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DETERMINISTIC AND NONDETERMINISTIC STRATEGIES FOR HINTIKKA GAMES IN FIRST-ORDER AND BRANCHING-QUANTIFIER LOGIC

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Abstract
Applications of game-theoretic semantics *à la Hintikka* can be extended from Lower Predicate Calculus to languages with branching quantifiers. When one does this, issues which in the LPC could be swept under the carpet suddenly cause unwelcome subtleties. It turns out that which formulæ of the branching quantifier logic one accounts true comes to depend on whether one requires that the winning strategies for Team ∃loïse in the Hintikka game be deterministic (or allows them to be nondeterministic). The set of valid formulæ is affected similarly.

Game-theoretic semantics for a logic $\mathcal{L}$ characterises the truth of formulæ of $\mathcal{L}$ in a model $\mathfrak{M}$ in terms of the existence of a winning strategy — in the usual ("Hintikka") game $\mathcal{G}(\mathfrak{M}, \phi)$ — for a player variously known as ∃loïse, `True` and *Eve*. I shall be calling her '∃loïse' throughout, because this constant harping on the existential quantifier will remind us of our concern here: not much attention has hitherto been paid to the question of whether these strategies are taken to be deterministic or nondeterministic and the existential case is where any difference is most likely to reveal itself. A deterministic strategy is one that says "When in situation x, do y"; a *non*deterministic strategy says "When in situation x, do one of the Y: it won't matter which".

Greek letters are dummies for complex expressions: Roman letters are predicate letters. Structures are denoted by letters in fraktur font, and their carrier sets by the corresponding upper-case Roman letter.

Since one of my purposes here is the elucidation of the rôle played by the axiom of choice in these games, I shall naturally not be assuming it.

It turns out that in Predicate Calculus the distinction between deterministic and nondeterministic strategies for player ∃loïse might affect our concept of truth-in-a-model but not our concept of satisfiability or validity. In the case of branching quantifier logic, the notion of validity *is* affected by our choice between deterministic and non-deterministic strategies for ∃loïse.

We have to rule either that

(1) ϕ is true in $\mathfrak{M}$ iff ∃loïse has a deterministic strategy to win $\mathcal{G}(\mathfrak{M}, \phi)$; or that

(2) ϕ is true in $\mathfrak{M}$ iff ∃loïse has a nondeterministic strategy to win $\mathcal{G}(\mathfrak{M}, \phi)$;

If we adopt (1), so that we require ∃loïse to have a deterministic winning strategy before we admit that — to take a germane example —

$$\forall x \exists y \phi(x, y) \qquad (A)$$

is true in a structure $\mathfrak{M} = \langle M, R\rangle$ (where R is the interpretataion of ϕ) then A might come out false under this interpretation if $\mathfrak{M}$ is a counterexample to DC, the axiom of dependent choices. That is to say, if for every m in M there is m' in M such that $\langle m, m'\rangle \in R$ but there is no ω-sequence $\langle m_i : i \in \mathbb{N}\rangle$ of elements of M with $\langle m_i, m_{i+1}\rangle \in R$ for all i. In these circumstances we would account (A) true if ∃loïse's strategies are allowed to be nondeterministic but false if they are required to be deterministic.

However, a choice of deterministic *versus* nondeterministic for ∃loïse's strategies will not affect our verdicts on whether or not a formula is *valid*. *Au fond* this is no more than the fact that Skolemisation preserves satisfiability.

It is clear that we can use the axiom of choice to show that if a formula of first-order logic is satisfiable then so is its skolemised version. However the preservation of satisfiability by skolemisation does not depend on the axiom of choice. Since there are inconvenient facts awaiting us later in this discussion which in contrast do rely on the axiom of choice, it is important to establish this early on.

The fact that Skolemisation preserves satisfiability is well known to the *cognoscenti*; and the rest of us can consult — for example — the entry by Avigad and Zach on the epsilon calculus in the Stanford online Encyclopædia of Philosophy. As a gesture in the direction of making this note self-contained, a sketch follows.

We will illustrate with a simple two-quantifier case. Our proof system will be sequent calculus; we will outline a proof of the contrapositive: if $\forall x\phi(x, f(x))$ is not satisfiable, then neither is $\forall x \exists y \phi(x, y)$.

Suppose we have a proof of

$$\vdash \exists x \neg\phi(x, f(x)) \qquad (1)$$

There might be more than one application of $\exists$-R with some contraction on the right but we will have got this from something like

$$\vdash \neg\phi(x_1, f(x_1)), \quad \neg\phi(x_2, f(x_2)), \quad \neg\phi(x_n, f(x_n))\ldots \tag{2}$$

where the various x_i are not necessarily variables but might be complex terms. Now any sequent proof of (2) can be transformed into a proof of

$$\vdash \neg\phi(x_1, z_1), \quad \neg\phi(x_2, z_2), \quad \neg\phi(x_n, z_n), \ldots \tag{3}$$

simply by replacing '$f(x_1)$', '$f(x_2)$', '$f(x_n)$' etc throughout by fresh variables z_i. (Since we know nothing about f it must destroy all information about its argument.) We can then do some $\forall$-R on the z_i to obtain

$$\vdash \forall y\neg\phi(x_1, y), \quad \forall y\neg\phi(x_2, y), \quad \forall y\neg\phi(x_n, y), \ldots \tag{4}$$

and further $\exists$-R and contraction-on-the-right to get

$$\vdash \exists x\forall y\neg\phi(x, y) \tag{5}$$

So if the skolemised version of the formula was refutable then the original formula was refutable, which is what we wanted.

The Branching-Quantifier Case

Let us now consider a branching-quantifier formula, such as the following:

$$\begin{pmatrix} \forall x\exists y \\ \forall x'\exists y' \end{pmatrix} \big((x = x' \to y = y') \wedge R(x, y) \wedge R(x', y')\big) \tag{6}$$

The intended meaning of this formula, in English is: for all x there is a y (depending only on x) and for all x' there is a y' (depending only on x') such that ...

The game-theoretic semantics for first-order logic extends smoothly to this new syntax. The difference now is that there is not a single $\exists$loïse as before, but one for each path through the prefix: a **team** of $\exists$loïses. The team are allowed a team-talk before the game to agree on strategies, but they may not communicate during the play of the game. This is to ensure that (to take the case above) "y depends only on x".

It makes no difference to the truth of this sentence in any given model whether or we demand that the team of $\exists$loïses have a pair of deterministic winning strategies or allow them to have nondeterministic winning strategies. If the two ∃loïses are to win, they have to pick the same witnesses. Since they are not allowed to confer the only way they can be sure of doing it is to have the same deterministic strategy. The extra leeway they apparently have when we allow them to play nondeterministically is of no use to them.

Consider now the conditional

$$\forall x \exists y R(x, y) \to$$
$$\begin{pmatrix} \forall x \exists y \\ \forall x' \exists y' \end{pmatrix} ((x = x' \to y = y') \wedge R(x, y) \wedge R(x', y')) \tag{7}$$

For the ∃loïses to have a strategy to win the conditional they had better have a strategy to win the consequent whenever they have a strategy to win the antecedent. There are two cases to consider.

If ∃loïse's strategies have to be deterministic, then they can have a winning strategy for the antecedent only if there is a choice function. If there is a choice function, then the consequent is true. Whenever they have a winning strategy for the antecedent, they also have a winning strategy for the consequent. On this reading formula (7) is valid *simpliciter*.

On the other hand, if the ∃loïse's strategies do not have to be deterministic then the situation is more complicated. In any model $\mathfrak{M}$ where there is a choice function for the interpretation of 'R' then the ∃loïses will have a winning strategy for the consequent and therefore (7) will be true in $\mathfrak{M}$. However if $\mathfrak{M}$ is a model in which the antecedent is true but there is no choice function for the interpretation of 'R' then the ∃loïses have no winning strategy for the consequent and (7) will come out false in $\mathfrak{M}$.

This means that — just as in the LPC case — a decision on the rule-makers' part as to whether the ∃loïse's strategies have to be deterministic or might be nondeterministic will affect the truth of formulæ in individual models. However — in contrast to the LPC case — it also means that the rule-makers' decision now affects which formulæ are *valid*. In particular the decision of whether go for deterministic or nondeterministic strategies for ∃loïse will determine whether formula (7) is valid *simpliciter* or valid if and only if the axiom of choice is true.

Coda

"Haven't we been here before?" people will say: "Isn't it known that we can cook up a formula of second-order logic that is valid iff the axiom of choice is true? Yes it is: the significance of this present note lies in the fact that although the choice between deterministic and nondeterministic strategies for ∃loïse does not affect the semantics in the first order case, it does in the branching quantifier case. As far as I am aware, nobody has noticed this before. Perhaps advocates of branching quantifier logics and their descendents will tell us which semantics they have in mind.

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