightarrow \vdash \neg B \rightarrow \neg A$	Suf.
T2. $\vdash \neg B \Rightarrow \vdash (A \rightarrow B) \rightarrow \neg A$	Pref.
T3. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	t12
T4. $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$	t14
T5. $\neg F$	A1
T6. $A \rightarrow \neg F$	A1, K

A B_{Km}-*model* is a quadruple $\langle K, S, R, \models\rangle$ where K, R and $\models$ are defined similarly as in a B_{K+} model and S is a non-empty subset of K. The clauses

$$\text{(v). } (a \leq b \;\&\; a \models F\,) \Rightarrow b \models F$$
$$\text{(vi). } a \models F \text{ iff } a \notin S$$

are added to (i)–(iv). A is B_{Km} valid ($\models_{B_{Km}} A$) iff $a \models A$ for all $a \in K$ in all models. Semantic consistency of B_{Km} follows immediately from that of B_{K+}. Moreover, we note that F is not valid (in fact, it is unsatisfiable). Let $\mathcal{M}$ be any model and $a \in S$. Then, $a \not\models F$.

Turning to completeness, we define the canonical model as the structure $\langle K^C, S^C, R^C, \models^C\rangle$ where K^C, R^C, $\models^C$ are defined similarly as in the B_{K+} canonical model, and S^C is interpreted as the set of all consistent prime non-null theories, a theory being consistent if $F \notin a$. In order to prove completeness, we have to prove that clauses (v) and (vi) are canonically valid and that S^C is not empty. Now, clauses (v) and (vi) are

$$\text{(v}'\text{). } (a \subseteq b \;\&\; F \in a\,) \Rightarrow F \in b$$
$$\text{(vi}'\text{).} F \in a \text{ iff } F \in a$$

when read canonically (cfr. definition of B_{K+} canonical model and lemma 6). So, there is nothing to prove. On the other hand, let B_{Km} be the set of its theorems. As $\not\models_{B_{Km}} F, \nvdash_{B_{Km}} F$ by the soundness theorem, i.e, $F \notin \mathrm{B}_{Km}$. Then, by lemma 2, there is a consistent prime theory x such that $F \notin x$. So, we have

Theorem 2: (Completeness of B_{Km}) *If* $\models_{B_{Km}} A$*, then* $\vdash_{B_{Km}} A$.

On the meaning of the constant F in B_{Km}, we prove

Proposition 1: *A theory a is inconsistent iff for some theorem $\neg B$, $B \in a$.*

Proof. (a) Suppose a inconsistent. Then, $F \in a$. But $\vdash \neg F$, by T5. (b) Suppose $B \in a$ for some theorem $\neg B$. By definition, $\vdash B \to F$. So, $F \in a$. □

In other words, a is inconsistent if it contains the argument of a negative formula that is a theorem.

7. *The logic* B_{Kcr}

The logic B_{Kcr} is B_{K+} plus weak constructive contraposition (A8, A9), constructive double negation as a rule (T10), and constructive reductio as a rule (T8, T9). It can be axiomatized by adding to B_{Km}

A8. $(A \to B) \to [(B \to F) \to (A \to F)]$
A9. $(B \to F) \to [(A \to B) \to (A \to F)]$

and the axiom of specialized reductio

A10. $[A \to (A \to F)] \to (A \to F)$

In addition to T1–T6, the following theses are some theorems of B_{Kcr}:

T7.	$\vdash A \Rightarrow\ \vdash (A \to \neg B) \to \neg B$	A10, K
T8.	$\vdash A \to \neg B \Rightarrow\ \vdash (A \to B) \to \neg A$	A10
T9.	$\vdash A \to B \Rightarrow\ \vdash (A \to \neg B) \to \neg A$	A8, A10
T10.	$\vdash A \Rightarrow\ \vdash \neg\neg A$	T9, K
T11.	$\vdash B \Rightarrow\ \vdash (A \to \neg B) \to \neg A$	T2, T10
T12.	$\neg A \to (B \to \neg A)$	A9, K
T13.	$\vdash A \Rightarrow\ \vdash (B \to \neg A) \to (A \to \neg B)$	T11, T12
T14.	$\neg(A \wedge \neg A)$	A2, T9
T15.	$(A \to B) \to \neg(A \wedge \neg B)$	A2, T8
T16.	$(A \to \neg B) \to \neg(A \wedge B)$	A2, T9
T17.	$[(A \to B) \wedge (A \to \neg B)] \to \neg A$	A8, T2, T14, t11
T18.	$[A \to (B \wedge \neg B)] \to \neg A$	T17, t11
T19.	$(\neg A \wedge \neg B) \to (\neg A \leftrightarrow \neg B)$	T12, t11

A B_{Kcr}-*model* is defined similarly as a B_{Km} model save for the addition of the postulates

P8. $R^2abcd \ \ \& \ \ d \in S \Rightarrow (\exists x \in K)\,(\exists y \in S)\,(Racx \ \ \& \ \ Rbxy)$
P9. $R^2abcd \ \ \& \ \ d \in S \Rightarrow (\exists x \in K)\,(\exists y \in S)\,(Rbcx \ \ \& \ \ Raxy)$
P10. $a \in S \Rightarrow (\exists x \in S)\, Raax$

$\vDash_{B_{Kcr}} A$ (A is B_{Kcr} valid) iff $a \vDash A$ for all $a \in K$ in all models. The postulates P8, P9 and P10 are, as we show below, the corresponding postulates for A8, A9 and A10, respectively. That is, given B_{K+} semantics, each axiom is shown valid by means of the respective postulate, and each postulate is shown valid with the respective axiom. Now, we note that in standard relevance logics the corresponding postulate for A10 is

P10(i). $(Rabc \;\&\; c \in S) \Rightarrow \;(\exists x \in S)\, R^2abbx$

Consider now the following postulate

P10(ii). $(Rabc \;\&\; c \in S) \Rightarrow \;(\exists x \in S)\, Rcbx$

It is interesting enough that in B_{K+} we have (the proof is left to the reader):

Proposition 2: *Given B_{K+} semantics, P10, P10(i) and P10(ii) are equivalent.*

Therefore, in B_{Kcr}, P10(i) or P10(ii) can be substituted by the weaker P10 in the semantics here presented, as it is the case.

In order to prove semantic consistency (soundness), we have to prove that A8–A10 are valid. Now, A8 and A9 are proved as in relevance models (see, e.g., [5]). On the other hand, we prove that A10 is valid:

Proof. Suppose $a \vDash A \to (A \to F)$, $a \nvDash A \to F$ for some $a \in K$ in some model. Then, $Rabc$, $b \vDash A$, $c \nvDash F$ (i.e, $c \in S$) for some b, $c \in K$. By P4, $b \leq c$, $c \vDash A$. Next, $c \vDash A \to F$ ($a \vDash A \to (A \to F)$, $b \vDash A$, $Rabc$). By P10, $Rccx$ for some $x \in S$. But we have $x \vDash F$ ($c \vDash A \to F$, $c \vDash A$, $Rccx$), i.e, $x \notin S$, by clause vi. □

As for completeness, the canonical model is defined similarly as the B_{Km} canonical model. Then, it is obvious that we just have to prove that the postulates P8, P9 and P10 are canonically valid. It is clear that this fact follows from the following lemma:

Lemma 12: (1) *Let $R^{T2}abcd$, a, b, c be non-null theories in K^T and d be a consistent theory in K^T. Then, there are some x in K^C and some y in S^C such that R^Tacx and R^Tbxy.*

(2) *Let $R^{T2}abcd$, a, b, c be non-null theories in K^T and d be a consistent theory in K^T. Then, there are some x in K^C and some y in S^C such that R^Tbcx and R^Taxy.*

(3) *Let a be a consistent non-null theory. Then, there is some x in S^C such that R^Taax.*

Proof. (1) Suppose $R^{T2}abcd$ (i.e, R^Tabz and R^Tzcd for some $z \in K^T$) and let a, b, c be non-null theories and d a consistent theory. Define the non-null theories u and w such that R^Tacu and R^Tbuw (cfr. lemma 8). We prove that w is consistent. Suppose it is not. Then $F \in w$. So, $B \to A \in a$, $A \to F \in b$ for some wffs A and $B \in c$. By A8, $(A \to F) \to (B \to F) \in a$; so, $B \to F \in z$ by R^Tabz. Therefore, $F \in d$ by R^Tzcd contradicting the hypothesis. Now (use lemma 2), there is some $y \in S^C$ such that $w \subseteq y$. So, clearly, R^Tbuy. Next (use lemma 4), there is some $x \in K^C$ such that $u \subseteq x$ and R^Tbxy. Obviously, R^Tacx. Thus, we have $x \in K^C$, $y \in S^C$ such that R^Tacx and R^Tbxy as it was required.

(2) The proof is similar to the proof of case 1. Use A9.

(3) Let a be a consistent non-null theory. Define the non-null theory y such that R^Taay. If y is not consistent, then $A \to F \in a$ for some $A \in a$. By T14, $[A \wedge (A \to F)] \to F$. Then, $F \in a$ contradicting the hypothesis. Next, use lemma 2 to extend y to a consistent non-null prime theory x such that R^Taax.

□

8. *The logic* $B_{K'cr}$

The logic $B_{K'cr}$ is the result of adding A8, A9 and A10 to $B_{K'+}$. The axiom A7 is not provable in B_{Kcr} (see Appendix B), though it is, of course, an "acceptable" implicative paradox in Lewis's sense. In addition to T1–T19, we have

T20. $\vdash A \Rightarrow \ \vdash \neg A \to \neg B$ A7, T10

and, most of all, the full constructive reductio axioms in the form

T21. $(A \to B) \to [(A \to \neg B) \to \neg A]$ A8, T7, T14, t17

T22. $(A \to \neg B) \to [(A \to B) \to \neg A]$ A8, T7, T14, t17

Regarding semantics, a $B_{K'cr}$ *model* is defined similarly as a B_{Kcr} model save for the addition of the postulate P7.

A is $B_{K'cr}$ valid ($\vDash_{B_{K'cr}} A$) iff $a \vDash A$ in all models.

Regarding the meaning of F in B_{Kcr} and $B_{K'cr}$, we note the following proposition:

Proposition 3: *If a is a theory containing the negation of a theorem, then a is inconsistent.*

Proof. Suppose $A \to F \in a$ for some theorem A. By T10, $\vdash (A \to F) \to F$. Then, $F \in a$. □

We note that (a) of course, proposition 1 is still provable and (b) the converse of proposition 3 is not provable (see Appendix B).

9. *Some remarks on the full reductio axioms*

The full (constructive) reductio axioms are T21, T22 of $\mathrm{B}_{K'cr}$. It is argued in [9] that these formulas cannot be introduced in B_+, the resources of the logic being insufficient to prove the corresponding semantical postulates for the axioms. Moreover, as it is discussed in [8], this seems to be so even in the case of strong full non-constructive axioms, i.e,

(a). $(\neg A \to \neg B) \to [(\neg A \to B) \to A]$
(b). $(\neg A \to B) \to [(\neg A \to \neg B) \to A]$
(c). $(A \to B) \to [(\neg A \to B) \to B]$
(d). $(\neg A \to B) \to [(A \to B) \to B]$

Now, in [8] and [9], it is proved that if the prefixing axiom

(e). $(B \to C) \to [(A \to B) \to (A \to C)]$

is added to B_+, the full reductio axioms (constructive and non-constructive) can be introduced in the resulting logic Bp_+.

On the other hand, the full reductio axioms T21 and T22 can be introduced, as we have seen, in $\mathrm{B}_{K'cr}$. And, what is more, we have a proof that if the constructive double negation axiom

(f). $A \to \neg\neg A$

is added to B_{Kcr}, the constructive reductio axioms T21, T22 can be defined, the prefixing axiom being not necessary. Nevertheless, it is our conjecture that T21 and T22 cannot be introduced in B_{Kcr} if (e) or (f) are not present. Consequently, in the following section, the logic Bp_{Kcr} is presented. It will be easy to build up a varied and large number of logics from Bp_{Kcr}.

10. *The logic* Bp_{Kcr}

The logic Bp_{Kcr} is the result of adding the axiom

A11. $(B \to C) \to [(A \to B) \to (A \to C)]$

to B_{Kcr}. We note that A11 is not derivable in $\mathrm{B}_{K'cr}$, and that A7 is provable (proof is left to the reader) with A11 and t17. So, $\mathrm{B}_{K'cr}$ is included in Bp_{Kcr} and, consequently, the full constructive reductio axioms can be introduced.

As for semantics, a Bp_{Kcr} *model* is defined similarly as a B_{Kcr} model except for the addition of the postulate

P11. $R^2abcd \Rightarrow \ (\exists x \in K)\,(Rbcx \ \&\ Raxd)$

Given B_+, the postulate P11 is the corresponding postulate for A11. And the canonical validity of P11 (and, therefore, the completeness of Bp_{Kcr}) can be derived immediately from the following lemma.

Lemma 13: *Let a, b, c be non-null elements in K^T, $d \in K^T$ and $R^T abcd$. Then, there is some non-null theory x such that $R^T bcx$ and $R^T axd$.*

Proof. Proof is left to the reader. See, e.g., [5]. □

11. *Strengthening the logics*

The logic Bp_{Kcr} can be strengthened without the K axiom and the different versions of ECQ being derivable. We briefly discuss some possibilities.

Consider the axioms *suffixing*

A12. $(A \to B) \to [(B \to C) \to (A \to C)]$

contraction

A13. $[A \to (A \to B)] \to (A \to B)$

and the rule of derivation *assertion*

A14. $\vdash A \Rightarrow \ \vdash (A \to B) \to B$

The logic TW_+ ("Contractionless positive Ticket Entailment") is Bp_+ (i.e, B_+ plus the prefixing axiom A11) plus A12. The logic T_+ ("Positive Ticket

Entailment") is TW_+ plus A13, and the logic E_+ ("Positive Logic of Entailment") is T_+ plus A14 (cfr. [2] for information about these logics). Therefore, TW_{K+}, T_{K+} and E_{K+} are TW_+, T_+ and E_+ plus the K rule, respectively.

Let us now define the semantics. Consider the following postulates

P12. $R^2abcd \Rightarrow\ (\exists x \in K)\,(Racx\ \&\ Rbxd)$

P13. $Rabc \Rightarrow\ R^2abbc$

P14. $(\exists x \in K)\,Raxa$

The postulates P12, P13 and P14 are, given the logic TW_{K+} and TW_{K+} semantics, the corresponding postulates for A12, A13 and A14, respectively. Well, the logic TW_{Kcr} is formulated by adding A12 to Bp_{Kcr}, the logic T_{Kcr}, by adding A13 to TW_{Kcr}, and, finally, the logic E_{Kcr} is T_{Kcr} plus A14. Consequently, TW_{Kcr} *models*, T_{Kcr} *models* and E_{Kcr} *models* are defined similarly as Bp_{Kcr} models except for the addition of P12, P13 and P14, respectively. Therefore, soundness and completeness of TW_{Kcr}, T_{Kcr} and E_{Kcr} are immediate from those of Bp_{Kcr} and the fact that P12, P13 and P14 are the corresponding postulates for A12, A13 and A14.

Appendix A. *Negationless theorems*

We note some theorems of B_+, B_{K+} and $B_{K'+}$.

A.1. *Theorems of* B_+

The following theses are, for example, theorems of B_+.

t1.	$(A \wedge B) \leftrightarrow (B \wedge A)$	A2, A3
t2.	$(A \vee B) \leftrightarrow (B \vee A)$	A4, A5
t3.	$[A \wedge (B \wedge C)] \leftrightarrow [(A \wedge B) \wedge C]$	A2, A3
t4.	$[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$	A4, A5
t5.	$A \leftrightarrow (A \wedge A)$	A1, A2, A3
t6.	$A \leftrightarrow (A \vee A)$	A1, A4, A5
t7.	$A \leftrightarrow [A \vee (A \wedge B)]$	A1, A2, A4, A5
t8.	$A \leftrightarrow [A \wedge (A \vee B)]$	A1, A2, A3, A4
t9.	$[A \vee (B \wedge C)] \leftrightarrow [(A \vee B) \wedge (A \vee C)]$	A2, A3, A4, A5, A6, T1
t10.	$[A \wedge (B \vee C)] \leftrightarrow [(A \wedge B) \vee (A \wedge C)]$	A2, A3, A4, A5, A6
t11.	$[A \to (B \wedge C)] \leftrightarrow [(A \to B) \wedge (A \to C)]$	A2, A3
t12.	$[(A \vee B) \to C] \leftrightarrow [(A \to C) \wedge (B \to C)]$	A4, A5
t13.	$[(A \to B) \vee (A \to C)] \to [A \to (B \vee C)]$	A4, A5
t14.	$[(A \to C) \vee (B \to C)] \to [(A \wedge B) \to C]$	A2, A5

A.2. *Theorems of* B_{K+}

In addition to t1–t14, the following theses are representative theorems of B_{K+}

t15.	$B \to (A \to A)$	A1, K
t16.	$(A \to B) \to [A \to (A \wedge B)]$	A1, A3, t15

A.3. *Theorems of* $B_{K'+}$

In addition to t1–t18, in $B_{K'+}$ we have, for example, the following theorems

t17.	$(A \to B) \to [(A \to C) \to [A \to (B \wedge C)]]$	A11, t1, t16
t18.	$(A \to C) \to [(B \to C) \to [(A \vee B) \to C]]$	A11, t2, t16
t19.	$(A \to B) \to [(A \wedge C) \to (B \wedge C)]$	A2, t1, t17
t20.	$(A \to B) \to [(A \vee C) \to (B \vee C)]$	A4, t2, t18

Appendix B. *Matrices*

The decidability of the logics here discussed being open, we present here simple matrix proofs of some facts claimed in the paper.

(1) Consider the following set of matrices where the only designated value is 3 and F is assigned the value 2.

$\to$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	2	2	3	3
3	0	2	0	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

This set satisfies the axioms and rules of B_{Kcr}, but falsifies A7 ($v(A) = 2$, $v(B) = 1$ and $v(C) = 3$) and A11 ($v(A) = v(B) = 2$ and $v(C) = 1$).

(2) Consider the following set of matrices where the only designated value is 2 and F is assigned the value 1.

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

This set satisfies the axioms and rules of E_{Kcr} but falsifies $A \to (B \to A)$ only when $v(A) = 1$ and $v(B) = 2$; and $(A \wedge \neg A) \to B$,

$A \to (\neg A \to B)$ and $\neg A \to (A \to B)$ only when $v(A) = 1$ and $v(B) = 0$.

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