*Logique & Analyse* **197** (2007), 5–41

# A GENERAL CAUCHY-COMPLETION PROCESS FOR ARBITRARY FIRST-ORDER STRUCTURES

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*Abstract*

A particular way to put a uniformity on a first-order structure is introduced and the natural Cauchy-completion is studied. The corresponding compactness problem leads to interesting extensions, to directed sets, of notions usually applied to cardinals : regularity, measurability, ramifiability, etc...

## 1. *Introduction*

The aim of this paper is to introduce a very general way to put a uniformity on an arbitrary first-order structure, using an adequate family of equivalences, and to study the natural corresponding Cauchy-completion.
Examples to this can be found here and there in algebra (and we mention some of them), but also in lambda-calculus (namely the "topological" Scott-models). More recently and actually more surprisingly, this construction also appeared in set theory and permitted to solve significant consistency problems ([2], [8], [14], [16], [17], [19], [23], [26]). The following example leads to the conviction that very interesting structures can appear by completion : an adequate completion of the very simple binary structure $(R_\omega, \in)$, where $R_\omega$ is the set of the hereditarily finite well-founded sets and $\in$ is restricted to $R_\omega$, produces a model for extensionality and "positive comprehension" (for details see section 4 and [8], [14], [17]).
This completion satisfies even an anti-foundation axiom (see section 4 and [8], [16]).
One can generalize the type of completion used in this "archetypical case" to arbitrary first-order structures.
In honor of the pioneer work of R.J. Malitz [23], who had the original idea

to use an adequate family of equivalences to put a uniformity on a set-theoretical universe, we will call this type of completion a "Malitz-completion". The problem of the compactness of such a completion leads to interesting properties of directed sets, namely generalisations of the notions of "regular", "measurable", "ramifiable", etc... (usually applied to cardinals).

## 2. *The construction*

One can give a very simple description of the construction, in terms of "$\mathcal{F}$-nets". We discuss the relation to "projective limits" and "uniformities" in section 3. So, let $\mathcal{L}$ be any first-order language, with : $R, R', \ldots$ relation symbols; $F, F', \ldots$ function symbols; $c, c', \ldots$ constant symbols. We admit arbitrarily many of these symbols.
A model for $\mathcal{L}$ will be a structure of type :

$$M = \big(A, R_M, R'_M, \ldots, F_M, F'_M, \ldots, c_M, c'_M, \ldots\big)$$

with universe $A$. Our metatheory will be $ZFC$, i.e. the Zermelo-Fraenkel set theory with axiom of choice.
Fix such a structure $M$ and consider a non-empty family $\mathcal{F}$ of equivalences on $A$. We adopt the natural order relation on $\mathcal{F}$, defined by : $\sim \leq_{\mathcal{F}} \sim'$ iff $\forall a, b \in A\, (a \sim' b \Rightarrow a \sim b)$.
As a relation is a set of couples, $\leq_{\mathcal{F}}$ is exactly the relation $\supseteq$ restricted to $\mathcal{F}$.
We will need some definitions and facts about "directed sets"; for $\leq$ an order relation on a set $D$ :

*Definition 2.1*: *$(D, \leq)$ is "directed" iff any finite subset of D has an upper bound in D, i.e. $\forall X \subseteq D\, (X$ is finite $\Rightarrow \exists d \in D \quad \forall x \in X \quad x \leq d)$.*

*Definition 2.2*: *A subset $Y$ of $D$ is "cofinal" in $(D, \leq)$ iff $\forall d \in D \quad \exists y \in Y \quad d \leq y$.*

*Definition 2.3*: *A "D-net" in a set $A$ is a function $x\ : D \to A$.*

*Notation :* $x = (x_d)_{d \in D}$ (so $x_d$ is $x(d)$).

*Definition 2.4*: *A "D-extractor" is a function $\sigma\ : D \to D$ such that $\forall d \in D\ d \leq \sigma(d)$.*

*Fact 2.5*: The following sentences are equivalent (once $\leq$ is an order relation on $D$) :

(i) $(D, \leq)$ is directed,
(ii) any finite partition of $D$ has a cofinal piece,
(iii) any cover of $D$ (by subsets of $D$) has a cofinal piece,
(iv) any cofinal subset in $(D, \leq)$ is directed (for the restriction of $\leq$ to this subset),
(v) there exists a directed cofinal subset in $(D, \leq)$

*Fact 2.6*: In a directed set $(D, \leq)$ the cofinal subsets are exactly the images of the $D$-extractors.

*Definition 2.7*: *A "$D$-subset" of a $D$-net $x$ (in a set $A$) is a $D$-net of type $x_\sigma = \left(x_{\sigma(d)}\right)_{d \in D}$ for some extractor $\sigma$.*

We can come back now to our family $\mathcal{F}$ of equivalences on $A$ (the universe of the first-order structure $M$). For the moment we will expect $(\mathcal{F}, \leq_{\mathcal{F}})$ to be directed and to satisfy the condition: $\forall a, b \in A\, [(\forall \sim \in \mathcal{F} \quad a \sim b) \Rightarrow a = b]$ (equivalently : $\cap\mathcal{F}$ is the equality relation on $A$).
Further, in order to avoid trivial cases we suppose also that $\mathcal{F}$ has no maximum element for $\leq_{\mathcal{F}}$. We introduce now some "topological" definitions concerning $\mathcal{F}$ :

*Definition 2.8*: *A "uniform Cauchy" $\mathcal{F}$-net (in $A$) is an $\mathcal{F}$-net $x$ such that $\forall \sim, \sim' \in \mathcal{F}$ ($\sim \leq_{\mathcal{F}} \sim' \Rightarrow x_\sim \sim x_{\sim'}$).*

*Definition 2.9*: *An $\mathcal{F}$-net $x$ (in $A$) has a "uniform limit" $z$ in $A$ iff $\forall \sim \in \mathcal{F}$ $x_\sim \sim z$.*

*Notation :* $\lim\limits_{\mathcal{F}} x = z$.

*Fact 2.10*: Any $\mathcal{F}$-net admitting a uniform limit is a uniform Cauchy $\mathcal{F}$-net and this uniform limit is unique.

*Definition 2.11*: *A subset $X$ of $A$ is "$\mathcal{F}$-closed" iff $X$ is closed under uniform limits, i.e.* $(\forall \sim \in \mathcal{F}\ x_\sim \in X\ \&\ \lim\limits_{\mathcal{F}} x = z) \Rightarrow z \in X$.

So we get a topology $\mathcal{T}_{\mathcal{F}}$ on $A$ by calling "$\mathcal{F}$-open" the complements (in $A$) of the $\mathcal{F}$-closed subsets of $A$.

*Definition 2.12*: *The set $\overline{A}_{\mathcal{F}}$ is the set of all uniform Cauchy $\mathcal{F}$-nets in $A$. The adequate equality on $\overline{A}_{\mathcal{F}}$ is the equivalence relation defined by :* $x \approx y \Leftrightarrow \forall \sim \in \mathcal{F}\ \ x_\sim \sim y_\sim$.

The canonical injection : $A \rightarrow \overline{A}_{\mathcal{F}}$ associates (in the obvious way) to each element $a \in A$ the constant $\mathcal{F}$-net $x$ such that $\forall \sim \in \mathcal{F}\ x_{\sim} = a$. So $A$ can be seen as a subset of $\overline{A}_{\mathcal{F}}$, and also of the quotient $\overline{A}_{\mathcal{F}}/\approx \overset{\text{def}}{=} \{[x]_{\approx} \mid x \in \overline{A}_{\mathcal{F}}\}$, where $[x]_{\approx} = \{y \mid y \approx x\}$. The constants of $M$ are easily "transferred" to $\overline{A}_{\mathcal{F}}$, as each $c_M$ can be seen as an element of $\overline{A}_{\mathcal{F}}$. The transfer of the relations is also easy : for $R$ a relation symbol in $\mathcal{L}$, $\overline{R}$ is defined on $\overline{A}_{\mathcal{F}}$ by :

$$\overline{R}(x, y, \ldots) \quad \text{iff} \quad \exists x' \approx x \quad \exists y' \approx y \ldots \forall \sim \in \mathcal{F} \quad R_M(x'_{\sim}, y'_{\sim}, \ldots)$$

Obviously $\approx$ is compatible with $\overline{R}$, i.e. :
$(x \approx a \,\&\, y \approx b \ldots) \Rightarrow (\overline{R}(x, y, \ldots) \Leftrightarrow \overline{R}(a, b, \ldots))$.
So $\overline{R}$ can also be seen as a relation on $\overline{A}_{\mathcal{F}}/\approx$ (and we keep the same notation "$\overline{R}$") i.e. $\overline{R}([x]_{\approx}, [y]_{\approx}, \ldots) \overset{\text{def}}{\Leftrightarrow} \overline{R}(x, y, \ldots)$

One has to be slightly more careful for the transfer of the functions. The most natural idea would be to assume that each $\sim \in \mathcal{F}$ is compatible with each $F_M$ (for $F$ a function symbol in $\mathcal{L}$), i.e.

$$\vec{x} \sim \vec{y} \Rightarrow F_M(\vec{x}) \sim F_M(\vec{y})$$

(where $\vec{x}$, $\vec{y}$ are $n$-tuples and $\vec{x} \sim \vec{y}$ means that $x_i \sim y_i$ for each component $i$).
We will suppose for the discussion that the function $F_M$ has one argument, i.e. that $F_M$ is a function : $A \rightarrow A$; our conclusions can be extended in the obvious way to the $n$-ary case.
So, if each $\sim \in \mathcal{F}$ is compatible with $F_M$, $F$ can naturally be extended to $\overline{A}_{\mathcal{F}}/\approx$, by the rule : $\overline{F}(x) = (F_M(x_{\sim}))_{\sim \in \mathcal{F}}$, for $x \in \overline{A}_{\mathcal{F}}$.
However, this condition of compatibility is too restrictive, and it suffices actually to suppose that $F_M$ is "$\mathcal{F}$-uniformly continuous", i.e. that there exists an $\mathcal{F}$-extractor $\sigma$ such that : $\forall a, b \in A\ (a\sigma(\sim)b \Rightarrow F_M(a) \sim F_M(b))$.
We call such a $\sigma$ a "uniformizer of $F_M$".
This condition is equivalent to : $\forall \sim \in \mathcal{F}\ \exists \sim' \in \mathcal{F}\ \forall a, b \in A\ (a \sim' b \Rightarrow F_M(a) \sim F_M(b))$ and corresponds exactly to the condition of "uniform continuity" of $F_M$, in the sense of the uniform spaces theory (see section 3). The extension $\overline{F}$ is defined now by : $\overline{F}(x) = \left(F_M(x_{\sigma(\sim)})\right)_{\sim \in \mathcal{F}}$ where $\sigma$ is some uniformizer of $F_M$.
One can easily verify that $\overline{F}$ is a well-defined function : $\overline{A}_{\mathcal{F}}/\approx \rightarrow \overline{A}_{\mathcal{F}}/\approx$.

What precedes leads to the following definitions :

*Definition 2.13*: *$\mathcal{F}$ is a "Malitz-family" for the first-order structure $M$ iff $\mathcal{F}$ is a non-empty directed set (for $\leq_{\mathcal{F}}$) of equivalences on $A$ (the universe of $M$), without maximum element, such that $\cap F$ is the equality on $A$ and each $F_M$ (for $F$ a function symbol in $\mathcal{L}$) is $\mathcal{F}$-uniformly continuous (i.e. $\forall \sim \in \mathcal{F}$ $\exists \sim' \in \mathcal{F}\, \forall \vec{a}, \vec{b} \in A^n$ $(\vec{a} \sim' \vec{b} \Rightarrow F_M(\vec{a}) \sim F_M(\vec{b}))$).*

*Definition 2.14*: *If $\mathcal{F}$ is a Malitz-family for $M$, then the structure $\overline{M}_{\mathcal{F}} = \left(\overline{A}_{\mathcal{F}}/\approx, \overline{R}, \overline{R'}, \dots, \overline{F}, \overline{F'}, \dots, [c]_{\approx}, \dots\right)$ is called the "Malitz-completion" of $M$ (modulo $\mathcal{F}$).*

We discuss now some basic facts about $\overline{M}_{\mathcal{F}}$.

*Basic fact 2.15*: We can see $A$ as a subset of $\overline{A}_{\mathcal{F}}/\approx$ by identifying each element of $A$ with the obvious correspondent $\mathcal{F}$-net. One can easily check that the restriction of $\approx$ to $A$ is exactly the equality on $A$.
Note that this canonical injection : $A \rightarrow \overline{A}_{\mathcal{F}}/\approx$ is not necessarily an *embedding* of $M$ into $\overline{M}_{\mathcal{F}}$, as in shown by this example : suppose $M$ is $(\mathbb{N} \cup \{\infty\}, R)$ where $\infty$ is some element not in $\mathbb{N}$ (the set of the integers) and $R = \left\{(n,n) \,\middle|\, n \in \mathbb{N}\right\}$; consider the Malitz-family $\mathcal{F} = \left\{\sim_k \middle| \, k \in \mathbb{N}\right\}$ where

$$x \sim_k y \quad \overset{\text{def}}{\Leftrightarrow} \quad [(x = y \in \mathbb{N} \;\;\&\;\; x < k \;\;\&\;\; y < k) \vee (x \geq k \;\;\&\;\; y \geq k)]\,.$$

Naturally the order "$\leq$" on $\mathbb{N} \cup \{\infty\}$ is the obvious one, i.e. the usual one on $\mathbb{N}$ and realizing $\forall x \in \mathbb{N} \cup \{\infty\} \quad x \leq \infty$.
Then $\overline{A}_{\mathcal{F}}/\approx$ is exactly $\mathbb{N} \cup \{\infty\}$, but $\infty\, \overline{R}\, \infty$ while $\neg \infty\, R\, \infty$.

*Basic fact 2.16*: The notions of uniform Cauchy-$\mathcal{F}$-net, uniform limit, $\mathcal{F}$-closed subset, etc... can easily be transferred to $\overline{A}_{\mathcal{F}}/\approx$ by extending each $\sim \in \mathcal{F}$ to an equivalence $\sim^\star$ on $\overline{A}_{\mathcal{F}}$ : (for $x, y \in \overline{A}_{\mathcal{F}}$)

$$x \sim^\star y \quad \overset{\text{def}}{\Leftrightarrow} \quad x_\sim \sim y_\sim \quad .$$

One can easily verify that $\overline{R}$ (for $R$ a $n$-ary relation symbol in $\mathcal{L}$) is a closed subset in $\left(\overline{A}_{\mathcal{F}}\right)^n$ (with the product topology) : actually it is the closure of $R_M$.

Further one can check that $\overline{F}$ is continuous (it is even uniform continuity), for each function symbol $F$ in $\mathcal{L}$. At last one can verify that $\overline{A}_{\mathcal{F}}/\approx$ is $\mathcal{F}$-Cauchy complete, i.e. any uniform Cauchy $\mathcal{F}$-net has a uniform limit. Note also that $A$ and $\overline{A}_{\mathcal{F}}/\approx$ are Hausdorff spaces (due to our assumption that $\cap\mathcal{F}$ is the equality on $A$)

*Basic fact 2.17*: If $Y$ is a cofinal subset of a Malitz family $\mathcal{F}$ (for $M$), then $\overline{M}_{\mathcal{F}}$ and $\overline{M}_Y$ are isomorphic in the strong sense : as first order structures *and* as uniform spaces.

## 3. *Remarks*

*Remark 3.1*: Does the use of Malitz-families instead of families of compatible equivalences really produce more, i.e. does there exist $M$, $\mathcal{F}$ such that $\forall \mathcal{F}'$ Malitz-family of equivalences *compatible* with the functions of $M$, we have that $\overline{M}_{\mathcal{F}}$ and $\overline{M}_{\mathcal{F}'}$, are *not* isomorphic as first-order structure ?

The answer is "yes" : consider $M = (\mathbb{N}, f)$ where $f$ is the function defined by :

$$f(n+1) = n \quad (\forall n \in \mathbb{N}) \quad \& \quad f(0) = 0 \quad .$$

Take $\mathcal{F} = \left\{ \sim_n \middle| \, n \in \mathbb{N} \right\}$ where

$$x \sim_n y \overset{\text{def}}{\Leftrightarrow} [(x \le n \ \& \ y \le n \ \& \ x = y) \vee (x > n \ \& \ y > n)] \quad .$$

An example of a "uniformizer" for $f$ is : $\sigma(\sim_n) = \sim_{n+1}$.
One can easily verify that $\overline{M}_{\mathcal{F}}$ looks like this :

$$(\mathbb{N} \cup \{\infty\} \quad , \quad f \cup \{(\infty, \infty)\}) \quad .$$

An additional slight effort permits to check that the only possible equivalences, compatible with $f$, are : $\mathbb{N}^2$ and $\simeq_n$ (for each $n \in \mathbb{N}$), defined by :

$$x \simeq_n y \Leftrightarrow [(x \le n \quad \& \quad y \le n) \vee (x = y > n)] \quad .$$

So the family of the compatible equivalences (on $M$) is a chain : $\mathbb{N}^2 \supset \cdots \supset \simeq_3 \supset \simeq_2 \supset \simeq_1 \supset \simeq_0$.
Any subfamily of this will have a maximum element (for the order $\supseteq$) so cannot be a Malitz-family.

*Remark 3.2*: The notion of "$\mathcal{F}$-Cauchy complete" (see basic fact 2.16) is equivalent to the classical notion of "Cauchy-complete", when $\mathcal{F}$ is taken as a base for the uniformity (see Kelley [22]).
Let us check the main point of this.
A Cauchy-net, in the sense of Kelley, is a net of type $(x_d)_{d \in D}$, where $D$ is some directed set, such that : $\forall \sim \in \mathcal{F}$ ($\mathcal{F}$ is the base of the uniformity) $\exists d \in D \, \forall i, j \in D \, [(i \ge d \quad \& \quad j \ge d) \Rightarrow x_i \sim x_j]$; this last sentence is

equivalent to : $\forall \sim\in \mathcal{F}\ \exists d \in D\ \forall i \in D\ (i \geq d \Rightarrow x_d \sim x_i)$.
So it is obvious, as $\mathcal{F}$ is a directed set, that Cauchy-completeness in the sense of Kelley implies $\mathcal{F}$-Cauchy-completeness.
For the other direction : suppose $\mathcal{F}$-Cauchy-completeness, i.e. each uniform Cauchy $\mathcal{F}$-net has a uniform limit. Take $(x_d)_{d\in D}$ a Cauchy-net in the sense of Kelley.
Choose for each $\sim\in \mathcal{F}$ one $d_\sim \in D$ such that $x_{d_\sim} \sim x_k$ for each $k \geq d_\sim$.
Define $y_\sim = x_{d_\sim}$. If $\sim'\in \mathcal{F}$ realizes $\sim\leq_{\mathcal{F}}\sim'$, take $d \in D$ an upper bound for $d_\sim$ and $d_{\sim'}$; then $y_\sim = x_{d_\sim} \sim x_d \sim' x_{d_{\sim'}} = y_{\sim'}$ and so $y_\sim \sim y_{\sim'}$. So obviously $(y_\sim)_{\sim\in\mathcal{F}}$ is a uniform Cauchy $\mathcal{F}$-net. By our hypothesis this $\mathcal{F}$-net has a uniform limit $z$ which is also the limit (in the sense of Kelley) of the net $(x_d)_{d\in D}$, i.e. $z$ realizes : for any open set 0 such that $z \in 0$ there exists $d \in D$ such that $\forall d' \geq d\ x_{d'} \in 0$ (the topology is, $\mathcal{T}_{\mathcal{F}}$ defined in 2.11; one can easily check that the equivalence classes $\{b|b \sim a\}$ (for $\sim\in \mathcal{F}$) form a base for $\mathcal{T}_{\mathcal{F}}$).

*Conclusion.* The particular type of uniform spaces considered here permits to restrict one's attention to $\mathcal{F}$-nets instead of general nets.

So the situation can be summarized by saying that a Malitz-completion is a Cauchy-completion for a first-order structure which universe has a uniformity base which is a Malitz-family for this structure.
We did not adopt this presentation initially, for two reasons :

(i) the "$\mathcal{F}$-nets" version is easy to handle and does not presuppose any knowledge about uniform spaces,
(ii) the intention of this type of construction is to start with a first-order structure (without any topology nor uniformity) and to find interesting Malitz-families.

The set of all possible Malitz-families (for a given structure $M$) is itself an interesting object (that can be empty), investigated in [20].

*Remark 3.3*: About the "projective limit" aspect : $\overline{A}_{\mathcal{F}}/\approx$ can indeed be seen as the projective limit of the quotient sets $A/\sim$ (where $A/\sim \stackrel{\text{def}}{=} \{[a]_\sim \mid a \in A\}$ and $[a]_\sim \stackrel{\text{def}}{=} \{b \in A \mid b \sim a\}$), in the usual sense (see f.ex. Douady [6]). Each relation $R_M$ is transferred to $A/\sim$ in the obvious way :

$$R_\sim([a]_\sim, [b]_\sim, \ldots) \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \exists a' \sim a\, \exists b' \sim b \ldots R_M(a', b', \ldots).$$

The projective system is the obvious one : the canonical projections are of type

$$\begin{aligned} & A/\sim' \to A/\sim : [a]_{\sim'} \mapsto [a]_{\sim} \quad (\text{for } \sim \leq_{\mathcal{F}} \sim') \\ & \overline{A}_{\mathcal{F}}/\approx \to A/\sim : [x]_{\approx} \mapsto [x_{\sim}]_{\sim} \quad . \end{aligned}$$

For the relations and the constants the situation is classical in the sense that the canonical projections are homomorphisms. We can even precize that these projections are "strong homomorphisms", in the sense of Chang & Keisler [4], which is not the case for arbitrary projective limits of structures (for some fixed language $\mathcal{L}$) : consider, for example, $M_n \stackrel{\text{def}}{=} (\mathbb{N}, R_n)$ where $R_n \stackrel{\text{def}}{=} \{(k, k+1) \mid k \geq n\}$ and the trivial projective system $M_n \to M_p : x \mapsto x$ $(n \geq p)$; each of these projections is a homomorphism, the projective limit is $\overline{M} = (\mathbb{N}, \emptyset)$, but no projection is a *strong* homomorphism. For the functions the situation is somewhat unusual in the sense that the factors of the projective limit don't stay necessarily in the category of the models for $\mathcal{L}$, however the limit itself is again in this category : for example, in the case of a Malitz-completion for a group $G$, with language $\mathcal{L} = \{., 1\}$, the factors $G/\sim$ are not necessarily groups because the function "." can cease to be a *function* on $G/\sim$.

*Remark 3.4*: We have also a system of canonical injections (once $A$ has been well-ordered) :

$$\begin{aligned} & A/\sim \to A \ : [x]_{\sim} \mapsto z \\ & A/\sim \to A/\sim' : [x]_{\sim} \mapsto [z]_{\sim} \qquad (\text{for } \sim \leq_{\mathcal{F}} \sim') \end{aligned}$$

where $z$ is the least element of $[x]_{\sim}$ for the well-ordering relation on $A$.
So each $A/\sim$ can be seen as a subset of $A$ and so "$\bigcup_{\sim \in \mathcal{F}} A/\sim$" $\subseteq A$.
Note that we don't have generally the equality for "$\bigcup_{\sim \in \mathcal{F}} A/\sim$" and $A$; consider $M = \mathbb{N} \cup \{\infty\}$ and the Malitz-family $\mathcal{F} = \{\sim_n \mid n \in \mathbb{N}\}$, where $x \sim_n y \stackrel{\text{def}}{\Leftrightarrow} [(x = y < n) \vee (x \geq n \,\&\, y \geq n]$ (with the usual conventions for $\infty$). We adopt the usual well-ordering on $M$.
Then "$\bigcup_{\sim \in \mathcal{F}} A/\sim$" is $\mathbb{N}$, while $A$ is $\mathbb{N} \cup \{\infty\}$.
The canonical system of injections is "in harmony" with the canonical system of projections, in the sense that :

$$\text{projection} \circ \text{injection} = \text{identity}$$

(whenever this composition makes sense).
Note that the canonical injections are *not* necessarily homomorphisms for the relations and the constants.

*Remark 3.5*: The Cauchy-completion of a first-order structure also makes sense for more general uniformities and have underwhile been investigated in [24]. The projective limit aspect however is lost when the uniformity has no base made of equivalences. The adequate compactness (see section 5) however is still linked to the tree property, like here (sections 5 and 6). The preservation problems (section 4) in that context have not (so far) been studied, but A. Rigo developed an interesting concept of "balanced formula" that allows predictions, about what will be satisfied in the completion, based on tests in the initial structure.
It should also be mentioned that "many" uniformities do admit a base made of equivalences, as those that are $\kappa$-uniformities (i.e. closed under $\kappa$-finite intersections) with $\kappa > \aleph_0$; then one is brought back to Malitz-completions.
All the projections $p$ are *strong* homomorphisms for the relations and the constants (the functions can cease to be functions on $M/\sim$ and $M/\sim'$ but can be considered as relations).
The injections are generally *not* homomorphisms, except the one from $M$ into $\overline{M}_{\mathcal{F}}$ (which however is generally *not* strong so could fail to be an embedding).

## 4. *Preservation*

Here we study briefly several "preservation" properties involving $M$ and $\overline{M}_{\mathcal{F}}$, where $\mathcal{F}$ is a Malitz-family for $M$. Some of these properties already appeared in more specialized contexts (see Forti & Hinnion [14], Esser [8]).
We will be interested in several classes of formulas in the language $\mathcal{L}$, depending on $M$ and $\mathcal{F}$.

*Definition 4.1*: *CONT is the class of the formulas (in $\mathcal{L}$) preserved under uniform $\mathcal{F}$-limits "from $M$ to $\overline{M}_{\mathcal{F}}$", i.e. : $\varphi(x, y, \ldots) \in$ CONT iff*

$$\forall a, b, \ldots \in \overline{A}_{\mathcal{F}} \left[ (\forall \sim \in \mathcal{F}\, M \vDash \varphi(a_\sim, b_\sim, \ldots)) \Rightarrow \overline{M}_{\mathcal{F}} \vDash \varphi([a]_\approx, [b]_\approx, \ldots) \right].$$

*Definition 4.2*: *$\overline{\mathit{CONT}}$ is the class of the formulas preserved under uniform $\mathcal{F}$-limits in $\overline{M}$, i.e. :*

$$\varphi(x, y, \ldots) \in \overline{\mathit{CONT}} \quad \mathit{iff} \quad \forall a, b \in \overline{A}_{\mathcal{F}}/\approx \quad \forall \left(x^{(\sim)}\right)_{\sim \in \mathcal{F}}$$

*$\mathcal{F}$-net with uniform limit a (in $\overline{M}_{\mathcal{F}}$) $\forall \left(y^{(\sim)}\right)_{\sim\in\mathcal{F}}$ $\mathcal{F}$-net with uniform limit b (in $\overline{M}_{\mathcal{F}}$)...*

$$\left[\left(\forall \sim\in \mathcal{F} \quad \overline{M}_{\mathcal{F}} \vDash \varphi\left(x^{(\sim)}, y^{(\sim)}, \ldots\right)\right) \Rightarrow \overline{M}_{\mathcal{F}} \vDash \varphi(a, b, \ldots)\right] \quad .$$

Let us recall that each $\sim\in\mathcal{F}$ has been extended to $\overline{A}_{\mathcal{F}}/\approx$ by the definition :

$$x \sim^{\star} y \Leftrightarrow x_{\sim} \sim y_{\sim} \quad \text{(see basic fact 2.16).}$$

We will simply write "$\sim$" also for the extension "$\sim^{\star}$", so that the definitions for "uniform Cauchy $\mathcal{F}$-net", "uniform $\mathcal{F}$-limit", etc... immediately apply to $\overline{M}_{\mathcal{F}}$; for example : "$z$ is the uniform $\mathcal{F}$-limit of the $\mathcal{F}$-net $\left(x^{(\sim)}\right)_{\sim\in\mathcal{F}}$ in $\overline{M}_{\mathcal{F}}$" means exactly : "$\forall \sim\in \mathcal{F} \quad x^{(\sim)} \sim z$", or more explicitly : "$\forall \sim\in \mathcal{F} \quad \left(x^{(\sim)}\right)_{\sim} \sim z_{\sim}$" ("$x^{(\sim)} \sim^{\star} z$" would be the very strict notation for "$x^{(\sim)} \sim z$").
Actually, $\varphi(x, y, \ldots)\in \overline{\text{CONT}}$ iff $\left\{(a, b, \ldots)\in (\overline{M}_{\mathcal{F}})^n \,\middle|\, \overline{M}_{\mathcal{F}} \vDash \varphi(a, b, \ldots)\right\}$ is closed in $(\overline{M}_{\mathcal{F}})^n$ (we suppose that $\varphi$ has $n$ free variables).

*Definition 4.3*: INV *is the class of the formulas which are "invariant" in the following sense :*

$\varphi(x, y, \ldots) \in$ *INV iff*
$\forall a, b, \ldots \in A(M \vDash \varphi(a, b, \ldots) \Rightarrow \overline{M}_{\mathcal{F}} \vDash \varphi([a]_{\approx}, [b]_{\approx}, \ldots)$.

*Definition 4.4*: *"$x$-*APPROX*" is the name of the class of the formulas "approximable at the variable $x$" (for simplicity we present here the case of the formulas with 2 free variables "$x$" and "$y$", but the general case can be obtained obviously by replacing "$x$" by an $n$-tuple "$\vec{x}$" and "$y$" by an $m$-tuple "$\vec{y}$"), i.e. :*
$\theta(x, y) \in x -$ *APPROX iff*
$\forall a, b \in \overline{A}_{\mathcal{F}}$ *such that* $\overline{M}_{\mathcal{F}} \vDash \theta([a]_{\approx}, [b]_{\approx})$
$\forall b' \in \overline{A}_{\mathcal{F}}$ *such that* $b \approx b'$
$\exists a' \in \overline{A}_{\mathcal{F}}$ *such that* $a \approx a'$
$\exists \sigma$ *extractor (on $\mathcal{F}$) such that* $\forall \sim\in \mathcal{F} \quad M \vDash \theta(a'_{\sigma(\sim)}, b'_{\sigma(\sim)})$.

*Definition 4.5*: *The class $x$-$\overline{APPROX}$ is defined by : $\theta(x, y) \in x$-$\overline{APPROX}$ iff*
$\forall a, b \in \overline{A}_{\mathcal{F}}/\approx$ *realizing* $\overline{M}_{\mathcal{F}} \vDash \theta(a, b)$
$\forall (v^{(\sim)})_{\sim\in\mathcal{F}}$ *$\mathcal{F}$-net in $\overline{M}_{\mathcal{F}}$ with uniform limit b.*

*$\exists(u^{(\sim)})_{\sim\in\mathcal{F}}$ $\mathcal{F}$-net in $\overline{M}_{\mathcal{F}}$ with uniform limit $a$.*
*$\exists\sigma$ extractor (on $\mathcal{F}$) such that $\forall\sim\in\mathcal{F}\quad \overline{M}_{\mathcal{F}}\vDash(u^{(\sigma(\sim))},v^{(\sigma(\sim))})$.*

*Definition 4.6*: *$\theta(x,y)\in x$-WAPPROX ("$w$" for "weak")*
*iff*
*$\forall a\in\overline{A}_{\mathcal{F}}\quad \forall b\in A$ such that $\overline{M}_{\mathcal{F}}\vDash\theta([a]_\approx,[b]_\approx)$*
*$\exists a'\in\overline{A}_{\mathcal{F}}$ such that $\forall\sim\in\mathcal{F}\quad M\vDash\theta(a'_\sim,b)$.*

Here follows a list of easy to prove facts :

*Fact 4.7*: CONT, $\overline{CONT}$, INV, do contain the atomic formulas and are closed under "&" and "$\vee$".

*Fact 4.8*: CONT and $\overline{\text{CONT}}$ are closed under "$\forall$" (i.e. $\varphi\in$ CONT $\Rightarrow$ $\forall z\quad\varphi\in$ CONT, and the same for $\overline{\text{CONT}}$).

*Fact 4.9*: INV is closed under "$\exists$".

*Fact 4.10*: CONT is closed under "$\forall$-APPROX" quantification, i.e. $\varphi\in$ CONT & $\theta(\vec{x},\vec{y})\in\vec{x}$-APPROX implies $\forall\vec{x}(\theta(\vec{x},\vec{y})\Rightarrow\varphi)$ is in the class CONT.

*Fact 4.11*: $\overline{\text{CONT}}$ is closed under "$\forall$-$\overline{\text{APPROX}}$" quantification.

*Fact 4.12*: If $\varphi\in$ CONT and $\theta(\vec{x},\vec{y})\in\vec{x}$-WAPPROX, then $\forall\vec{x}(\theta(\vec{x},\vec{y})\Rightarrow\varphi)$ is in the class INV.
In particular, if $\varphi\in$ CONT, then $\forall\vec{x}\varphi\in$ INV.

*Fact 4.13*: If $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact, then CONT and $\overline{\text{CONT}}$ are closed under "$\exists$". Naturally, "$\mathcal{F}$-compact" means that any $\mathcal{F}$-net admits an $\mathcal{F}$-subnet which has a uniform limit in the space considered; this notion is studied in section 5.

*Fact 4.14*: If $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact, then CONT, $\overline{\text{CONT}}$, INV do contain the class of the positive formulas (i.e. the formulas obtained from the atomic ones under $\vee$, &, $\exists$, $\forall$).

In order to illustrate the relations between the classes of formulas studied here, we discuss now briefly the "archetypical example" : start with the (simple) structure $M=(R_\omega,\in)$, where $R_0=\emptyset$, $R_{n+1}=\mathcal{P}R_n$ ($\mathcal{P}$ is the power set operation), $R_\omega=\bigcup_{n\in\omega}R_n$ ($\omega$ is the set of the von Neumann integers), the

"$\in$" in $M$ is restricted to $R_\omega$. We consider the following Malitz-family for $M$ : $\mathcal{F} = \{\sim_n \mid n \in \mathbb{N}\}$, where $\sim_0 = R_\omega \times R_\omega$ and $\sim_{n+1} = (\sim_n)^+$; this "+" operation is the classical one (see [1], [14], [18]) :

$$x \sim^+ y \quad \overset{\text{def}}{\Leftrightarrow} \quad \left[\left(\forall t \in x \ \ \exists t' \in y \ \ t \sim t'\right) \& \ \ \left(\forall z \in y \ \ \exists z' \in x \ \ z \sim z'\right)\right]$$

(for $\sim$ any equivalence relation on $R_\omega$).
Actually this family $\mathcal{F}$ is a chain :

$$\sim_0 \supset \sim_1 \supset \sim_2 \supset \cdots$$

The completion $\overline{M}_\mathcal{F}$ has been studied by various authors (who used different presentations) : it is noted $\overline{\mathcal{M}}_\omega$ in Weydert [26], $N_\omega$ in Forti & Hinnion [14], Forti & Honsell [16], $X_\omega$ in Hinnion [17] and $M$ in Esser [8].
One can show that the "sets" of this set-theoretical universe $\overline{M}_\mathcal{F}$ are exactly the closed subsets of $\overline{(R_\omega)_\mathcal{F}}/\approx$, that $\overline{M}_\mathcal{F}$ is $\mathcal{F}$-compact and that $\overline{M}_\mathcal{F}$ is a model for : the axiom of extensionality, the anti-foundation axiom $X_1$ (introduced by Forti & Honsell [15]) and the "generalized positive comprehension" principle. We don't discuss these highly interesting aspects here, but just mention them in order to be able to justify some details in what follows.

We can summarize the general situation for CONT, $\overline{\text{CONT}}$, INV by

*Proposition 4.15*:

$$\text{CONT} \subseteq \text{INV} \quad \& \quad \overline{\text{CONT}} \cap \text{INV} \subseteq \text{CONT}$$

In the case of $\overline{(R_\omega, \in)}_\mathcal{F}$ precedingly described, the areas $1 =$ INV $\setminus$ CONT, $2 =$ CONT $\setminus$ $\overline{\text{CONT}}$, $3 =$ CONT $\cap$ $\overline{\text{CONT}}$, $4 =$ $\overline{\text{CONT}}$ $\setminus$ CONT are non-empty; this is witnessed by the following formulas :

Area 1 : $x \neq y$ ($\in$ INV $\setminus$ CONT)
Indeed, $x \neq y \in$ INV is obvious.
Further $x \neq y \notin$ CONT, as
$\forall n \in \omega \quad n \neq n+1$, so $M \vDash x_n \neq x_{n+1}$ for $x_n \overset{\text{def}}{=} n$. But one can check that $x = (x_n)_{n\in\omega}$ and $y = (x_{n+1})_{n\in\omega}$ are uniform Cauchy $\mathcal{F}$-nets such that $x \approx y$. So $\overline{M}_\mathcal{F} \vDash x = y$.

Area 2 : $x = \{x, y\} \& \ x \neq y$ ($\in$ CONT $\setminus\overline{\text{CONT}}$).
This formula is trivially CONT because $x = \{x, y\}$ is impossible to

realize in $R_\omega$.

Further, consider the $\mathcal{F}$-net $x$ defined by : $x_0 = \emptyset$ & $x_{k+1} = \{x_k\}$ and construct, for each $k \in \omega$, an element $y_k$ in $\overline{M}_\mathcal{F}$ such that $\overline{M}_\mathcal{F} \vDash y_k = \{y_k, x_k\}$.

Fundamentally, this is possible because $\overline{M}_\mathcal{F}$ satisfies the anti-foundation axiom $X_1$ (Forti & Honsell [16], Esser [8]), but one can explain this directly by the following construction : define $z_k^{(n)}$ (in $M$) by : $z_k^{(0)} = \emptyset$ & $z_k^{(n+1)} = \left\{z_k^{(n)}, x_k\right\}$. One can easily check that the formula $v = \{u, w\}$ is CONT and that $\left(z_k^{(n)}\right)_{n\in\omega}$ is a uniform Cauchy $\mathcal{F}$-net, which we call $y_k$ (naturally, as we work in $\overline{A}_\mathcal{F}/\approx$, this $y_k$ has to be understood modulo $\approx$); so we get $\overline{M}_\mathcal{F} \vDash y_k = \{y_k, x_k\}$.

Further $\overline{M}_\mathcal{F} \vDash y_k \neq x_k$ because any $y_k$ is a proper pair in $\overline{M}_\mathcal{F}$, which is not the case of $x_k$ (except for $k = 2$).

So we have $\forall k \in \omega \quad \overline{M}_\mathcal{F} \vDash (y_k = \{y_k, x_k\} \quad \& \quad y_k \neq x_k)$. If the formula $x = \{x, y\}$ & $x \neq y$ is $\overline{\text{CONT}}$, and $a$ is the limit of the $x_k$ and $b$ is the limit of the $y_k$, we should have : $\overline{M}_\mathcal{F} \vDash b = \{b, a\}$ & $b \neq a$.

Note that $\overline{M}_\mathcal{F} \vDash a = \{a\}$ because $v = \{u\}$ is CONT and $M \vDash x_{k+1} = \{x_k\}$.

But it is also known that $\overline{M}_\mathcal{F}$ is strongly extensional in the sense of Aczel [1], i.e. : any bisimulation on $\overline{M}_\mathcal{F}$ is the equality on $\overline{M}_\mathcal{F}$ (a bisimulation on a binary structure $N = (B, \in_N)$ is an equivalence relation $\simeq$ on $N$ such that $\simeq \subseteq \simeq^+$).

Here, $\simeq \overset{\text{def}}{=} \{(a, b), (b, a)\} \cup \left\{(t, t) \mid t \in \overline{M}_\mathcal{F}\right\}$ is a bisimulation on $\overline{M}_\mathcal{F}$, so that $\overline{M}_\mathcal{F} \vDash a = b$, contradicting what precedes.

Area 3 : $x \in y$ ($\in$ CONT $\cap\overline{\text{CONT}}$).

Obvious, as "$x \in y$" is atomic (see fact 4.7).

Area 4 : $\forall x \quad x \notin x$ ($\in \overline{\text{CONT}}\setminus$ CONT).

This formula is trivially $\overline{\text{CONT}}$.

Further it is not CONT because $M \vDash \forall x \quad x \notin x$ while $\neg\overline{M}_\mathcal{F} \vDash \forall x \quad x \notin x$ (consider for example $x = (x_k)_{k\in\omega}$ defined as in the case "Area 2").

We can summarize the general situation for the "approx" classes by

*Proposition 4.16*:

$x$-*APPROX* $\subseteq$ $x$-*WAPPROX* & $x$-*WAPPROX*$\cap x-\overline{\textit{APPROX}} \subseteq x-$*APPROX*

In the case of $\overline{M}_{\mathcal{F}}$ precedingly described ($M = (R_\omega, \in)$), the areas 1, 2, 3, 4 are non-empty; this is witnessed by :

Area 1 : $y = \{y\} \vee y \in x$ ($\in x$-WAPPROX $\backslash x$-APPROX).

Suppose $\overline{M}_{\mathcal{F}} \vDash y = \{y\} \vee y \in \overline{x}$, with $y \in M$ & $\overline{x} \in \overline{M}_{\mathcal{F}}$. So $\overline{M}_{\mathcal{F}} \vDash y \in \overline{x}$ because no $y$ in $M$ can realize $\overline{M}_{\mathcal{F}} \vDash y = \{y\}$.

Take $(x_n)_{n\in\omega}$ any $\mathcal{F}$-net with uniform limit $\overline{x}$. Define $x'_n \overset{\text{def}}{=} x_n \cup \{y\}$.

Then $\overline{x}$ is still the limit of the $x'_n$ and $\overline{M}_{\mathcal{F}} \vDash y \in x'_n$. So the formula "$y = \{y\} \vee y \in x$" is $x$-WAPPROX.

Further, consider $\Omega =$ the unique auto-singleton in $\overline{M}_{\mathcal{F}}$ (= the limit of $x_k$ for $x_0 = \emptyset$ & $x_{n+1} = \{x_n\}$) and any $\mathcal{F}$-net $(z_n)_{n\in\omega}$ in $M$, with limit $\emptyset$. Then $\overline{M}_{\mathcal{F}} \vDash \Omega = \{\Omega\} \vee \Omega \in \emptyset$. But no $\mathcal{F}$-net $(t_n)_{n\in\omega}$ in $M$ can realize (for some extractor $\sigma$ on $\omega$) :

$$M \vDash t_{\sigma(n)} = \left\{t_{\sigma(n)}\right\} \vee t_{\sigma(n)} \in z_{\sigma(n)}$$

because "$v = \{v\}$" is impossible in $M$ and $\lim\limits_{n\in\omega} z_n = \emptyset$ implies $z_n = \emptyset$ for $n \geq 1$.

So "$y = \{y\} \vee y \in x$" is not $x$-APPROX.

Area 2 : $x = y = \{y\} \vee \forall z(z = \{z\} \Rightarrow z \notin z)$ ($\in x$-APPROX $\backslash \overline{x\text{-APPROX}}$).

Suppose $\overline{M}_{\mathcal{F}} \vDash \overline{x} = \overline{y} = \{\overline{y}\} \vee \forall z(z = \{z\} \Rightarrow z \notin \overline{z})$ and $\overline{y}$ is the limit of $y_n$ in $M$.

Then $x_n \overset{\text{def}}{=} y_n$ trivially realizes $M \vDash x_n = y_n = \{y_n\} \vee \forall z(z = \{z\} \Rightarrow z \notin y_n)$ just because $M \vDash \forall z \quad z \neq \{z\}$.

So the initial formula is $x$-APPROX.

Consider now $x_0 = \emptyset$ & $x_{k+1} = \{x_k\}$, $\Omega = \{\Omega\}$ and $y_n = \{\Omega, x_n\}$. The limit $\overline{y}$ of $y_n$ is again $\Omega$ because the limit of $x_k$ is $\Omega$, and so $\overline{y} = \{\Omega, \Omega\} = \{\Omega\} = \Omega$.

So, for $\overline{x} = \overline{y} = \Omega$ we have :

$$\overline{M}_{\mathcal{F}} \vDash \overline{x} = \overline{y} = \{\overline{y}\} \vee \forall z(z = \{z\} \Rightarrow z \notin \overline{y}).$$

However no $(x_n)_{n\in\omega}$ with limit $\overline{x}$ can realize $\overline{M}_{\mathcal{F}} \vDash x_n = y_n = \{y_n\}$ because each such $y_n$ is a proper pair while $\{y_n\}$ is a singleton, nor can it realize $\overline{M}_{\mathcal{F}} \vDash \forall z$
$(z = \{z\} \Rightarrow z \notin y_n)$ because $\Omega$ is the only $z$ in $\overline{M}_{\mathcal{F}}$ realizing $z = \{z\}$ and $\Omega \in y_n$.

Area 3 : $y \in x$ ($\in x$-APPROX $\cap x$-$\overline{\text{APPROX}}$)

Just take $x_n \overset{\text{def}}{=} \overline{x} \cup \{y_n\}$, for $\overline{y}$ limit of $y_n$. Then $\overline{x}$ is the limit of $x_n$.

Area 4 : $y \neq \{y\}$ & $x = \{x\}$ ($\in x$-$\overline{\text{APPROX}} \setminus x$-WAPPROX).

Suppose $\overline{M}_{\mathcal{F}} \vDash \overline{y} \neq \{\overline{y}\}$ & $\overline{x} = \{\overline{x}\}$, and $\overline{y}$ is the limit of $y_n$ (in $\overline{M}_{\mathcal{F}}$). Then, for some extractor $\sigma$ on $\omega$, $\overline{M}_{\mathcal{F}} \vDash y_{\sigma(n)} \neq \{y_{\sigma(n)}\}$ because otherwise infinitely many $n \in \omega$ would realize $\overline{M}_{\mathcal{F}} \vDash y_n = \{y_n\}$ and then $\overline{M}_{\mathcal{F}} \vDash \overline{y} = \{\overline{y}\}$. Take $x_n \overset{\text{def}}{=} \overline{x}$. So $\forall n \in \omega$ $\overline{M}_{\mathcal{F}} \vDash y_{\sigma(n)} \neq \left\{y_{\sigma(n)}\right\}$ & $x_{\sigma(n)} = \left\{x_{\sigma(n)}\right\}$ and this shows that our formula is $x$-$\overline{\text{APPROX}}$.

Further, suppose that $y_0 \in M$ and $\overline{x} \in \overline{M}$ realize $\overline{M}_{\mathcal{F}} \vDash y_0 \neq \{y_0\}$ & $\overline{x} = \{\overline{x}\}$.

Then each $\mathcal{F}$-net $(x_n)_{n\in\omega}$ in $M$, with limit $\overline{x}$, realizes $\overline{M}_{\mathcal{F}} \vDash x_n \neq \{x_n\}$ and so $\neg\overline{M}_{\mathcal{F}} \vDash y_0 \neq \{y_0\}$ & $x_n = \{x_n\}$, so that our formula is not $x$-WAPPROX.

## 5. *The compactness problem*

As we will see in section 6, the $\mathcal{F}$-compactness is sometimes essential (but not always). So we study this problem here, via 3 approaches.
The first one which comes in mind is to try to relate $\mathcal{F}$-compactness to usual "cover-compactness" and use some "Tychonoff" theorem. The second approach was already suggested by Malitz [23] and uses adapted ultrafilters as "deus ex machina". The third approach is the direct one, where one tries to see how one can extract a uniform Cauchy $\mathcal{F}$-subnet from an arbitrary $\mathcal{F}$-net. Before we can develop these approaches, we have to clarify some basic facts about $\mathcal{F}$-compactness and also to introduce some definitions and results concerning directed sets.

*Definitions 5.1*: *($(D, \leq)$ is supposed to be a directed set without maximum and $\mathcal{F}$ is supposed to be a Malitz-family for a structure $M$ with universe $A$).*

*5.1.1 $D$ is "$\theta$-directed" (where $\theta$ is an infinite cardinal) iff each $\theta$-finite subset of $D$ has an upper bound in $D$ ("$\theta$-finite" means "of cardinal $< \theta$").*

*5.1.2 a "strictly well-ordered chain" in $D$ is a $\delta$-net in $D$ (for some ordinal $\delta$) : $(d_\alpha)_{\alpha<\delta}$ such that*

$$\forall \alpha, \beta < \delta \quad (\alpha < \beta \Leftrightarrow d_\alpha < d_\beta)$$

*(equivalently : the order-type of this $\delta$-net is $\delta$ itself).*

*5.1.3 a "basic chain" in $D$ is a strictly well-ordered chain in $D$ : $(d_\alpha)_{\alpha<\delta}$ such that $\{d_\alpha \mid \alpha < \delta\}$ has no strict upper bound in $D$.*

*5.1.4 $\delta_D$ is the least infinite ordinal $\delta$ such that some $\delta$-net in $D$ is a basic chain in $D$.*

*5.1.5 $\kappa_{\mathcal{F}}$ is the least strict upper bound of the cardinals $|A/\sim|$, for $\sim \in \mathcal{F}$ (where $|X| \overset{def}{=}$ the cardinal of $X$)).*

*Basic fact 5.2: (the proofs are easy).*

*5.2.1 $D$ is $\theta$-directed is equivalent to each of the following sentences :*

(i) *each partition of $D$ into strictly less that $\theta$ pieces admits a cofinal piece,*
(ii) *each cover of $D$ with strictly less than $\theta$ pieces admits a cofinal piece,*
(iii) *each cofinal subset of $D$ is $\theta$-directed,*
(iv) *$D$ admits a cofinal, $\theta$-directed subset.*

*5.2.2 $\delta_D$ is a regular (infinite) cardinal.*

*5.2.3 $D$ is $\delta_D$-directed.*

*5.2.4 $D$ is not $\delta_D^+$-directed ("$\theta^+$" is the usual notation for the successor cardinal of $\theta$.)*

*5.2.5 $\kappa_{\mathcal{F}} \leq |A|^+$.*

*Comment :* as each $A/\sim$, for $\sim \in \mathcal{F}$, can be canonically injected in $A$ (remark 3.4), we have $|A/\sim| \leq |A|$ and so $\kappa_{\mathcal{F}} \leq |A|^+$.
The situations $\kappa_{\mathcal{F}} = |A|^+$, $\kappa_{\mathcal{F}} = |A|$, $\kappa_{\mathcal{F}} < |A|$ are all possible : for $\kappa_{\mathcal{F}} = |A|^+$, take $M = \mathbb{N}$ and $\mathcal{F} = \{\sim_n \mid n \in \mathbb{N}\}$, with $a \sim_n b \Leftrightarrow [(a \leq n \,\&\, b \leq n) \vee (a = b > n)]$, and so $|A| = \aleph_0$ while $|A|^+ = \aleph_1 = \kappa_{\mathcal{F}}$; for $\kappa_{\mathcal{F}} = |A|$, take $M = \mathbb{N}$ and $\mathcal{F} = \{\sim \mid A/\sim \text{ is finite}\}$, and so $|A| = \aleph_0 = \kappa_{\mathcal{F}}$; for $\kappa_{\mathcal{F}} < |A|$, take $M = \aleph_1$ and $\mathcal{F} = \{\sim \mid A/\sim \text{ is finite}\}$, and so $|A| = \aleph_1$ while $\kappa_{\mathcal{F}} = \aleph_0$.

*5.2.6 $\delta_{\mathcal{F}} \leq |A|$.*

*Proof.* Consider a strictly well-ordered chain $(\sim_\nu)_{\nu<\theta}$ in $(\mathcal{F}, \leq_{\mathcal{F}})$. Modulo the canonical injections (remark 3.4), the family of sets $(A/\sim_\nu)_{\nu<\theta}$ can be seen as a strictly increasing chain for inclusion, with $A$ as upper bound :

$$A/\sim_0 \subset A/\sim_1 \subset \cdots \subset A/\sim_\nu \subset \cdots \subset A$$

No $A/\sim_\nu$ can "be" $A$ because $\mathcal{F}$ has no maximum. So, we get :

$$\alpha_0 < \alpha_1 < \cdots < \alpha_\nu < \cdots < \mathcal{E}$$

where $\alpha_\nu$ is the (well-) order type of $A/\sim_\nu$ (for $\nu < \theta$) and $\mathcal{E}$ is the (well-) order type of $A$. As $\mathcal{E} < |A|^+$ and $(\theta, <)$ is embedded in $(\mathcal{E}, <)$ by the function : $\nu \mapsto \alpha_\nu$, we get : $\theta \le \mathcal{E} < |A|^+$ and so $\theta < |A|^+$ ($\alpha_\nu$, $\mathcal{E}$ are ordinals).
So, if we take for $\theta$ one of the *cardinals* $\delta_\mathcal{F}$ or $\beta_\mathcal{F}$, we get :

$$\delta_\mathcal{F} \le |A| \quad \& \quad \beta_\mathcal{F} \le |A|$$

□

*5.2.7* $\quad \delta_\mathcal{F} \le \kappa_\mathcal{F}$ & $\beta_\mathcal{F} \le \kappa_\mathcal{F}$.

*Proof.* Like for 5.2.7, except that "$\mathcal{E}$" is replaced by "$\kappa_\mathcal{F}$"; so one gets that $\theta \le \kappa_\mathcal{F}$ for each strictly well-ordered chain $(\sim_\nu)_{\nu<\theta}$ in $(\mathcal{F}, \le_\mathcal{F})$. □

*5.2.8 If $\delta_\mathcal{F} = \kappa_\mathcal{F}$, then any basic chain (in $\mathcal{F}$) has order type $\delta_\mathcal{F}$, i.e. $\theta = \delta_\mathcal{F}$ for any basic chain $(\sim_\nu)_{\nu<\theta}$ in $\mathcal{F}$.*

*Proof.* If $(\sim_\nu)_{\nu<\theta}$ is a basic chain in $\mathcal{F}$, then $\delta_\mathcal{F} \le \theta \le \kappa_\mathcal{F}$ (by proof of 5.2.8 & the definition of $\delta_\mathcal{F}$); so $\delta_\mathcal{F} = \kappa_\mathcal{F}$ implies $\theta = \delta_\mathcal{F}$. □

*5.2.9 $\delta_\mathcal{F} = \kappa_\mathcal{F}$ iff $\mathcal{F}$ is $\kappa_\mathcal{F}$-directed.*

*Proof.* Use basic facts 5.2.3, 5.2.4, 5.2.7. □

*5.3* We come back now to $\mathcal{F}$-compactness : first, let us recall that "$\overline{M}_\mathcal{F}$ is $\mathcal{F}$-compact" means that each $\mathcal{F}$-net (in $\overline{M}_\mathcal{F}$) admits a uniform Cauchy $\mathcal{F}$-subnet converging to some element of $\overline{M}_\mathcal{F}$. As $\overline{M}_\mathcal{F}$ is always "$\mathcal{F}$-complete" in the sense that any uniform Cauchy $\mathcal{F}$-net has a limit in $\overline{M}_\mathcal{F}$, it is clear that "$\mathcal{F}$-compactness" (for $\overline{M}_\mathcal{F}$) is equivalent to the fact that each $\mathcal{F}$-net *in* $M$ admits a uniform Cauchy $\mathcal{F}$-subnet (Kelley [22] would call $M$ "totally bounded", while Malitz [23] uses the term "crowded").

We examine now the very important "necessary condition for $\mathcal{F}$-compactness" :

*Lemma 5.3.1* : *If $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact, then $\mathcal{F}$ is $\kappa_{\mathcal{F}}$-directed (and so $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$).*

*Proof.* Suppose that $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact but that $\mathcal{F}$ is *not* $\kappa_{\mathcal{F}}$-directed. Then there exists a partition of $\mathcal{F}$ into $\theta < \kappa_{\mathcal{F}}$ pieces : $(P_\alpha)_{\alpha<\theta}$, such that no piece $P_\alpha$ is cofinal in $\mathcal{F}$. So $|M/\sim_0| \geq \theta$ for some $\sim_0 \in \mathcal{F}$, because $\kappa_{\mathcal{F}}$ is the least *strict* upper bound of the $|A/\sim|$, for $\sim \in \mathcal{F}$. Choose $(a_\alpha)_{\alpha<\theta}$ such that $a_\alpha \in A$ and $\forall \alpha, \alpha' < \theta\ (\alpha \neq \alpha' \Rightarrow \neg a_\alpha \sim_0 a_{\alpha'})$. Further, define for each $\sim \in \mathcal{F}$ : $a^{(\sim)}$ = the unique $a_\alpha$ such that $\sim \in P_\alpha$. Then the $\mathcal{F}$-net $(a^{(\sim)})_{\sim\in\mathcal{F}}$ in $M$ can't admit any uniform Cauchy $\mathcal{F}$-subnet. Indeed : if $(a^{\sigma(\sim)})_{\sim\in\mathcal{F}}$ is a uniform Cauchy $\mathcal{F}$-net (for some extractor $\sigma$ on $\mathcal{F}$), then $\forall \sim \geq \sim_0\ a^{\sigma(\sim)} \sim_0 a^{\sigma(\sim_0)}$ and so $\{\sigma(\sim) \mid \sim \geq \sim_0\}$, which is cofinal in $\mathcal{F}$, would be a subset of $B = \{\sigma(\sim) \mid a^{\sigma(\sim)} \sim_0 a^{\sigma(\sim_0)}\}$, forcing $B$ to be also cofinal in $\mathcal{F}$. As no $P_\alpha$ is cofinal in $\mathcal{F}$, $B \subset P_\alpha$ is false for each $\alpha < \theta$ and so $B$ has to meet at least 2 of them, i.e. $\exists \alpha, \alpha' < \theta\ (\alpha \neq \alpha'\ \&\ B \cap P_\alpha \neq \emptyset\ \&\ B \cap P_{\alpha'} \neq \emptyset)$. Take some $\sigma(\sim) \in B \cap P_\alpha$ and $\sigma(\sim') \in B \cap P_{\alpha'}$. Then

$$a_\alpha = a^{\sigma(\sim)} \sim_0 a^{\sigma(\sim_0)} \sim_0 a^{\sigma(\sim')} = a_{\alpha'}$$

and so $a_\alpha \sim_0 a_{\alpha'}$, contradicting the choice of the $(a_\alpha)_{\alpha<\theta}$. □

*Comment 5.3.2*

Remember that : $\mathcal{F}$ is $\kappa_{\mathcal{F}}$-directed iff $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$, and that : $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}} \Rightarrow$ any basic chain in $\mathcal{F}$ has order type $\delta_{\mathcal{F}}$ (see basic facts 5.2.10, 5.2.9). This eliminates (a priori) a lot of types of directed sets as candidates for a "compactifying" Malitz-family; for example : no Malitz-family $\mathcal{F}$ such that $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact can be isomorphic to this directed set $(D, \leq)$ : $D \overset{\text{def}}{=} \aleph_0 \times \aleph_1$ & $\leq$ on $D$ is defined by:

$$(\alpha, \beta) \leq (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha' \& \ \beta \leq \beta' \quad .$$

This is a directed set, with $\delta_D = \aleph_0$ : a witness basic chain is (for example) $((\alpha, 0))_{\alpha<\aleph_0}$. But one can find basic chains of order type $\aleph_1$ as well :

$$((0, \beta))_{\beta<\aleph_1} \quad .$$

*Comment 5.3.3*

Lemma 5.3.1 mentions a necessary condition for $\mathcal{F}$-compactness, but not a sufficient one, i.e. : $\mathcal{F}$ is $\kappa_{\mathcal{F}}$-directed $\not\Rightarrow \overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact. Before we give a counter-example, we have to recall the definition of the "tree-property" for an infinite cardinal $\theta$ (see Chang & Keisler [4]) : a "tree" is an ordered set $(X, \leq)$ such that $X$ has a minimum element and any $\{y \in X \mid y < x\}$ is well-ordered by $\leq$ (for each $x \in X$).
A subset $B \subset X$, well-ordered by $\leq$, is called a "branch". The "order" of an element $x$ of $X$ is the order type of $\{y \in X \mid y < x\}$ and the "order" of $X$ is the supremum of the orders of $x$ in $X$. The level $\alpha$ ($\alpha$ is an ordinal) is the set : $L_\alpha = \{x \mid x \text{ is of order } \alpha \text{ in } X\}$.
An infinite cardinal $\theta$ is said to have the "tree-property" (Chang & Keisler [4]) or equivalently to be "ramifiable" (Comfort & Negrepontis [5]) iff any tree of order $\theta$ with $\theta$-finite levels (i.e. levels of cardinal $< \theta$) has a branch of order $\theta$.
Let us mention at once that the tree-property is a strong one, implying weak inaccessibility ($\theta$ is weakly inaccessible iff $\theta$ is a regular cardinal & a limit cardinal); so it is a "large cardinal" condition.
So here follows our example :
$\aleph_1$ is not ramifiable (because it is a successor cardinal), so there exists a tree $(X, \leq)$ of order $\aleph_1$, with $\aleph_1$-finite levels and without any branch of order $\aleph_1$. If we drop the undesirable elements in $X$, we can even choose $X$ such that each $\{y \in X \mid y \geq x\}$ is itself a tree of order $\aleph_1$.
Define the following Malitz-family $\mathcal{F}$ for $(X, \leq)$ :

$$\begin{cases} \mathcal{F} & \stackrel{\text{def}}{=} & (\sim_\alpha)_{\alpha < \aleph_1} \\ a \sim_\alpha b & \stackrel{\text{def}}{\Leftrightarrow} & [(\exists z \in L_\alpha \, (z \leq a \,\&\, z \leq b)) \vee a = b] \end{cases}$$

One can easily check that $\kappa_{\mathcal{F}} = \delta_{\mathcal{F}} = \aleph_1$, so $\mathcal{F}$ is $\kappa_{\mathcal{F}}$-directed.
But $\overline{M}_{\mathcal{F}}$ is not $\mathcal{F}$-compact. Indeed, consider an $\mathcal{F}$-net $(x_\alpha)_{\alpha<\aleph_1}$ in $M = (X, \leq)$ such that $x_\alpha$ is of order $\alpha$ in $X$. If such a net admits a uniform Cauchy $\mathcal{F}$-subnet $(x_{\sigma(\alpha)})_{\alpha<\aleph_1}$ (for $\sigma$ an extractor on $\aleph_1$), then one could construct a branch $B$ of order $\aleph_1$, in $X$ : $B \stackrel{\text{def}}{=} \{z_\alpha \mid \alpha < \aleph_1\}$, with $z_\alpha \stackrel{\text{def}}{=}$ the unique $z \in L_\alpha$ such that $z \leq x_{\sigma(\alpha)}$.
Actually this type of counterexample can be generalized and permits to prove that : $\forall \kappa$ regular (infinite) but non ramifiable cardinal $\exists \mathcal{F}$ a Malitz-family for a structure $M$ such that $\overline{M}_{\mathcal{F}}$ is not $\mathcal{F}$-compact.

### *5.4 The first approach*

As we try here to use some version of Tychonoff's theorem, we have to translate the notion of "$\mathcal{F}$-compactness" in terms of more usual "cover-compactness". Let us recall that the topology induced by the uniformity on $\overline{M}_{\mathcal{F}}$ (for which $\mathcal{F}$ is a "uniformity base" in the sense of Kelley [22]) has $\{[x]_\sim \mid x \in \overline{A}_{\mathcal{F}}/_\approx \ \& \ \sim\in \mathcal{F}\}$ as a base of open sets.
Remember that $\mathcal{F}$ has been "extended" to $\overline{A}_{\mathcal{F}}$ by the rule : $x \sim y \Leftrightarrow x_\sim \sim y_\sim$ and so also is extended to $\overline{A}_{\mathcal{F}}/\approx$.
Each element $[x]_\sim$ of the base of the topology on $\overline{A}_{\mathcal{F}}/\approx$ is actually a clopen set (closed & open). As we are interested in $\mathcal{F}$-compactness here, we will suppose that $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$ (see lemma 5.3.1 and comment 5.3.2). So the topology on $\overline{M}_{\mathcal{F}}$ is actually a "$\kappa_{\mathcal{F}}$-topology", i.e. a topology closed under $\kappa_{\mathcal{F}}$-finite intersections (equivalently : the intersection of $< \kappa_{\mathcal{F}}$ open sets is an open set).
The notion of "base of a $\kappa$-topology" is just the usual notion of "base of a topology", i.e. $\mathcal{B}$ is a base for the topology $\mathcal{T}$ iff each element of $\mathcal{T}$ (each "open set") is the union of elements of $\mathcal{B}$.
But the adequate notion of "subbase" for a $\kappa$-topology is : $S$ is a $\kappa$-subbase for the $\kappa$-topology $\mathcal{T}$ iff $\mathcal{T}$ is the "closure" of $S$ under arbitrary unions and $\kappa$-finite intersections (of elements of $S$).
Naturally, for $\kappa = \aleph_0$ one gets back the usual notion of subbase for a topology (as in Kelley [22]).
We already mentioned (remark 3.3) that $\overline{A}_{\mathcal{F}}/\approx$ can be seen as a subset of the cartesian product $\prod_{\sim\in\mathcal{F}} A/\sim$. The natural "$\kappa_{\mathcal{F}}$-product topology" on $\prod_{\sim\in\mathcal{F}} A/\sim$ is the one obtained by adopting

$$S = \left\{ \left\{ y \in \prod_{\sim\in\mathcal{F}} A/\sim \mid x_{\sim_0} = y_{\sim_0} \right\} \Big| \ x \in \prod_{\sim\in\mathcal{F}} A/\sim \ \& \ \sim_0 \in \mathcal{F} \right\}$$

as a $\kappa_{\mathcal{F}}$-subbase.
Note that, except for $\kappa_{\mathcal{F}} = \aleph_0$, this $\kappa_{\mathcal{F}}$-product topology is generally *not* the simple "product topology" (as defined in Kelley [22]), because for this product topology $S$ is only a subbase (i.e. an $\aleph_0$-subbase). One can check that the $\kappa_{\mathcal{F}}$-topology we adopted on $\overline{A}_{\mathcal{F}}/\approx$ is exactly the one induced by the $\kappa_{\mathcal{F}}$-product topology on $\prod_{\sim\in\mathcal{F}} A/\sim$ (when $\overline{A}_{\mathcal{F}}/\approx$ is seen as a subset of $\prod_{\sim\in\mathcal{F}} A/\sim$), and that $\overline{A}_{\mathcal{F}}/\approx$ is closed in $\prod_{\sim\in\mathcal{F}} A/\sim$.

*Definition 5.4.1 : (for $\kappa$ an infinite regular cardinal). A $\kappa$-topological space is said to be "$\kappa$-compact" iff any cover of the space (by open sets) admits a $\kappa$-finite subcover.*

One can prove in the very standard way that any closed subset of a $\kappa$-compact space is also $\kappa$-compact.

The last point we have to check before we discuss Tychonoff's theorem in this context is :

*Property 5.4.2 : If $\mathcal{F}$ is a Malitz-family on $M$ such that $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$, then $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact iff $\overline{M}_{\mathcal{F}}$ is $\kappa_{\mathcal{F}}$-compact.*

*Proof.*

(a) As $\mathcal{B} = \left\{[x]_\sim \;\middle|\; \sim \in \mathcal{F} \,\&\, x \in \overline{M}_{\mathcal{F}}\right\}$ is a base for the topology on $\overline{M}_{\mathcal{F}}$, we have : $\overline{M}_{\mathcal{F}}$ is $\kappa_{\mathcal{F}}$-compact $\Leftrightarrow$ any cover of $\overline{M}_{\mathcal{F}}$ *by elements of* $\mathcal{B}$ admits a $\kappa_{\mathcal{F}}$-finite subcover.

(b) We need the following fact :
if an $\mathcal{F}$-net $(x_\sim)_{\sim\in\mathcal{F}}$ in $\overline{M}_{\mathcal{F}}$ has no uniform Cauchy $\mathcal{F}$-subnet, then $\forall z \in \overline{M}_{\mathcal{F}}\ \exists \sim \in \mathcal{F}$ such that $\{\sim' \in \mathcal{F} \mid x_{\sim'} \sim z\}$ is *not* cofinal in $\mathcal{F}$.
Indeed : otherwise $\exists z \in \overline{M}_{\mathcal{F}}\quad \forall \sim \in \mathcal{F}\ \{\sim' \in \mathcal{F} \mid x_{\sim'} \sim z\}$ is cofinal in $\mathcal{F}$.
Then $\{\sim' \mid \sim' \geq \sim \ \&\ x_{\sim'} \sim z\}$ is also cofinal in $\mathcal{F}$. Choose for each $\sim \in \mathcal{F}$ some $\sim' \geq \sim$ realizing $x_{\sim'} \sim z$, and call this $\sim'$ : $\sigma(\sim)$. Then $(x_{\sigma(\sim)})_{\sim\in\mathcal{F}}$ would be a uniform Cauchy $\mathcal{F}$-subnet of $(x_\sim)_{\sim\in\mathcal{F}}$.

(c) Suppose now that $\overline{M}_{\mathcal{F}}$ is $\kappa_{\mathcal{F}}$-compact, but not $\mathcal{F}$-compact. Then some $\mathcal{F}$-net $(x_\sim)_{\sim\in\mathcal{F}}$ in $\overline{M}_{\mathcal{F}}$ has no uniform Cauchy $\mathcal{F}$-subnet, and so (by fact (b)) :
$\forall z \in \overline{M}_{\mathcal{F}}\quad \exists \sim \in \mathcal{F}$ such that $\{\sim' \in \mathcal{F} \mid x_{\sim'} \sim z\}$ is not cofinal in $\mathcal{F}$. Choose for each $z$ in $\overline{M}_{\mathcal{F}}$ one $\sim$ like that and call it $\sim_z$.
Define $Y_z = \{\sim' \in \mathcal{F} \mid x_{\sim'} \sim z\}$.
Then $\left\{[z]_{\sim_z} \;\middle|\; z \in \overline{M}_{\mathcal{F}}\right\}$ is an open cover of $\overline{M}_{\mathcal{F}}$. So this admits a $\kappa_{\mathcal{F}}$-finite subcover $S = \left\{[z_\alpha]_{\sim_{(z_\alpha)}} \;\middle|\; \alpha < \theta\right\}$, with $\theta < \kappa_{\mathcal{F}}$.
Then the family $\left(Y_{(z_\alpha)}\right)_{\alpha<\theta}$ covers $\mathcal{F}$, has no cofinal piece but is $\kappa_{\mathcal{F}}$-finite : this contradicts our assumption that $\mathcal{F}$ is $\kappa_{\mathcal{F}}$-directed (modulo basic fact 5.2.1 (ii)).

(d) Suppose now that $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact, but not $\kappa_{\mathcal{F}}$-compact. By point (a), there exists an open cover $\mathcal{C}$ of $\overline{M}_{\mathcal{F}}$ such that $\mathcal{C}$ is a subset of the base $\mathcal{B}$ and $\mathcal{C}$ admits no $\kappa_{\mathcal{F}}$-finite subcover. Let us "saturate" $\mathcal{C}$, i.e. replace $\mathcal{C}$ by
$\mathcal{C}' = \{Y' \in \mathcal{B} \mid \exists Y \in \mathcal{C} \quad Y' \subseteq Y\}$.
Then $\mathcal{C}'$ has also no $\kappa_{\mathcal{F}}$-finite subcover. As each $A/\sim$ is of cardinal $< \kappa_{\mathcal{F}}$, there should exist, for each $\sim \in \mathcal{F}$, some $x \in \overline{M}_{\mathcal{F}}$ avoiding each $[v]_\sim \in \mathcal{C}'$, i.e. some $x \in \overline{M}_{\mathcal{F}}$ such that $\forall Y \in \mathcal{C}' : Y$ is of type $[v]_\sim$ (for some $v \in \overline{M}_{\mathcal{F}}$) implies $x \notin Y$. Actually such an $x$ avoids also any $Y \in \mathcal{C}'$ of type $[v]_{\sim'}$, for any $\sim' \leq \sim$ (because, if $x \in [v]_{\sim'} \in \mathcal{C}'$ and $\sim' \leq \sim$, then $[x]_\sim \subset [v]_{\sim'} \in \mathcal{C}'$ and so $[x]_\sim \in \mathcal{C}'$, contradicting the definition of $x$). Choose for each $\sim \in \mathcal{F}$ one $x$ avoiding each $[v]_\sim \in \mathcal{C}'$ and call it "$x_\sim$". Then $\left(x_{\sigma(\sim)}\right)_{\sim \in \mathcal{F}}$ will have a uniform limit $y$ in $\overline{M}_{\mathcal{F}}$, for some extractor $\sigma$ (because $\overline{M}_{\mathcal{F}}$ is supposed to be $\mathcal{F}$-compact). As $\mathcal{C}'$ covers $\overline{M}_{\mathcal{F}}$, $y$ is in some $[v]_{\sim_0} \in \mathcal{C}'$, for some $v \in \overline{M}_{\mathcal{F}}$ & $\sim_0 \in \mathcal{F}$. As $x_{\sigma(\sim_0)} \sim_0 y$, we get : $x_{\sigma(\sim_0)} \in [y]_{\sim_0} = [v]_{\sim_0} \in \mathcal{C}'$. As $\sim_0 \leq \sigma(\sim_0)$ and $x_{\sigma(\sim_0)}$ avoids any $[v]_{\sim'} \in \mathcal{C}'$ (for any $\sim' \leq \sim$), this is impossible.

□

*Conclusion 5.5* : What precedes shows that it suffices to get $\prod_{\sim \in \mathcal{F}} A/\sim$ $\kappa_{\mathcal{F}}$-compact if we want $\overline{M}_{\mathcal{F}}$ to be $\mathcal{F}$-compact. For $\kappa_{\mathcal{F}} = \aleph_0$, the problem is solved by Tychonoff's theorem, stating that a product of compact spaces (i.e. $\aleph_0$-compact spaces) is compact (for the product topology). The proof of this uses Alexander's theorem, which states that a space is compact iff each cover by members of a subbase $S$ has a finite subcover (see Kelley [22]).
So, for $\kappa_{\mathcal{F}} = \aleph_0$, the problem is solved :

*Theorem 5.5.1* : *If $\mathcal{F}$ is a Malitz-family for $M$ and $\kappa_{\mathcal{F}} = \aleph_0$, then $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact.*

What can be said when $\kappa_{\mathcal{F}} > \aleph_0$ ? The problem is that the proof of Alexander's theorem uses $\kappa_{\mathcal{F}} = \aleph_0$ in a way that cannot be (obviously) generalized to $\kappa_{\mathcal{F}} > \aleph_0$ (see Kelley [22]), so that the analogues of Tychonoff's theorem and Alexander's theorem necessitate some "deus ex machina", namely an "Alexander cardinal", together with extra conditions which we explicitate now (details can be found in Comfort & Negropontis [5]).

*Definition 5.5.2* : *An "Alexander cardinal" $\alpha$ (equivalently : "a cardinal $\alpha$ having Alexander's property") is an infinite cardinal such that any $\alpha$-topological space $X$ admitting an $\alpha$-subbase $S$ of cardinality $\leq \alpha$, such* *that every cover of $X$ by elements of $S$ has an $\alpha$-infinite subcover, is necessarily $\alpha$-compact.*

Theorem 8.23 in [5] states (inter alia) that an Alexander cardinal is exactly a strongly inaccessible, ramifiable cardinal. The notion of "ramifiable" has been recalled in comment 5.3.3. The notion of "strongly inaccessible" is the usual one, i.e. : $\alpha$ is "strongly inaccessible" iff $\alpha$ is weakly inaccessible (i.e. regular & limit cardinal) and realizes : $\forall \beta < \alpha \; 2^\beta < \alpha$.
Further, lemma 8.21 in [5] states that, if $(X_i)_{i \in I}$ is a family of $\alpha$-compact ($\alpha$-topological) spaces, such that $\forall i \in I \; X_i$ admits a base (for the topology on $X_i$) of cardinality $\leq \alpha$, $|I| \leq \alpha$, and $\alpha$ is an Alexander cardinal, then $\prod_{i \in I} X_i$ is $\alpha$-compact (for the $\alpha$-product topology).
In our situation, $I$ will be $\mathcal{F}$, $\alpha$ will be $\kappa$, and each $X_i$ will admit a base of cardinality $\leq \alpha$ (as $|A/\sim| < \kappa_{\mathcal{F}}$ for each $\sim \in \mathcal{F}$ and each $A/\sim$ has the trivial discrete topology).

So we deduce :

*Lemma 5.5.3* : *If $\mathcal{F}$ is a Malitz-family for $M$, $\kappa_{\mathcal{F}} (> \aleph_0)$ is an Alexander cardinal (i.e. strongly inaccessible & ramifiable) and $|\mathcal{F}| \leq \kappa_{\mathcal{F}} = \delta_{\mathcal{F}}$ then $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact.*

But this is not at all interesting, because in that case $\mathcal{F}$ itself, as a directed set, is ramifiable (see section 5.7), so that we get it as a very particular consequence of theorem 5.7.8 ! Actually theorem 5.5.1 (i.e. the case where $\kappa_{\mathcal{F}} = \aleph$) can also be seen as a consequence of theorem 5.7.8, because we know from remark 5.6.5 that $\mathcal{F}$ will be measurable and from [9] that this implies that $\mathcal{F}$ is ramifiable. Even more generally can one show that when $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$ is a strongly compact cardinal, $\mathcal{F}$ is necessarily ramifiable (see [9]). Conclusion : the Tychonoff-approach is only of some interest in the case $\kappa_{\mathcal{F}} = \aleph_0$, as there it provides a simple direct proof of theorem 5.5.1.

### *5.6 The second approach*

This approach uses adapted ultrafilters as "deus ex machina" producing the expected "miracle". Let us first recall that a cardinal $\kappa$ is "measurable" iff there exists a non-principal, $\kappa$-complete ultrafilter $\mathcal{U}$ over $\kappa$ (see Chang & Keisler [4] or Comfort & Negropontis [5]).
If $\kappa$ is a measurable cardinal, one can easily check that each $U \in \mathcal{U}$ (the appropriate ultrafilter) is cofinal in $\kappa$ (seen as a directed set). We generalize now the notion of "measurable" :

*Definition 5.6.1 :* *A directed set $D$ is "measurable" iff $D$ is a regular directed set (i.e. any basic chain in $D$ has order type $\delta_D$) and there exists a $\delta_D$-complete, non-principal ultrafilter $\mathcal{U}$ over $D$ such that each $U \in \mathcal{U}$ is cofinal in $D$.*

Obviously a measurable cardinal is also a measurable directed set.

*Theorem 5.6.2 :* *If $\mathcal{F}$ is a measurable Malitz-family for $M$ and realizes $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$, then $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact.*

*Proof.* Suppose $\mathcal{U}$ is a $\kappa_{\mathcal{F}}$-complete, non principal ultrafilter over $\mathcal{F}$, such that any $U \in \mathcal{U}$ is cofinal in $\mathcal{F}$. We will show that $M$ is $\mathcal{F}$-crowded, i.e. that each $\mathcal{F}$-net in $M$ admits a uniform Cauchy $\mathcal{F}$-subnet (and this implies that $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact).
So, fix any $x$, an $\mathcal{F}$-net in $M$.
Then, any $\sim \in \mathcal{F}$ induces an equivalence $\sim^\star$ on $\mathcal{F}$, defined by : $\sim_1 \sim^\star \sim_2$ iff $x_{\sim_1} \sim x_{\sim_2}$.
Obviously, $|\mathcal{F}/\sim^\star| < \kappa_{\mathcal{F}}$, so one of the classes of $\sim^\star$ in $\mathcal{F}$ has to be an element $U$ of $\mathcal{U}$, because any partition (of $\mathcal{F}$) of size $< \kappa_{\mathcal{F}}$ has a piece which is an element of $\mathcal{U}$ (see Lemma 4.2.3 in Chang & Keisler [4]).
Call this "$U$" : "$U_\sim$". By our assumptions on $\mathcal{U}$, this $U_\sim$ is cofinal in $\mathcal{F}$, so we can choose some element $\sim' \geq \sim$ such that $\sim' \in U_\sim$, and call this "$\sim'$" : "$\sigma(\sim)$".
So obviously : $U_\sim = \left\{\sim'' \in \mathcal{F} \;\middle|\; x_{\sim''} \sim x_{\sigma(\sim)}\right\}$.
We verify now that $\sigma$ is an extractor on $\mathcal{F}$, such that $x_\sigma \overset{\text{def}}{=} \left(x_{\sigma(\sim)}\right)_{\sim \in \mathcal{F}}$ is a uniform Cauchy $\mathcal{F}$-subnet of $x$ : suppose $\sim_1 \leq \sim_2$ and take $\sim'' \in U_{\sim_1} \cap U_{\sim_2}$ (note that $U_{\sim_1} \cap U_{\sim_2} \neq \emptyset$ because $U_{\sim_1} \cap U_{\sim_2} \in \mathcal{U}$); then $x_{\sigma(\sim_1)} \sim_1 x_{\sim''} \sim_2 x_{\sigma(\sim_2)}$ and so $x_{\sigma(\sim_1)} \sim_1 x_{\sigma(\sim_2)}$. □

*Remark 5.6.3 :* Suppose $\mathcal{U}$ is a $\delta_D$-complete, non principal ultrafilter on the regular directed set $D$.
Then : $\forall U \in \mathcal{U}\; U$ is cofinal in $D$ iff $\forall d \in D\; \{d' \in D \mid d \leq d'\} \in \mathcal{U}$.
Another equivalence can easily be proved, using the family

$$\mathcal{E} \overset{\text{def}}{=} \left\{B(X) \;\middle|\; X \subseteq D \,\&\, |X| < \delta_D\right\}$$

where

$$B(X) \overset{\text{def}}{=} \left\{y \in D \;\middle|\; \forall x \in X \quad y \geq x\right\}.$$

This family $\mathcal{E}$ is closed under intersections of strictly less than $\delta_D$ members (of $\mathcal{E}$), and : $\forall U \in \mathcal{U}\; U$ is cofinal in $D$ iff $\mathcal{E} \subseteq \mathcal{U}$.

So a "good" ultrafilter $\mathcal{U}$ should extend $\mathcal{E}$.

*Remark 5.6.4* : It is well-known that any non-principal ultrafilter $\mathcal{U}$ (over a set $X$) admits a cardinal $\theta_{\mathcal{U}}$ such that $\mathcal{U}$ is $\theta_{\mathcal{U}}$-complete but *not* $\theta_{\mathcal{U}}^{+}$-complete, and that this cardinal $\theta_{\mathcal{U}}$ *is* measurable (Chang & Keisler [4], proposition 4.2.7).
So, if $D$ is a measurable directed set and $\mathcal{U}$ a corresponding ultrafilter over $D$, we have obviously : $\delta_D \leq \theta_{\mathcal{U}}$.
Let us show that $\delta_D < \theta_{\mathcal{U}}$ is excluded : remember that any $\{y \in D \mid y \geq x\}$ has to be an element of $\mathcal{U}$; so, if $\delta_D < \theta_{\mathcal{U}}$, take $(x_\nu)_{\nu<\delta_D}$ a basic chain in $D$ and consider $Y = \bigcap_{\nu<\delta_D} \{y \in D \mid y \geq x_\nu\}$; as we used a basic chain, $Y$ should obviously be $\emptyset$, but on the other hand $Y$ should be an element of $\mathcal{U}$ !

We can conclude from this, that $\delta_D = \theta_{\mathcal{U}}$, and so that :

$D$ is a measurable directed set $\Rightarrow$ $\delta_D$ is a measurable cardinal.

*Remark 5.6.5* : In the case of $\delta_D = \aleph_0$ (and $D$ regular) however, we know by the ultrafilter theorem (proposition 4.1.3, Chang & Keisler [4]), that some ultrafilter $\mathcal{U}$ extends $\mathcal{E}$ (which has the finite intersection property) and this $\mathcal{U}$ witnesses that $D$ is measurable.

So we find again back theorem 5.5.1. And, as already mentioned at the end of section 5.5, more generally : any regular $D$ with $\delta_D$ strongly compact will be measurable (see [9]). The fact that "measurable" implies "ramifiable" (for directed sets) lets appear this second section approach as unnecessary, if only "compactification" is concerned. It is however still agreeable to have a simple direct proof of those cases where we know $\mathcal{F}$ to be measurable. Further did the concept itself of "measurable directed set" provide results that probably cannot be extended to "ramifiable directed sets" (see [20]).

### *5.7 The third approach*

This is the "direct" approach for $\mathcal{F}$-compactness and leads to a generalized notion of "tree-property" (also called "ramifiability") : we try to understand how to extract a uniform Cauchy $\mathcal{F}$-subnet from an arbitrary $\mathcal{F}$-net in $M$.
So, consider $\mathcal{F}$ a Malitz-family for $M$, realizing $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$ and take $x$ an $\mathcal{F}$-net in $M$. This $x$ induces, for each $\sim \in \mathcal{F}$, an equivalence $\sim^\star$ on $\mathcal{F}$, defined by :

$$\sim_1 \sim^\star \sim_2 \Leftrightarrow [(x_{\sim_1} \sim x_{\sim_2} \& \sim_1 \geq \sim \& \sim_2 \geq \sim) \vee \neg(\sim_1 \geq \sim \vee \sim_2 \geq \sim)]$$

(this is a refinement of the "$\sim^\star$" used in the proof of theorem 5.6.2).
Define $\mathcal{F}^\star = \{(\sim, Y) \mid Y \in \mathcal{F}/\sim^\star \ \& \ \sim \in \mathcal{F} \& Y \text{ is cofinal in } \mathcal{F}\}$. So any element $Y$ of $\mathcal{F}/\sim'$ will be of type $[\sim_1]_{\sim^\star} \stackrel{\text{def}}{=} \{\sim_2 \in \mathcal{F} \mid \sim_1 \sim^\star \sim_2\}$ and any element of $\mathcal{F}^\star$ is an ordered pair $(\sim, Y)$, with $Y \in \mathcal{F}/\sim^\star$.
Obviously : $|\mathcal{F}/\sim^\star| < \kappa_{\mathcal{F}}$. So, as $\mathcal{F}$ is $\kappa_{\mathcal{F}}$-directed, at least one of the equivalence classes of $\sim^\star$ in $\mathcal{F}$ has to be cofinal in $\mathcal{F}$, i.e. : $\forall \sim \in \mathcal{F} \quad \exists Y \subseteq \mathcal{F} \quad (\sim, Y) \in \mathcal{F}^\star$.
Put on $\mathcal{F}^\star$ the strict order relation "$<^\star$" defined by :

$$(\sim, Y) <^\star (\sim', Y') \quad \stackrel{\text{def}}{\Leftrightarrow} \quad (\sim < \sim' \ \& \, Y' \subseteq Y).$$

The corresponding order relation "$\leq^\star$" is naturally defined by :

$$(\sim, Y) \leq^\star (\sim', Y') \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \left[(\sim, Y) <^\star (\sim', Y') \vee (\sim, Y) = (\sim', Y')\right].$$

A straightforward investigation shows that the structure $(\mathcal{F}^\star, \leq^\star)$ has the following

*Basic properties 5.7.1*

(1) $\forall \sim \in \mathcal{F} \quad \exists Y \subseteq \mathcal{F} \quad (\sim, Y) \in \mathcal{F}^\star$ (already mentioned),

(2) $\forall (\sim, Y) \in \mathcal{F}^\star \quad \forall \sim' > \sim \quad \exists Y' \subseteq \mathcal{F} \quad (\sim, Y) <^\star (\sim', Y')$ & $(\sim', Y') \in \mathcal{F}^\star$
(due to the fact that any $\kappa_{\mathcal{F}}$-finite partition of a cofinal $Y$ in $\mathcal{F}$ admits a cofinal piece $Y' \subseteq \mathcal{F}$).

(3) $\forall (\sim', Y') \in \mathcal{F}^\star \quad \forall \sim < \sim' \quad \exists! Y \subseteq \mathcal{F} \quad (\sim, Y) <^\star (\sim', Y')$
(due to the fact that each equivalence class of $\sim'$ is contained in exactly one equivalence class of $\sim$, when $\sim < \sim'$).

(4) $\forall \sim_0 \in \mathcal{F} \quad \left|\{X \in \mathcal{F}^\star \mid \exists Y \subseteq \mathcal{F} \quad X = (\sim_0, Y)\}\right| < \kappa_{\mathcal{F}}$
(due to : $|\mathcal{F}/\sim_0^\star| < \kappa_{\mathcal{F}}$).

*Fact 5.7.2* : We can now easily get a uniform Cauchy $\mathcal{F}$-subnet of $x$, if we suppose the existence of a "faithful" embedding $h$ of $(\mathcal{F}, <)$ into $(\mathcal{F}^\star, <^\star)$, i.e. of an embedding $h$ such that : $\forall \sim \in \mathcal{F} \quad h_1(\sim) = \sim$ (where $h_1(\sim)$ is the first component of the ordered pair $h(\sim)$). Indeed : choose, for any $\sim \in \mathcal{F}$, some $\sim' \in h_2(\sim)$ (where $h_2(\sim)$ is the second component of $h(\sim)$) and call this $\sim'$ : "$\sigma(\sim)$". If $\sim_1 < \sim_2$ (in $\mathcal{F}$), then $h(\sim_1) <^\star h(\sim_2)$ and so $h_2(\sim_1) \supseteq h_2(\sim_2)$. As $\sigma(\sim_1) \in h_2(\sim_1)$ and $\sigma(\sim_2) \in h_2(\sim_2)$, this implies

that $\sigma(\sim_1) \sim_1^\star \sigma(\sim_2)$ and so $x_{\sigma(\sim_1)} \sim_1 x_{\sigma(\sim_2)}$.
This shows, that $x_\sigma \stackrel{\text{def}}{=} (x_{\sigma(\sim)})_{\sim\in\mathcal{F}}$ is a uniform Cauchy $\mathcal{F}$-net.

*Remark 5.7.3* : Actually any function $f : \mathcal{F} \to \mathcal{F}^\star$ realizing : $\forall \sim\in \mathcal{F} \quad f_1(\sim) \geq\sim$ and $\forall \sim,\sim'\in \mathcal{F} \quad (\sim\leq\sim'\Rightarrow f(\sim) \leq^\star f(\sim'))$ suffices to construct an adequate extractor $\sigma$. This, however, does not produce a stronger result (in Fact 5.7.2) because the existence of such an $f$ implies the existence of a "faithful" embedding $h$; to get $h$ from $f$, just define : $h(\sim) =$ the unique $(\sim, Y)$ realizing $(\sim, Y) \leq^\star f(\sim)$. The basic properties (5.7.1) of $(\mathcal{F}^\star, \leq^\star)$ guarantee that $h$ is well-defined : $\mathcal{F} \to \mathcal{F}^\star$, and is a faithful embedding : $(\mathcal{F}, \leq) \to (\mathcal{F}^\star, \leq^\star)$. This is why we prefer the "faithful embedding" version.

What precedes suggests the following abstract definitions (for a directed set $(D, \leq)$, without maximum element) :

*Definition 5.7.4 : A "D-arborescence" is an ordered set $(D^\star, \leq^\star)$, together with a surjective homomorphism $s : (D^\star, \leq^\star) \to (D, \leq)$, such that the three following conditions are verified :*

(i) $\forall d \in D \qquad \left|s^{-1}(d)\right| < \delta_D$

(ii) $\forall u \in D^\star \quad \forall d \in D \quad (d \geq s(u) \Rightarrow \exists v \in s^{-1}(d) \quad v \geq^\star u)$

(iii) $\forall u \in D^\star \quad \forall d \in D \quad (d \leq s(u) \Rightarrow \exists! v \in s^{-1}(d) \quad v \leq^\star u)$.

*Remark 5.7.5* : One can easily check that $(\mathcal{F}^\star, \leq^\star)$ is an $\mathcal{F}$-arborescence, where $s : \mathcal{F}^\star \to \mathcal{F} : (\sim, Y) \mapsto\sim$.
Condition (i) corresponds to (4) (in Basic properties 5.7.1), (ii) to (2) and (iii) to (3). The very important condition (iii) will be called the "downwards uniqueness property".

*Definition 5.7.6 : D has the (generalized) "tree-property" (equivalently : "D is ramifiable") iff D is regular and can be "faithfully" embedded in each D-arborescence, i.e. : $\forall D^\star$ (with $\leq^\star, s$) D-arborescence, there exists an embedding $h : (D, \leq) \to (D^\star, \leq^\star)$, realizing $\forall d \in D \quad h(d) \in s^{-1}(d)$.*

*Remark 5.7.7* : In fact 5.7.2, the surjection $s : \mathcal{F}^\star \to \mathcal{F}$ is the one mapping $(\sim, Y)$ onto $\sim$, so that $h(\sim) \in s^{-1}(\sim)$ exactly expresses that $h_1(\sim) =\sim$.

Our final conclusion is summarized by

*Theorem 5.7.8* : *If $\mathcal{F}$ is a ramifiable Malitz-family for $M$, realizing $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$, then $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact.*

*Proof.* Just combine Fact 5.7.2, Definitions 5.7.4, Remark 5.7.5 and Definition 5.7.6. □

*Remark 5.7.9* : The (generalized) tree-property (or ramifiability) is not a necessary condition for $\mathcal{F}$-compactness, i.e. $\mathcal{F}$ is a Malitz-family for $M$ & $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact does not imply that $\mathcal{F}$ is ramifiable.

*Counter-example :* $M = \aleph_1 \cup \{\aleph_1\}$ with $\mathcal{F} = \{\sim_\alpha \big| \ \alpha < \aleph_1\}$, where $x \sim_\alpha y \ \ \overset{\text{def}}{\Leftrightarrow} (x < \alpha \& \, y < \alpha \& \, x = y) \vee (x \geq \alpha \& \, y \geq \alpha)$.
One can easily check that $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact. But $\mathcal{F}$ is not ramifiable, because $(\mathcal{F}, \leq_{\mathcal{F}})$ is isomorphic to $(\aleph_1, \leq)$ and $\aleph_1$ (as a cardinal) is not ramifiable (for the correspondence between "ramifiable" for directed sets and "ramifiable" for cardinals see Proposition 6.1 in next section).

*Remark 5.7.10* : Call "$D$-tree" an ordered set $(D^\star, \leq^\star)$ satisfying only conditions (i) and (iii), in Definition 5.7.4; the concept of "ramifiable $D$" (Definition 5.7.6) is still the same if one replaces "$D$-arboresence" by "$D$-tree" (see [9]). For cardinals, such "arboresences" are called usually "well-pruned trees".

*Remark 5.7.11* : There is an analogue of theorem 5.7.8 for general uniformities, due to A. Rigo [24].

## 6. *About the tree-property*

*Proposition 6.1*: *If $\kappa$ is an infinite cardinal, then :*

*$\kappa$ (as a directed set) is ramifiable $\Leftrightarrow$ $\kappa$ (as a cardinal) is ramifiable (equivalently : "has the tree-property").*

*Proof.*

(i) Suppose $\kappa$ (as a directed set) is ramifiable. Then $\kappa$ is a regular cardinal (remember that a directed set $D$ is regular iff each basic chain is of order type $\delta_D$). Take a tree $T$ of order $\kappa$, with levels of size $< \kappa$. Consider the sub-tree $T^\star$ obtained by selecting the elements $x$ of $T$ realizing that the sub-tree $\{y \in T \ \big| \ y \geq x\}$ is still of order $\kappa$. Then $T^\star$ will be a $D$-arborescence (definition 5.7.4), for $D = \kappa$. This is easy to verify, but uses the fact that $\kappa$ is regular (as a cardinal).

As $\kappa$ is ramifiable (as a directed set), there exists a "faithful" embedding $h$ :
$(\kappa, \leq) \to (T^\star, \leq)$, which image is a branch of order $\kappa$, in $T$.

(ii) Suppose $\kappa$ (as a cardinal) is ramifiable. It is well-known that this implies that $\kappa$ is a regular cardinal, so that $\kappa$ is a regular directed set also.
Suppose $D^\star$ is a $\kappa$-arborescence, i.e. a $D$-arborescence for $D = \kappa$. Just add to $D^\star$ an artificial least element : the new structure will exactly be a tree of order $\kappa$, with $\kappa$-finite levels. As $\kappa$ is a ramifiable cardinal, this tree has a branch of order $\kappa$, which induces obviously a "faithful" embedding : $\kappa \to D^\star$.

$\square$

*Remark 6.2*: The definition of "$D$ is a ramifiable directed set" (5.7.6) mentions explicitly that $D$ has to be regular. If we call "weakly ramifiable" a directed set $D$, which is embeddable in each $D$-arborescence (but is not a priori supposed to be regular), we get another formulation of *Proposition 6.1 : If $\kappa$ is an infinite cardinal, then $\kappa$ is regular and weakly ramifiable (as a directed set) iff $\kappa$ is ramifiable (as a cardinal). Note that $\aleph_\omega$ (for example) is weakly ramifiable (as a directed set) but not ramifiable (as a cardinal).*

*Proposition 6.3*: *If $D$ is a regular, infinite directed set admitting some cofinal ramifiable subset, then $D$ itself is a ramifiable directed set.*

*Proof.* Suppose $Y \subseteq D$ is cofinal in $D$ and ramifiable (when $Y$ is seen as directed by the restriction of $\leq$ (on $D$) to $Y$). So $Y$ is the image of some extractor $\sigma$ on $D$. Note that $\delta_Y = \delta_D$ because $D$ is regular. Take $D^\star$ some $D$-arborescence and let $Y^\star$ be the obvious part of $D^\star$ corresponding to $Y$. Then $Y^\star$ is a $Y$-arborescence, so there exists a faithful embedding $h \ : Y \to Y^\star$, which we extend to $D$ by the rule : $\overline{h}(z) \ \stackrel{\text{def}}{=}$ the unique (by the "downwards uniqueness property") element $v \in s^{-1}(z)$ realizing $v \leq^\star h(\sigma(z))$.
Naturally $s$ is the surjection appearing in the definition of "$D$-arborescence" 5.7.4.
In order to convince the reader that $\overline{h}$ is a faithful embedding : $D \to D^\star$, we check now the only non-trivial point, namely :

$$\forall z, z' \in D \quad (z \leq z' \Rightarrow \overline{h}(z) \leq^\star \overline{h}(z')).$$

Take $z \leq z'$, in $D$, and choose some $t \in Y$ realizing $\sigma(z) \leq t$ & $\sigma(z') \leq t$. The "downwards uniqueness property" (remark 5.7.5) guarantees that there exists exactly one $b \in s^{-1}(z)$ such that $b \leq^\star \overline{h}(z')$. But then $b$ realizes also

$b \leq^\star h(t)$, because $\sigma(z') \leq t$ and so $\overline{h}(z') \stackrel{\text{def}}{=} h(\sigma(z')) \leq^\star h(t)$. As $\overline{h}(z)$ also realizes $\overline{h}(z) \in s^{-1}(z)$ & $\overline{h}(z) \leq^\star h(t)$, the "downwards uniqueness" implies that $\overline{h}(z) = b$. So we conclude : $\overline{h}(z) \leq^\star \overline{h}(z')$. □

*Proposition 6.4*: *If $D$ is an infinite, ramifiable directed set, then $\delta_D$ is a ramifiable cardinal.*

*Proof.* Suppose $D$ is infinite and ramifiable. Take $(c_\alpha)_{\alpha<\delta_D}$ a basic chain in $D$.
We will show that $C = \{c_\alpha | \alpha < \delta_D\}$ is ramifiable (as a directed set), and so $\delta_D$ is a ramifiable cardinal (by lemma 6.1). So, let $C^\star$ be a $C$-arborescence and $s$ be the corresponding adequate surjection : $C^\star \to C$. Note that $\delta_D = \delta_C$.
We construct a $D$-arborescence $D^\star$, extending $C^\star$ : define, for each $d \in D$, $\alpha_D \stackrel{\text{def}}{=}$ the least ordinal $\alpha < \delta_{\mathcal{F}}$ such that $\neg c_\alpha \leq d$. Note that $\alpha_d = 0$ if no $c_\alpha \leq d$; otherwise, $\alpha_d$ is the *strict* supremum (i.e. the least *strict* upper bound) of $\{\alpha < \delta_{\mathcal{F}} \mid c_\alpha \leq d\}$. Obviously : $d \leq d' \Rightarrow \alpha_d \leq \alpha_{d'}$.
We define the "level $d$" by : $L_d \stackrel{\text{def}}{=} s^{-1}(c_{(\alpha_d)}) \times \{d\}$; $L_d$ is just a copy of $s^{-1}(c_{(\alpha_d)})$. Further we define : $D^\star \stackrel{\text{def}}{=} \bigcup_{d\in D} L_d$, and $\leq^\star$ on $D$ : $(x, d) \leq^\star (y, d') \stackrel{\text{def}}{\Leftrightarrow} d \leq d'$ & $x \leq^\star y$ (where $\leq$ is the order on $D$, while $\leq^\star$, in "$x \leq^\star y$", is the order on $C^\star$).
One extends $s$ to $D^\star$ by :

$$\overline{s} \ : D^\star \to D \ : (x, d) \mapsto d \quad .$$

One can easily check that $D^\star, \leq^\star, \overline{s}$ is a $D$-arborescence. So there exists a faithful embedding $h \ : D \to D^\star$, and the restriction of $h$ to $C$ is the desired faithful embedding : $C \to C^\star$.
□

*6.5* Ramifiability for directed sets has been further investigated (see [9], [10], [11], [13], [20], [21], [24]) or used (see [7], [13], [24]); and the concept has even been extended to (general) partial orders (see [10], [11], [12]).

## 7. *Some examples*

### *7.1 In set theory*

The "archetypical" example already mentioned in Section 4, namely the completion of $(R_\omega, \in)$, is a particular case of more general structures $N_\kappa$

investigated by Forti & Honsell in [16].
These $N_\kappa$ are Cauchy-complete and $\kappa$-compact structures. The involved Malitz-family is $\mathcal{F} = \{\sim_\alpha \big| \ \alpha < \kappa\}$, where $\sim_0 = N_\kappa \times N_\kappa$, $\sim_{\alpha+1} = (\sim_\alpha)^+$ (the "+" is defined in Section 4), $\sim_\gamma = \bigcap_{\alpha<\gamma} \sim_\alpha$. The condition on $\kappa$ is the one we presented here in Section 5, Theorem 5.5.5, namely : *"$\kappa$ is an Alexander cardinal (equivalently : $\kappa$ is strongly inaccessible and ramifiable)"*.
In this case $\mathcal{F}$ will be a ramifiable directed set realizing $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}} = \kappa$.

*7.2* In non-classical logics, namely in "partial logic" and in "paradoxical logic", adequate Malitz-completions provide models for naive set theory with extensionality. In Hinnion [19], the structure $X_\omega$ (in section 5.2), for example, can be seen to be $\mathcal{F}$-complete and $\mathcal{F}$-compact for the Malitz-family $\mathcal{F} = \{\sim_n \big| \ n \in \mathbb{N}\}$, where : $a \sim_n b$ iff $a_n = b_n$ (for $a, b \in X_\omega$).

As in 7.1, the compactness plays an essential role here.

*7.3* Each infinite structure $M$, which functions have exactly *one* variable, admits Malitz-families $\mathcal{F}$.
For example : $\mathcal{F} = \{\sim \mid \sim$ in an equivalence on $A$ & $|A/\sim| < \aleph_0\}$, where $A$ is the universe of $M$.
The obvious "uniformizer" for a function $F_M$ (with 1 variable) is $\sigma$ defined by :

$$\sigma(\sim) = \sim_{F_M} \cap \sim \quad ,$$

where

$$x \sim_{F_M} y \quad \overset{\text{def}}{\Leftrightarrow} \quad F_M(x) \sim F_M(y) \quad .$$

One can give more refined examples by selecting "definable" equivalences (and naturally several notions of "definable" can be used). Families of more or less definable equivalences (which however are not necessarily exactly Malitz-families) have been used in Hinnion [18] and can suggest possible paths for the creation of Malitz-families.

Notice that some structures have no Malitz-family at all (see [20]).

*7.4 In lambda-calculus*

The well-known "topological" models for lambda-calculus (see for example Scott [25]) are examples of $\mathcal{F}$-complete structures (for an adequate Malitz-family $\mathcal{F}$).
These models are obtained like this (see [25] for details) : $D_0$ is a continuous lattice, $D_{n+1}$ is the set of the continuous functions from $D_n$ to $D_n$. One chooses a continuous injection $i \ : D_0 \to D_1$ and a continuous surjection

$s : D_1 \to D_0$, such that $s \circ i$ is the identity function on $D_0$. These $s$ and $i$ can be extended to the other "levels", as functions : $s : D_{n+1} \to D_n$ & $i : D_n \to D_{n+1}$. $D_\infty$ is the projective limit of the $D_n$ and is a model for extensional lambda-calculus.
The system of injections permits to see the sequence $D_0, D_1, D_2, \dots$ as a chain $D_0 \subset D_1 \subset D_2 \dots$
Take $A =$ the "union" of this chain and let $\rho(a)$ be the "rank" of $a \in A$, i.e. the least $n < \aleph_0$, such that $a \in D_n$.
Define $s^m = s \circ s \circ \cdots \circ s$ ($m$ times).
Put on $A$ the "product" defined by :

$$a \cdot b \stackrel{\text{def}}{=} \begin{cases} \text{the minimum element of } D_0, \text{ if } \rho(a) = 0; \\ a(s^{k+1}(b)), \text{if } \rho(a) = n+1 \text{ \& } \rho(b) = n+1+k. \\ (s^k(a))(b), \text{if } \rho(b) = n \text{ \& } \rho(a) = n+1+k. \end{cases}$$

One can check that $\mathcal{F} = \{\sim_n \big| \ n < \aleph_0\}$ is a Malitz-family for $M = (A, \cdot)$, where

$$a \sim_n b \stackrel{\text{def}}{\Leftrightarrow} [(\rho(a) < n\& \ \rho(b) < n\& \ a = b) \\ \vee (s^{\rho(a)-n}(a) = s^{\rho(b)-n}(b)\& \ \rho(a) \geq n\& \ \rho(b) \geq n)].$$

$D_\infty$ is exactly the Malitz-completion $\overline{M}_{\mathcal{F}}$. This shows that compactness is not always necessary to get interesting completions.
Notice that Scott [25] himself mentions (p. 98) that "$D_\infty$ is the completion of the union $A$" but that $D_\infty$ is more easily described as the inverse (i.e. projective) limit of the $D_n$".

*7.5 In algebra* We restrict our attention here to groups, but many algebraic properties are first-order, more or less "positive" formulas and so will be "preserved" under adequate completions.

*7.5.1* If we see a group $G$ as a first-order structure of type $M = (G, \cdot, {}^{-1}, 1)$, $M$ has to satisfy the axioms :

(1) $\forall a, b, c \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$
(2) $\forall x \quad x \cdot 1 = x = 1 \cdot x$
(3) $\forall x \quad x \cdot x^{-1} = 1 = x^{-1} \cdot x$

The preservation rules studied in Section 4, and more precisely Fact 4.7 and Fact 4.12, guarantee that these axioms are INV, so that any Malitz-completion $\overline{M}_{\mathcal{F}}$ will again be a group.

However, if we see a group $G$ as a first-order structure of type $M = (G, \cdot, 1)$, then $M$ has to satisfy the axioms : (1) and (2) (as in 7.5.1), and

$$(3') \qquad \forall x \quad \exists y \qquad x \cdot y = 1 = y \cdot x$$

Axioms (1) and (2) will again be INV for any Malitz-family $\mathcal{F}$, but, in order to force (3') to be INV, it seems that we have to suppose that $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact, because we need Fact 4.13 which involves $\mathcal{F}$-compactness.
So a priori $\overline{M}_{\mathcal{F}}$ will be a group *if* $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact.

M. Boffa however found the following (unpublished) improvement : if $\mathcal{F}$ is a Malitz-family for a group $(G, \cdot, 1)$, then $\mathcal{F}$ is also a Malitz-family for the group $(G, \cdot, ^{-1}, 1)$. So that the hypothesis of compactness is not necessary.

*Proof.* Consider a uniformizer $\sigma$ for $\cdot$; it suffices to show that there exists a uniformizer for $^{-1}$. Actually $\sigma^2$ (i.e. $\sigma \circ \sigma$) is adequate, because (successively) :
if $x\ \sigma^2(\sim)\ y$, then :

- $x^{-1}.x\ \sigma(\sim)\ x^{-1}.y$
- $1\ \sigma(\sim)\ x^{-1}.y$
- $1.y^{-1}\ \sim\ x^{-1}.y.y^{-1}$
- $y^{-1}\ \sim\ x^{-1}$

□

*7.5.2* The "profinite completion" $\hat{G}$ of an arbitrary group $G$ is defined as the projective limit (for the obvious projections) of the $G/N$ where $N$ is a normal subgroup of $G$ and $|G/N| < \aleph_0$ (Douady [6]). The topology on $\hat{G}$ is the one induced by the product topology, when the topology on each $G/N$ is the discrete one. This $\hat{G}$ is exactly the Malitz-completion $\overline{M}_{\mathcal{F}}$, for $M = (G, \cdot, 1)$ and $\mathcal{F} = \{\sim_N \mid N$ is a normal subgroup of $G$ and $G/N$ is finite$\}$, where $x \sim_N y \overset{\text{def}}{\Leftrightarrow} x \cdot N = y \cdot N$, at least when $\cap\, \mathcal{F}$ is the equality on $G$. Here $\kappa_{\mathcal{F}} = \aleph_0$, $\overline{M}_{\mathcal{F}}$ is $\mathcal{F}$-compact and the elements of $\mathcal{F}$ are compatible equivalences (so that each $G/\sim_N$ is also a group).

*7.5.3* If $\mathcal{F}$ is a Malitz-family for the group $G$ seen as $M = (G, \cdot, ^{-1}, 1)$, and $\kappa_{\mathcal{F}} = \aleph_0$, then $\overline{M}_{\mathcal{F}}$ is a topological group which universe is a "profinite" topological space (i.e. a projective limit of finite discrete topological spaces), and so $\overline{M}_{\mathcal{F}}$ is necessarily a "profinite group" (i.e. a projective limit of discrete finite groups, or, equivalently, a group $G$ coinciding with its own profinite completion $\hat{G}$; see Douady [6] for the different equivalent presentations of the notion of "profinite group").

8. *About compatible equivalences*

*Definition 8.1*: *Let us call a Malitz-family $\mathcal{F}$ (for a structure $M$) "soft" if there exists a Malitz-family $\mathcal{F}^\star$ (for $M$) which elements are compatible equivalences (see Section 2, before definition 2.13), such that $\overline{M}_{\mathcal{F}}$ and $\overline{M}_{\mathcal{F}^\star}$ are isomorphic (as first-order structures). A Malitz-family which is not "soft" will be called "hard".*

In remark 3.1, we showed that there do exist hard Malitz families. We investigate this problem more in detail now.
So, let $\mathcal{E}_M$ be the number of functional symbols in the language of $M$ (remember that $\mathcal{E}_M$ might be an infinite cardinal).

*Theorem 8.2*: *Any Malitz-family $\mathcal{F}$ (for $M$), satisfying $\delta_{\mathcal{F}} > \aleph_0$ & $\delta_{\mathcal{F}} > \mathcal{E}_M$, is soft.*

*Proof.* Suppose that $(\mathcal{T}^{(\alpha)})_{\alpha<\mathcal{E}}$ enumerates uniformizers for the functions of $M$ (where $\mathcal{E}$ is $\mathcal{E}_M$).
For each $\sim \in \mathcal{F}$, construct :

$$\begin{cases} \sim_0 = \sim \\ \sim_{n+1} = \text{some upper bound for } \left\{\mathcal{T}^{(\alpha)}(\sim_n) \;\middle|\; \alpha < \mathcal{E}\right\} \\ \sim^\star = \cap \left\{\sim_n \middle| \; n < \aleph_0\right\} \end{cases}$$

Consider $\mathcal{F}^\star = \left\{\sim^\star \middle| \sim \in \mathcal{F}\right\}$.
One can easily check that $\mathcal{F} \cup \mathcal{F}^\star$ is also a Malitz-family for $M$ and that both $\mathcal{F}$ and $\mathcal{F}^\star$ are cofinal in $\mathcal{F} \cup \mathcal{F}^\star$.
Further, the elements of $\mathcal{F}^\star$ are compatible equivalences. So, by basic fact 2.17, we get :

$$\overline{M}_{\mathcal{F}} \cong \overline{M}_{\mathcal{F}\cup\mathcal{F}^\star} \cong \overline{M}^\star_{\mathcal{F}} \quad .$$

So $\mathcal{F}$ is soft.
The example in remark 3.1 shows that there exist hard Malitz-families realizing $\delta_{\mathcal{F}} = \aleph_0$ (even with $\mathcal{E}_M = 1$), so that "$(\delta_{\mathcal{F}} \geq \aleph_0$ & $\delta_{\mathcal{F}} > \mathcal{E}_M) \Rightarrow \mathcal{F}$ is soft" is false. The next theorem shows that "$(\delta_{\mathcal{F}} > \aleph_0$ & $\delta_{\mathcal{F}} \geq \mathcal{E}_M) \Rightarrow \mathcal{F}$ is soft" is also false. □

*Theorem 8.3*: *For any regular cardinal $\delta$, there exists a hard Malitz-family $\mathcal{F}$ (for some structure $M$), satisfying $\delta = \delta_{\mathcal{F}} = \mathcal{E}_M$.*

*Proof.* The universe of $M$ will simply be $\delta$ and $M$ will admit $\delta$ functions, denoted : $f^{(\alpha,\beta)}$ (for $0 < \beta < \alpha < \delta$). The action of $f^{(\alpha,\beta)}$ is defined by :

$$f^{(\alpha,\beta)}(x) = \begin{cases} \beta & \text{if} \quad x = \alpha \\ x & \text{if} \quad x \neq \alpha \end{cases}$$

Further $\mathcal{F}$ will be the set of the "final" equivalences on $\delta$, i.e. :

$$\mathcal{F} = \left\{ \sim_\alpha \middle| \, \alpha < \delta \right\},$$

where $\sim_\alpha$ is defined by :

$$x \sim_\alpha y \Leftrightarrow [(x = y < \alpha) \vee (x \geq \alpha \,\&\, y \geq \alpha)] .$$

One can easily check that $\mathcal{F}$ is a Malitz-family for $M$, satisfying $\delta = \delta_{\mathcal{F}} = \mathcal{E}_M$. A routine investigation shows that the compatible equivalences on $M$ are exactly the "initial" equivalences $\simeq_\alpha$ (for $0 < \alpha < \delta$) defined by :

$$x \simeq_\alpha y \Leftrightarrow [(x < \alpha \,\&\, y < \alpha) \vee (x = y \geq \alpha)] .$$

Any set $\mathcal{F}^\star$ of compatible equivalences will have a maximum element, namely $\cap\mathcal{F}^\star$, so that no $\mathcal{F}^\star$ can be a Malitz-family. So $\mathcal{F}$ is indeed hard. □

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