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WHO IS WITTGENSTEIN'S WORST ENEMY?: STEINER ON WITTGENSTEIN ON GÖDEL

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In his recent "Wittgenstein as his Own Worst Enemy: The Case of Gödel's Theorem,"[1] Mark Steiner argues that Wittgenstein's (*RFM* App. III) remarks on Gödel's Theorem are 'indefensible,' but that this is not really problematic for Wittgenstein since his "remarks on Gödel's theorem are an aberration" (Steiner, 2001, 258). Wittgenstein is "his own worst enemy," according to Steiner, for given his view that "that mathematics is... a 'family resemblance' concept," Wittgenstein "should have said" that "Gödel's theorem had made it impossible to identify mathematical truth with provability in any one system, which should have encouraged the conclusion that mathematical truth is multicolored" (Steiner, 2001, 261, 273). Given the organic relation between "'true' and 'Tarski true'," argues Steiner, and the fact Wittgenstein 'shows' that the 'concepts' "number, proof, truth [and possibly mathematical truth]... are... 'family resemblance' concepts" (Steiner, 2001, 260), Wittgenstein 'should' or will "want[] to extend the 'Russell notion' of truth... to cover the undecidable sentence P," which compels him to "adopt Tarski truth as the extension of 'true' in light of Gödel's theorem."

Steiner's argument is an enhanced compatibility argument. He aims to show that Wittgenstein would be truer to his central tenets and far better off if, in the light of Gödel's Theorem, he adopted Tarski truth for mathematics in place of the conception of mathematical truth he articulates (i.e., "true in calculus Γ" means "proved/provable in calculus Γ"[2]). If Steiner is right, his argument has implications that far outreach Wittgenstein's Philosophy of Mathematics, for if Steiner is right, his conclusion applies with equal force to any variant of Finitism, or Formalism, or Constructivism that similarly

[1] Mark Steiner, "Wittgenstein as his Own Worst Enemy: The Case of Gödel's Theorem," *Philosophia Mathematica* (3) Vol. 9 (2001), pp. 257–279.

[2] See my (1999a), p. 177: "There is, admittedly, an apparent vacillation as regards 'proved' and 'provable' in Wittgenstein's remarks on GIT ["Gödel's First Incompleteness Theorem"], but, as we shall see, this 'vacillation' reflects at times Wittgenstein's own view (e.g., §6 and §8), and at times Wittgenstein considering GIT *on its own terms* (e.g., §7 and §17)."

maintains an identification of "true in calculus Γ" and "proved/provable in calculus Γ."

My aim in this paper is to show that Steiner's argument fails because he fails to show that Wittgenstein (and anyone holding similar views about mathematical truth and provability) is *compelled* to extend his conception of mathematical truth. Steiner, as I shall show, is right about quite a lot,[3] but he is mistaken in claiming that Wittgenstein is his own worst enemy and, in particular, he is mistaken in claiming (or intimating) that Wittgenstein views "mathematical truth" (or even truth) as a family resemblance concept and therefore should extend his conception of mathematical truth. Wittgenstein, qua finitist, formalist, and radical constructivist, has very good reasons to demur: he believes that 'truth' and 'falsity' can be *eliminated* from mathematics without any substantive loss.

1. *Steiner on Wittgenstein's Two Mistakes*

According to Steiner, Wittgenstein makes two errors: (1) He takes the undecidable Gödelian proposition 'P' to be a "self-contradictory" or 'paradoxical' sentence, and (2) he attempts to refute Gödel's 'proof.' Steiner points out the first mistake by analyzing (*RFM* App. III, §8).

> I imagine someone asking my advice; he says: "I have constructed a proposition (I will use 'P' to designate it) in Russell's symbolism, and by means of certain definitions and transformations it can be so interpreted that it says: 'P is not provable in Russell's system'. Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable.

[3] There is much that is interesting and/or correct in Steiner's paper, but I do not have the space here to discuss it in detail. In addition to my agreement with Steiner (2001, 263) that Wittgenstein uses a self-referential *interpretation* of 'P' in all of his arguments, I agree that Wittgenstein claims that "the mere *interpretation* of a sentence can never... make it unprovable in the Russell system of rules and axioms," and that §14 "is strong evidence against [Floyd's 1995] interpretation" (though I disagree with Steiner's reason for saying this; see my (1999a), §2.4). I also agree that there is a significant difference between Gödel's proof and "classical impossibility proofs" in that the former is "finitistic and constructive" whereas the latter "are not limited in this way" (p. 259). Cf. my (1999a), pp. 189–190.

> Just as we ask: "'provable' in what system?, so we must also ask: "'true' in what system?" 'True in Russell's system' means, as was said: proved in Russell's system; and 'false in Russell's system' means: the opposite has been proved in Russell's system. — Now what does your "suppose it is false" mean? *In the Russell sense* it means 'suppose the opposite is proved in Russell's system'; *if that is your assumption*, you will now presumably give up the interpretation that it is unprovable. And by 'this interpretation' I understand the translation into this English sentence. — If you assume that the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and the interpretation "P is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. Further: if the proposition is supposed to be false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's sense. (What is called "losing" in chess may constitute winning in another game.)

As I argued in my (1999a), in the first paragraph of (§8) Wittgenstein's interlocutor runs two *reductios*, proving that 'P' must be true by assuming that 'P' must either be true or false in Russell's system, and proving that 'P' must be unprovable by assuming that 'P' must either be provable or unprovable in Russell's system. In running these two Gödelian *reductios*, Wittgenstein mistakenly believes Gödel's proof *requires* a self-referential interpretation of 'P' and '$\neg P$' (i.e., that Gödel's proof *requires* that 'P' "be so interpreted that it *says*: 'P is not provable in Russell's system"'). Indeed, Wittgenstein's *Gödelian* reasoning throughout (*RFM* App. III) — especially at (§8), (§10), (§11), (§17) and (§18) — *mistakenly* 'derives' contradictions by 'interpreting' 'P' to mean "P is not provable in Russell's system."[4] On this important matter Steiner agrees, for, on his construal, Wittgenstein thinks "that the Gödel proof proceeds, not by deducing a contradiction from the assumption that sentence P is provable, but by interpreting P itself as a paradoxical sentence..., a self-contradiction" (Steiner, 2001, 262–63). Steiner is right, and in agreement with my interpretation, for Wittgenstein's Gödelian reasoning does not deduce a contradiction *within PM* on the assumption of either 'P' or '$\neg P$' — Wittgenstein's Gödelian reasons, rather, that a proof of 'P' and a proof of '$\neg P$' both lead to contradictions *by way of*

[4] See my (1999a), p. 181, where I interpret (§8, par. 1) in precisely this way. On pp. 181–183, I argue that in (§§10 & 17), Wittgenstein also seems to believe that Gödel's proof requires a natural language, self-referential interpretation of 'P' and '$\neg P$' — that Gödel's proof *requires* that 'P' "be so interpreted that it *says*: 'P is not provable in Russell's system"' (§8).

the self-referential *interpretation* of 'P.' This is Wittgenstein's first mistake, according to Steiner.

Wittgenstein's second mistake, on Steiner's evaluation, is to "attempt to refute Gödel's proof" (Steiner, 2001, 258). Given that Wittgenstein misunderstands Gödel's reasoning, he really attacks "an *argument* Gödel never made," but since this argument is "an informal version of a 'semantical' proof of Gödel's theorem..., which can be made mathematically precise," "Wittgenstein... attempted to refute an informal version of a mathematical proof" (Steiner, 2001, 263). It is a mistake for Wittgenstein to attack this argument, according to Steiner, first because it is a rigorous mathematical proof, and second, because Wittgenstein has very good reasons to accept the semantical version of Gödel's proof and its implications for mathematical truth.

2. *Steiner on What Wittgenstein "Should Have Said"*

Thus, despite Wittgenstein's mistaken interpretation of 'P' and his mistaken attempt to refute "an informal version of a 'semantical' proof of Gödel's theorem," Steiner wishes to show that Wittgenstein is not *entirely* mistaken or *entirely* foolish. To achieve this end, Steiner endeavours to demonstrate that Wittgenstein's (*RFM* App. III) 'indefensible' "remarks on Gödel's theorem are an aberration" (Steiner, 2001, 258), that they conflict with certain central views Wittgenstein has about mathematics and philosophy. Wittgenstein is "his own worst enemy," according to Steiner, for given his view that "that mathematics is... a 'family resemblance' concept," he should never have set out to "refute Gödel's theorem." Instead, Wittgenstein "should have said" that "Gödel's theorem had made it impossible to identify mathematical truth with provability in any one system, which should have encouraged the conclusion that mathematical truth is multicolored" (Steiner, 2001, 261, 273). Since Wittgenstein "connects the notion of mathematical truth with mathematical proof," he *should* flexibly accept a conception of mathematical truth whereby mathematical truth *grows*, in Gödelian cases, non-arbitrarily, by a kind of 'forcing,' which, in turn, gives Wittgenstein a very good reason to adopt Tarski-truth for mathematical calculi.

Steiner's attempt to show that Wittgenstein's remarks on Gödel's theorem are "an aberration" fails, however, for two closely related reasons: first, it ignores the clear fact that Wittgenstein's attack on Gödel issues primarily from Wittgenstein's *own* conception of mathematical truth; and second, it conflicts with Wittgenstein's view that a mathematical language-game is a formal, purely syntactical calculus whose only 'interpretation' is the application we have given it to the real world (e.g., in physics).

3. *Wittgenstein's Conception of Mathematical Truth*

Taking the first point first, Steiner's argument fails because he does not recognize — or he is unwilling to grant — that Wittgenstein's response to Wittgenstein's own mistaken interpretation of Gödel's proof is based on Wittgenstein's identification of "true in calculus Γ" and "proved/provable in calculus Γ." *It is anything but an aberration!* This failure or reluctance causes Steiner to misinterpret the main thrust of, especially, (§§5–8, 19), wherein Wittgenstein *answers* the questions of (§§5–6) with the answers of (§§6–8). In particular, Wittgenstein answers the question of (§8, par. 1) — "Must I not say that this proposition on the one hand is true, and on the other hand is unprovable?" — *negatively* simply by elaborating his *own* view (§8, par. 2) that "'True in Russell's system' means, as was said: proved in Russell's system; and 'false in Russell's system' means: the opposite has been proved in Russell's system." The crucial point here is that it does not matter that Wittgenstein misunderstands how *Gödel's proof works*, because, on the basis of his own explication of "mathematical truth," Wittgenstein is rejecting *any* argument [e.g., (§8, par. 1, §10, and §17, par. 3)] purporting to establish the existence of true but unprovable mathematical propositions. Indeed, the main point of (§8, par. 2) is: *Given* that "'True in Russell's system' means...: proved in Russell's system," *no reasoning* could establish the existence of a true but unprovable proposition *of* PM. Similarly, the point of (§7) — indeed (§§6–8) — is that 'a proposition which cannot be proved in Russell's system is "true" or "false" in a different sense from a proposition of *Principia Mathematica*' (§8: "if the proposition is supposed to be false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's sense").

One is inclined to think that Steiner recognizes, to some extent, that Wittgenstein equates "true in calculus Γ" and "proved/provable in calculus Γ," for he quotes and discusses Wittgenstein's (§§6, 7, 8),[5] where Wittgenstein says in three separate but related ways that 'a proposition which cannot be proved in Russell's system is "true" or "false" in a different sense from a proposition of *Principia Mathematica*.' Specifically, Steiner quotes (Steiner,

[5] Steiner says that "[s]ince the formula ['schema'] has infinitely many such instances, it cannot serve as a definition of the concept of truth" (Footnote #36), but even if this is true, it does not follow that Wittgenstein did not *intend* it as a definition or explication of "mathematical truth." Clearly, Wittgenstein *does* intend it this way, as is indicated by the very passage in §8 quoted by Steiner. (A small note: In Footnote #36, Steiner says that Wittgenstein's §6 " ' "*p*" is true = *p*' cannot serve as such a definition anyway," since "[t]he expression '*p*' on the left-hand side does not refer to the sentence, but to the sixteenth letter of the alphabet." In MS 118 and TS 223, the text is: "*p is true* = *p*.")

2001, 265) Wittgenstein's §6, "'p' is true = p," he correctly states (273, Footnote #36) that this is a 'schema,' and he correctly notes (273) that Wittgenstein *uses* this schema to claim that the fundamental questions are "[U]nder what circumstances do we assert a [mathematical] proposition?," "[H]ow is the assertion of the proposition used in the language-game?," and "Under what circumstances is a proposition asserted in Russell's game?" Despite these quotations and his comments on them, Steiner's remarks and criticisms militate against a genuine recognition of Wittgenstein's clearly articulated position. For example, Steiner quotes (p. 265) "'True in Russell's system' means, as was said: proved in Russell's system" from §8, and then immediately adds that "Wittgenstein's view" here "is not correct, if 'true' means 'satisfiable by all sequences of natural numbers." This, however, is clearly *not* Wittgenstein's conception of "mathematical truth," for Wittgenstein clearly answers his own clear questions at §6, as we shall shortly see, and he nowhere claims (nor could he claim) that "mathematically true" means "satisfiable by all sequences of natural numbers." Similarly, after quoting the first line of §6, Steiner says that if "Wittgenstein [had] been after a 'definition' of truth, he would have looked precisely for an account of every utterance of a proposition," and that "[t]he formula "'p" is true = p' means only that left and the right sides have the same assertability criteria" (Steiner, 2001, 274). But these points misunderstand Wittgenstein's aims, for Wittgenstein is offering a 'theory' or explication of (only) "mathematical truth," *not* 'truth' or 'Truth,' and hence, on his view, it is sufficient to give "assertability criteria" for mathematical propositions (i.e., within systems) and for claims about the *mathematical truth* of *mathematical propositions* (i.e., within systems).

We can best see that Steiner has misinterpreted Wittgenstein's own view of mathematical truth if we examine Wittgenstein's own *unequivocal* answer to the question "Under what circumstances is a proposition asserted in Russell's game?" Wittgenstein says (§6, par. 3), as clearly as one might say, that "the answer is: at the end of one of his proofs, or as a 'fundamental law' (Pp.). There is no other way in this system of employing asserted propositions in Russell's symbolism." This answer is the *crucial* part of (§6), since it *answers* Wittgenstein's own questions and it *explicates* "'p' is true = p" by presenting Wittgenstein's own view of intra-systemic, mathematical truth. Steiner quotes (§6, par. 1) *and* (§6, par. 2), but, strikingly, he does not quote (§6, par. 3). I do not know why Steiner does not quote the crucial (§6, par. 3), especially when he quotes the rest of §6, but if Steiner *had* quoted this passage, it is hard to see how he could have made various claims (e.g., (275), "according to Wittgenstein the growth of mathematics is accompanied by the growth of mathematical truth (this is what is meant by "'P" is true = p')") and how he could have launched his principal arguments.

Despite the clear and incontrovertible evidence of (§§6, 7, 8) — and especially the clarificatory importance of the omitted (§6, par. 3) — Steiner sides with Floyd in saying "that Wittgenstein had no thought of giving either a theory or even a definition... of truth" (Steiner, 2001, 273).[6] Steiner grants (274) that "[i]n mathematics, sentences are standardly asserted in the context of proofs," that "[t]his is certainly correct in the formalized context of *Principia Mathematica*," that "[t]he 'equivalence' '*p*' is true = *p*... allows Wittgenstein to claim that an undecidable sentence of *Principia Mathematica*, if true at all, is true in a different sense than are the theorems of *Principia Mathematica*," and that Wittgenstein uses this 'equivalence' to mount a "'refutation' of Gödel's theorem" (Steiner, 2001, 274), but Steiner effectively ignores Wittgenstein's own position by saying:

> But this [§8] riposte ignores Wittgenstein's own views about 'family resemblance' concepts and about mathematical proof. Wittgenstein writes as though Gödel had constructed a trivially new interpretation of the 'Russell' formalism — like calling a tail a leg, and then saying that a cow has five legs. And therefore, Gödel was free to call anything he wanted 'true'. (Steiner, 2001, 274–75)

"Ironically," Steiner continues, "Wittgenstein himself was far from this kind of conventionalism." Wittgenstein says "mathematical truth is... created, not discovered," according to Steiner (2001, 275), but "[t]his does not make Wittgenstein a conventionalist." But *how*, one wonders, is this possible? From 1929 through at least 1944, Wittgenstein consistently maintains that "[t]he mathematician is an inventor, not a discoverer," (*RFM* I, 168; Appendix II, §2), that "one cannot discover any connection between parts of mathematics or logic that was already there without one knowing" (*PG* 481), that "[i]n mathematics *everything* is algorithm and *nothing* is meaning ['Bedeutung']" (*PG* 468), and that "the proof... makes new connexions," "[i]t does not establish that they are there; they do not exist until it makes them" (*RFM* III, §31). How, given this (and more) textual evidence, can Steiner deny, as he apparently does, that Wittgenstein makes the quintessential Conventionalist move of identifying "true in Russell's system" with "proved/provable in

[6] See: Juliet Floyd, "Prose versus Proof: Wittgenstein on Gödel, Tarski, and Truth," *Philosophia Mathematica* (3) Vol. 9, 2001, pp. 280–307; Juliet Floyd and Hilary Putnam, 'A Note on Wittgenstein's "Notorious Paragraph" about the Gödel Theorem,' *The Journal of Philosophy*, Volume XCVII, Number 11, November, 2000, pp. 624–632; and my "Misunderstanding Gödel: New Arguments about Wittgenstein and New Remarks by Wittgenstein," *Dialectica*, Vol. 57, No. 3, 2003, pp. 279–313, where I criticize the arguments of the Floyd and Putnam paper.

Russell's system"?[7] How can Steiner deny that, at §6, Wittgenstein identifies "true propositions of Russell's system" with "asserted propositions of Russell's system" and "asserted propositions of Russell's system" with "axioms ('fundamental law[s]') and (proved) propositions of Russell's system"?[8] Steiner's answer is that:

> In the case of undecidable sentences,... Wittgenstein slipped into the kind of conventionalism he usually condemned. ... The 'picture' of truth as the winning position of a game suggests to [Wittgenstein] the following idea: extending the 'Russell system' by adding undecidable sentences is like defining the rules of a new game, and is not like discovering new terrain. ... [Wittgenstein] even slid into thinking that this insight refutes Gödel's theorem itself, believing as he did (however briefly) that the theorem depends on the concept of truth. (Steiner, 2001, 275)

In short, Wittgenstein *slipped*. Despite Wittgenstein's claim that "mathematical truth is created, not discovered," and despite the fact that Wittgenstein's (*RFM* App. III) attack on Gödel is largely driven by *Wittgenstein's* identification of "true in Russell's system" with "proved/provable in Russell's system," on Steiner's account this identification and the concomitant attack is but a series of 'slips.' On Steiner's interpretation, what appear to be clear and explicit statements of Wittgenstein's view of mathematical truth are, evidently, inexplicable slips from an otherwise consistent philosophy of mathematics.

[7] Part, but not all, of the answer is that Steiner (sometimes) views 'Conventionalism' in the sense of Dummett's "Radical Conventionalism," which is clear when he shows that Wittgenstein is not a Conventionalist (i.e., a Radical Conventionalist) by correctly saying that, on Wittgenstein's view, "even 75 + 26 = 101 is a rule, but that does not mean we could adopt the rule 75 + 26 = 102" (p. 275). What confuses is that Steiner then says that Wittgenstein "slipped into... Conventionalism" in part because he thought that "[t]o call an undecidable sentence 'true' is an arbitrary decision" (p. 275). Wittgenstein does think this, but *not* because he is a Radical Conventionalist, but rather because he is a Conventionalist (specifically, a Radical Constructivist) in holding that: (1) we invent mathematical calculi, (2) mathematical truths are the axioms or theorems of some existent system, and (3) as such, mathematical truths are *conventional*, since they are derived from conventionally accepted 'propositions' using conventionally accepted rules. As with (*RFM* III, §31), this is clearly and consistently Wittgenstein's position in *PR* and *PG*. See, e.g.: (*PR* §202): "A mathematical *proposition* can only be either a stipulation, or a result worked out from stipulations in accordance with a definite method." See my (1997) for Wittgenstein's account of what constitutes a mathematical proposition.

[8] In short, this is how Wittgenstein eliminates the need for 'truth' and 'falsity' in mathematics. In Section 4, we will see that, at (*RFM* App. III, §4), Wittgenstein makes the closely related point that we do not need the notion of a *mathematical proposition*.

4. *Family Resemblance Concepts and Mathematics-By-Invention*

The reason Steiner turns a blind eye to all of this evidence, it seems, is that he is struck by Wittgenstein's notion of a "family resemblance" concept and he thinks it holds the key to how Wittgenstein "should have" responded to Gödel's Theorem. Steiner's principal claim is that Wittgenstein "should have" said that "Gödel's theorem had made it impossible to identify mathematical truth with provability in any one system, which should have encouraged the conclusion that mathematical truth is multicolored" (Steiner, 2001, 273). In support of this assertion, Steiner makes the *crucial* claim that "[o]ne of Wittgenstein's main strategies [involves] showing that [the] concepts: number, proof, truth... are... 'family resemblance' concepts" (Steiner, 2001, 260).[9] Steiner similarly says that "Gödel's theorem makes plausible, that any attempt to formalize all mathematics (as a recursively axiomatized system) is doomed," which "fits perfectly the Wittgenstein doctrine that mathematics is a 'motley' of techniques of proof (*RFM*, III, §46), that is, a 'family resemblance' concept" (Steiner, 2001, 261).

Given his claim that Wittgenstein 'show[s]' that truth and/or mathematical truth is/are a family resemblance concept(s), it is surprising that Steiner makes no attempt to establish the truth of this claim by providing proof text from Wittgenstein's writings or by citing such text. Instead, Steiner references (Floyd, 1995, 405), which, however, provides absolutely no support for his claim, for, at (1995, 405–06), Floyd does *not* claim that 'truth' is a family resemblance concept, she does not quote proof text from Wittgenstein's writings, and she does not cite such text.[10] This is an *absolutely crucial* lacuna in Steiner's argument, since Steiner argues that Wittgenstein *should* adopt Tarski-truth because he *should* extend "true in PM" to "true in PM_2," etc., *because*, ultimately, he views mathematical truth (or truth) as a family resemblance concept. If Wittgenstein does *not* view mathematical truth as a family resemblance concept — as is the case — then Steiner's

[9] Similarly, on p. 274, Steiner says that, according Wittgenstein, "[t]ruth is a family-resemblance concept, just like 'game'." Cf. Steiner, p. 260: "Indeed, in *PI*, he explicitly gives the concept of 'number' as an example of a family resemblance. Even truth is an example, as Floyd points out."

[10] Floyd makes some remarks about Wittgenstein's views on truth, mathematical truth, and provability, but there is nothing on pp. 405–406 or in the Endnotes ##108 and 109, there referenced, which constitutes evidence for Steiner's claim. In her *reply* to Steiner (Floyd, 2001, 287), Floyd does speak of "Wittgenstein's repeated emphasis on the contextual or 'family-resemblance' character of our notions of *truth*, *proof*, *consistency*, and *number*," but, again, she provides no evidence that Wittgenstein viewed truth or mathematical truth as a family resemblance concept.

argument collapses. And it is not just that Steiner fails to provide any evidence for this pivotal claim. The matter is much more serious, for the claim is simply false: Wittgenstein's always views contingent or empirical truth as truth-by-correspondence and he always *contrasts* mathematical truth with contingent or empirical truth.

In this connection, it should suffice to show that in (*RFM* App. III), the focus of Steiner's claims, Wittgenstein *contrasts* contingent/empirical truth and "mathematical truth" in very strong terms. Most importantly, Wittgenstein asserts that, unlike contingent/empirical propositions, we do not *use* mathematical propositions to make assertions about existent entities or facts in the universe. For example, at (*RFM* App. III, §4) he asks and answers a very important question.

> 4: Might we not do arithmetic without having the idea of uttering arithmetical *propositions*, and without ever having been struck by the similarity between a multiplication and a proposition?
>
> Should we not shake our heads, though, when someone shewed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining? — Yes; and here is a point of connexion. But we also make gestures to stop our dog, e.g. when he behaves as we do not wish.
>
> We are used to saying '2 times 2 is 4,' and the verb 'is' makes this into a proposition, and apparently establishes a close kinship with everything that we call a 'proposition.' Whereas it is a matter only of a very superficial relationship.

The answer to the question of the first paragraph is 'Yes,' we *could* "do arithmetic without having the idea of uttering arithmetical *propositions*, and without ever having been struck by the similarity between a multiplication and a proposition." As Wittgenstein says at (*RFM* IV, §15):

> People can be imagined to have an applied mathematics without any pure mathematics. They can e.g. — let us suppose — calculate the path described by certain moving bodies and predict their place at a given time. For this purpose they make use of a system of co-ordinates, of the equation of curves (*a form of description of actual movement*) and of the technique of calculating in the decimal system. The idea of a proposition of pure mathematics may be quite foreign to them.

At (*RFM* App. III, §4), Wittgenstein grants that there "is a point of connexion" between our *behaviour* relative to (mistaken) calculations and our *behaviour* relative to (mistaken) contingent assertions: in both cases we nod our heads when the calculation or assertion is correct, and shake our heads when it is incorrect. It does not follow, however, that the word 'correct' means the same (or a strongly similar) thing in both contexts. Indeed, on

Wittgenstein's account, it does not, since a correct calculation is correct by virtue of its agreement with stipulated rules, whereas a correct contingent assertion is correct (i.e., true) by virtue of its agreement with a state of affairs or fact in the world.

But, Steiner might ask, doesn't this 'connexion' constitute a family resemblance between, on the one hand, contingent/empirical propositions and the truth they have, and, on the other hand, mathematical propositions and the truth they have? Wittgenstein answers this question negatively by saying (*RFM* App. III, §4): "We are used to saying '2 times 2 is 4,' and the verb 'is' makes this into a proposition, and apparently establishes a *close kinship* with everything that we call a 'proposition.' Whereas it is a matter only of *a very superficial* relationship" (italics mine). Put differently, the 'propositions' '2 times 2 is 4' and "'$2 \times 2 = 4$' is true" bear "only... a very superficial relationship," *not* a medium-to-strong family resemblance, to contingent/empirical propositions — in both cases we say 'correct' and 'incorrect' and we speak of a 'proposition' and its opposite (i.e., its negation), but contingent/empirical truth is a completely different animal from mathematical truth. From this it follows that in his discussion of Gödel in (*RFM*, App. III), not only does Wittgenstein *not* suggest that contingent/empirical truth and mathematical truth are family resemblance concepts, he *strongly suggests* that there is *no* family resemblance between the two,[11] and he always views mathematical truth, even in the *Tractatus*, as symbolic/syntactical in nature. Furthermore, if one insists that since Wittgenstein views contingent/empirical truth and mathematical truth as *similar* things (i.e., concepts with "a point of connexion"), he therefore views 'truth' as a family resemblance concept, this erroneous inference still does not give Steiner what he needs, since Wittgenstein's view on *mathematical truth* remains constant, especially from 1929 through at least 1944 (when he did almost all of his work on mathematics), which means that he does not view *mathematical truth* as a family resemblance concept.[12] That is, even if Wittgenstein viewed truth as a family resemblance concept (which he does

[11] Thus here, in the very text Steiner analyzes, Wittgenstein *clearly* states that, although many think that there is a "close kinship" (a family resemblance?) between a contingent/empirical proposition and a mathematical proposition, the relationship is "very superficial," which strongly indicates that Wittgenstein *denies* a family resemblance between contingent/empirical truth and mathematical truth.

[12] Again, Steiner seems partially aware of this when he says (273): "Another source of Wittgenstein's confused reaction to Gödel's theorem has to do with *his view concerning* [*mathematical*] truth..." (italics mine). I have inserted 'mathematical' because, at §6 (and §7, §8, etc.), Wittgenstein is *clearly* discussing *only mathematical truth* when he makes the original question of §6 more precise by asking "Under what circumstances is a proposition asserted in Russell's game?", which he then answers *unequivocally*.

not), Steiner's argument requires that Wittgenstein view *mathematical truth* as a family resemblance concept. Since all of the evidence indicates that Wittgenstein *never* viewed mathematical truth as family resemblance concept, Steiner's central *forcing* argument cannot work on Wittgenstein.[13]

This brings us to Steiner's claim that the 'plausible' assertion "that any attempt to formalize all mathematics (as a recursively axiomatized system) is doomed" "fits perfectly the Wittgenstein doctrine that mathematics is a 'motley' of techniques of proof (*RFM*, III, §46), that is, a 'family resemblance' concept" (Steiner, 2001, 261).[14]

[13] In Wittgenstein's *Nachlaß*, there are *no* close occurrences of "mathematical truth" and "family resemblance" (or 'family' or 'resemblance') (i.e., no hits when searching for "mathematischen Wahrheit" or "mathematische Wahrheit" or "mathematischer Wahrheit" *and* 'Familie'). Indeed, there are only 6 occurrences of either "mathematischen Wahrheit" or "mathematische Wahrheit," and none of these, such as (*RFM* IV, §44: MS 125; 62v, September 15, 1942), includes any use of 'Familie' or 'Ähnlichkeit.' Even more fundamentally, there are no close occurrences of 'Wahrheit' and 'Familie' in the *Nachlaß*, and only six somewhat close occurrences of 'wahr' and 'Familie' (i.e., none in the same sentence), none of which are helpful to Steiner's argument. More generally, there are only four occurrences of "family resemblance" in the English translations of Wittgenstein's published works, only nine occurrences of 'Familienähnlichkeit' in the *Nachlaß* (five of which are versions of the Spengler passage at *Culture and Value*, p. 21), and only twelve occurrences of 'Familienähnlichkeiten' in the *Nachlaß*, six of which are versions of (*PI* §67). There are *no* (even medium-sized) passages that contain "ähnlich," "Familie," *and* either 'wahr' *or* 'mathematisch' (or the inflected forms of 'mathematisch'). Obviously, a more thorough investigation can be conducted, but this reasonably thorough search has turned up absolutely no evidence for the claim that Wittgenstein *ever* viewed truth or mathematical truth as a family resemblance concept.

[14] Steiner's much weaker claim that Wittgenstein viewed 'mathematics' as a family resemblance concept — that, e.g., mathematical calculi and 'activities' have a family resemblance — is much more plausible than his claim that Wittgenstein viewed truth or mathematical truth as a family resemblance concept. Still, there are only three close occurrences of 'family' and 'mathematics' in *RFM*: (1) (*RFM* V, §15), where Wittgenstein asks "Why should I not say that what we call mathematics is a family of activities with a family of purposes?," and shortly after (two pages later in MS 126, written on the same day) he says (*RFM* V, §16) that "[w]e might speak of a kind of alchemy in mathematics"; (2) (*RFM* VII, §33), where Wittgenstein says "Mathematics is, then, a family; but *that is not to say that we shall not mind what is incorporated into it*" (italics mine) and then immediately adds "We might say: if you did not understand *any* mathematical proposition better than you understand the Multiplicative Axiom, then you would *not* understand mathematics"; and (3) (*RFM* VII, §42), where Wittgenstein says "When I said that the propositions of mathematics determine concepts, that is *vague*... [for] there is a family of cases." Interestingly, Steiner seems to think that the two occurrences of 'motley' in the English translation of *RFM* (*RFM* III, §46, §48) indicate that mathematics is 'multicolored,' that 'mathematics' is a family resemblance concept, and that mathematical truth is a multicolored, family resemblance concept. What is strange about this is that 'motley' and even 'multicolored' indicate a *variety* or *diversity*, whereas a family resemblance is meant to indicate a particular kind of *similarity*. The German terms translated as 'motley' are "buntes Gemisch" and 'Buntheit,' respectively. The former might be translated as "multi-coloured mix" or "variety of colour," and the latter could be similarly

To support this claim, Steiner interestingly invokes Wittgenstein's (Conventionalist) view that "[e]very new proof, every new calculation even, adds to mathematics," that "mathematical truth is therefore created, not discovered" (Steiner, 2001, 275).[15] "[F]or a philosopher [i.e., Wittgenstein] who connects the notion of mathematical truth with mathematical proof," Steiner tells us, "Gödel's theorem suggests that the concept of mathematical truth admits of flexibility, is not written in stone" (Steiner, 2001, 260–61). How, one wonders, can this be? Steiner's ingenious answer is, I believe, that since Wittgenstein "connects the notion of mathematical truth with mathematical proof," he need only flexibly accept a conception of mathematical truth whereby mathematical truth *grows*, in Gödelian cases, non-arbitrarily, by a kind of 'forcing.' Gödel's Theorem gives Wittgenstein a very good reason to *extend* his conception of "true in Russell's system" by adding 'P' to PM to yield the new calculus PM_2, and, concomitantly, creating the required *new* conception of "true in PM_2" for the new calculus. This good reason to extend PM will, naturally, also apply to PM_2, and to PM_3, and so on, which gives Wittgenstein a good reason to adopt Tarski-truth for mathematical calculi.

translated. What is puzzling is that Steiner speaks of "the Wittgenstein doctrine that mathematics is a 'motley' of techniques of proof (*RFM*, III, §46), that is, a 'family resemblance' concept" *and* he argues, seemingly borrowing 'multicolored' *from* 'motley,' that Wittgenstein should have concluded "that mathematical truth is multicolored." Even if "mathematics is a MOTLEY of techniques of proof," though it follows that mathematical proof techniques are *diverse* (with, *perhaps*, a family resemblance, since they are all techniques of *proof*), it does *not* follow (with or without Gödel's theorem) that *mathematical truth* is multicolored or a family resemblance concept.

[15] It is worth noting that this correct treatment of Wittgenstein's (Conventionalist) conception of mathematical calculi, the growth of a mathematical calculus, and the growth of mathematics as a whole is very hard to reconcile with Steiner's claim that Wittgenstein "slipped into the kind of conventionalism he usually condemned" by thinking that "extending the 'Russell system' by adding undecidable sentences is like defining the rules of a new game, and *is not like discovering new terrain*" (italics mine). As Steiner says, on Wittgenstein's account we create a new calculus (i.e., "add to mathematics") by *every* new proof and *every new calculation* — we don't discover "new terrain" and, in particular, we don't discover the pre-*existing* terrain *that* such-and-such proposition was all along provable in our calculus. (Though Steiner might not wish to state Wittgenstein's Radical Constructivism in such radical terms, how can he say, with Wittgenstein, that a new calculation and a new proof "adds to mathematics" if there *pre-existed a fact* (i.e., some terrain) that the proved proposition was provable before it was proved?). This radical view *is* Wittgenstein's view and, though Steiner calls it an uncharacteristic 'slip' on p. 275, he articulates it on p. 274 and there uses it to argue that Wittgenstein should accept the *growth* of intra-systemic mathematical truth (and, thereby, Tarski truth for mathematics). At footnote #41, Steiner even grants that at the time of writing (*RFM* App. III) Wittgenstein held the radical view of autonomous calculi I have presented here and elsewhere. See Section 6.

> A mathematical characterization of truth is given for Arithmetic (Tarski's), according to which all theorems of Arithmetic are true. We want to extend this concept of truth to other sentences than the theorems of Arithmetic, while leaving the *laws* of truth intact. We are offered the choice of calling the Gödel sentence, or its negation, true. Gödel's theorem in its semantic version shows that calling the negation of the sentence true will violate one of the fundamental laws of arithmetic truth — which is that a generalization of ϕ is true if and only if ϕ itself is satisfied by every natural number. Why? Each substitution instance of the Gödel theorem, we can show, is provable in Arithmetic; therefore, each substitution instance is Tarski-true. But since it is also a theorem (provable in set theory) that every number is designated by some numeral, the truth of each substitution instance implies the truth of the generalization, *i.e.*, the Gödel sentence, G. (Steiner, 2001, 277–78)

In short, "what is true in one system may be mathematically fixed by what is provable in another" (Steiner, 2001, 278).

Interestingly, Steiner considers an objection to this argument presented to him by Burton Dreben and Juliet Floyd.

> Suppose Wittgenstein were to appear before us and say, 'It is correct that a Tarski-type truth definition can be used for a mathematical proof of Gödel's theorem. So I was wrong... But I still maintain that Tarski-truth is not truth. The mathematical notion "Tarski truth" gives no enlightenment whatever *concerning truth as it is used in mathematics* [italics mine]. Thus, we have a dilemma: if we stick to the customary notion of mathematical truth, Argument A ["an informal version of a 'semantical' proof of Gödel's theorem"] is in fact invalid. And if we use an "artificial" notion, like Tarski's, we can prove the theorem, but get no enlightenment.' (Steiner, 2001, 268)

In response to this, Steiner argues:

> [T]o raise this dilemma, *i.e.*, to make a sharp boundary (on Wittgenstein's behalf) between 'true' and 'Tarski true', is to overlook the organic relation between the two which is illuminated brightly by Gödel's theorem. Namely, the mathematician who has proved Gödel's theorem, and who wants to extend the 'Russell notion' of truth (truth as provability in *Principia Mathematica*, or, as in our treatment, PA) to cover the undecidable sentence P, has no choice whatever — he *must* adopt Tarski truth as the extension of 'true' in light of Gödel's theorem! (Steiner, 2001, 268)

Steiner's argument is ingenious, in my opinion, because it (seemingly) enables Wittgenstein to accommodate Gödel's Theorem *and* Wittgenstein's own unification of proof and mathematical truth by utilizing Wittgenstein's notions of family resemblance and mathematics-by-invention. On the positive side, there can be no doubt that Wittgenstein saw the organic growth of the concept of number (*PI* §67), from natural to rational to irrational to imaginary, in this manner (though, as I have argued in my (1999b) and (2000a), he also viewed pseudo-irrationals, lawless irrationals, and transfinite cardinals as violations of this organic growth). But would or should Wittgenstein be compelled by Steiner's reasoning?

5. *Wittgenstein's Formalism*

The answer is 'No,' which brings us to the second reason that Steiner's argument fails, namely, because it conflicts with Wittgenstein's view that a mathematical language-game is a formal, purely syntactical calculus whose only 'interpretation' is the *application* we have given it to the real world (e.g., in physics). On Wittgenstein's account, the problem with Gödel's First *Incompleteness* Theorem, as it is standardly called and interpreted, is that it presupposes a distinction between syntax and semantics *within* mathematics — between, e.g., mathematical proof and mathematical truth, mathematical term and mathematical object, mathematical calculus and mathematical model — when, in fact, mathematical terms and propositions are not *about* anything ("[i]n mathematics... *nothing* is meaning"). When, e.g., Steiner says that "since it is also a theorem (provable in set theory) that every number is designated by some numeral, the truth of each substitution instance implies the truth of the generalization, *i.e.*, the Gödel sentence, *G*" (Steiner, 2001, 278),[16] Wittgenstein would *begin* by saying (*PR* §109) that "[a]rithmetic doesn't talk about numbers, it works with numbers."[17] It, therefore, is misleading to say, with Steiner,[18] that Wittgenstein's anti-Platonism prompts his opposition to Gödel's Theorem, since the very idea that a mathematical calculus is *incomplete* is anathema to Wittgenstein.[19] Even on the broadest

[16] See (MS 163; 37v–38r; July 8, 1941): "The mathematical fact that we have here an arithmetical proposition which can neither be proved nor disproved in P does not interest me."

[17] Cf. (*PR* §§157 & 159), (*PG* 333), and (*RFM* V, §16, par. 2–3).

[18] Steiner (2001, 273): "Wittgenstein... fell prey to Gödel phobia, first, because Gödel's theorem had become an icon of mathematical realism — the idea that mathematical truth is distinct from the activity of proving theorems."

[19] See, e.g., (*PR* §158): "Where a connection is now known to exist which was previously unknown, there wasn't a gap before, something incomplete which has now been filled in!" "That is why I have said there are no gaps in mathematics. This contradicts the usual view."

interpretation of Wittgenstein's philosophy of mathematics, the only propositions *of* a mathematical calculus are those that are provable or refutable *within* that calculus. Thus, for Wittgenstein, it is a mistake to talk, on the one hand, of the *language* of, e.g., PM, and on the other hand, of the *propositions* provable or refutable *in* PM. This is *precisely* why, at (§7), Wittgenstein says that *if* there are "true propositions which are written in [Russell's] symbolism, but are not provable in Russell's system," they must be "true in *another* system, i.e. can rightly be asserted [i.e., proved or assumed as axioms] in another game." As I have said, a true but unprovable proposition of PM is a contradiction-in-terms for Wittgenstein, *but so is* a true mathematical proposition that is not presently provable (or proved) in *any* existent mathematical calculus. This is why Wittgenstein says that *if* there exists a true mathematical proposition that is not provable in PM, it must be provable in some other existent mathematical 'system.'

What is especially interesting about Steiner's argument is that, at least on my interpretation of Wittgenstein's Philosophy of Mathematics, Wittgenstein says that every new proof creates a new calculus — that (*PG* 371) "a mathematical proof incorporates the mathematical proposition into a new calculus, and alters its position in mathematics." In the middle period, Wittgenstein says, e.g., that "[i]f I find a formula for the roots of an equation, I've constructed a new calculus; I haven't filled in a gap in an old one" (*PG* 373). Later, at (*RFM* V, §9; MS 126, 1942), Wittgenstein famously says that "the further expansion of an irrational number is a further expansion of mathematics." What is deceiving, in light of Steiner' argument, is that in these cases we have a calculus, say Γ_1, which we *extend* in accordance with the rules of Γ_1, yielding a new calculus, Γ_2. On Wittgenstein's view, it is appropriate to call this an 'extension' because we have proved a proposition, and created a new calculus, *without altering the rules or axioms of the original calculus*. We have, therefore, the creation of a new calculus as the *organic extension* of another. Indeed, on my interpretation,[20] Wittgenstein *needs* this account to enable him to identify "true in Γ_1" with "proved in Γ_1" (*not* "provable in

"Mathematics cannot be incomplete; any more than a sense can be incomplete." Wittgenstein's views on the completeness of mathematical calculi (and the creation of new calculi by extending existent calculi, bit-by-little-bit) are closely related to his opposition to mathematical continuity and the notion of a gapless mathematical continuum consisting of *all* the real numbers. Cf. (*PR* §181; *PG* 472–73): "What criterion is there for the irrational numbers being complete?" "This shows clearly that an irrational number isn't the extension of an infinite decimal fraction, it's a law." "'If π were an extension, we would never feel the lack of it', i.e. it would be impossible for us to observe a gap." Cf. (*RFM* V, §34).

[20] See my "Wittgenstein's Anti-Modal Finitism," *Logique et Analyse*, Vol. 43, No. 171–172, 2000 (published January 2003), pp. 266ff.

Γ_1"), while accommodating the *rational search* for decisions of mathematical conjectures (e.g., Goldbach's Conjecture). The crucial point, however, is that this is *not* what Steiner is talking about. We have a proof that a certain concatenation of symbols is independent of, say, PA, if PA is consistent, and we have Wittgenstein's identification of "true in PM" with "proved/provable in PM"; we do not have a new proof of a proposition using the (unaltered) rules of PM. Steiner grants that one is not compelled to extend PM to PM_2 — since one can simply "refus[e] to extend the notion [of "true in *PM*"]"[21] — but he argues that Wittgenstein *should* allow the extension because of the organic relation between "'true' and 'Tarski true'." Given this organic relation, Wittgenstein should or will "want[] to extend the 'Russell notion' of truth... to cover the undecidable sentence P,"[22] which gives him "no choice whatever — he *must* adopt Tarski truth as the extension of 'true' in light of Gödel's theorem."

The key difference between the two cases of calculus extension/creation is that, in Wittgenstein's case, we accept the proved proposition as a true proposition in the newly created calculus Γ_2 (i.e., as "true in Γ_2") because it was proved using the rules and axioms of Γ_1. The "organic relation" in this

[21] Steiner, op. cit., p. 268: "The only way to avoid adopting Tarski's notion of truth (other than, of course, refusing to extend the notion at all) is to attack the theorem, so we return to my original conclusion: Wittgenstein's 'prose' is *in effect* ([...]) an attack on Gödel's theorem..." We shall see in Section 6 that Steiner's inability to see that Wittgenstein's attack on Gödel's theorem is, at bottom, nothing more than the claim that "a true but unprovable mathematical proposition" is a contradiction-in-terms from one (i.e., *Wittgenstein's*) standpoint — which is really a "refus[al] to extend the notion at all" — creates a tension that permeates Steiner's paper.

[22] In her (2001), Floyd says (301): "Indeed, Steiner holds that Wittgenstein *should* have said about Gödel's proofs what I claim he *did* say." This is highly misleading, since Floyd only agrees that Wittgenstein would say that 'truth' is a family resemblance concept (300). On p. 300, Floyd poses Steiner's challenge as: "[I]s it false to say that Gödel's first incompleteness theorem is a result about *mathematical* (or at least *arithmetical*) *truth* — namely, that the totality of such truths cannot be recursively axiomatized in a single formal system?" Since Steiner thinks Wittgenstein '*should*' answer this question negatively, and Floyd says she and Steiner agree about 'should' viz. '*did*,' this suggests that Floyd claims that Wittgenstein would answer the question negatively. On the other hand, Floyd says that (304) "Wittgenstein would deny that ['Tarski's model-theoretic'] account resolves the philosophical questions at stake in arguments about the nature and scope of mathematics" and that "he would have questioned whether Tarski's model-theoretic account of truth definitions for formalized languages yields a philosophical account of our notion of *mathematical truth*." Does this mean that Floyd thinks Wittgenstein would answer the original (300) question *positively*? If so, since Floyd does not say *why* Wittgenstein would answer this question positively or why Wittgenstein would 'deny' or 'question' "Tarski's model-theoretic account," we are never told what Wittgenstein's position is (or what Floyd's interpretation of Wittgenstein *on this matter* is).

case is so clear and strong that most mathematicians and philosophers would say that we have *not* created a new calculus, but rather we have proved the proposition in Γ_1. In Steiner's case, however, we have a conditional proof that 'P' is neither provable in PM not refutable in PM. If someone finds a proof of 'P' or its negation in PM, we cannot simply say we have created a new (extended) calculus, PM_2, since PM_2 (and PM) is inconsistent. If PM is consistent, we will not be able to use the axioms and rules of PM to prove 'P.' The two cases could not be more dissimilar. What reason, then, would Wittgenstein have for *wanting* to call 'P' either true or false — for wanting, more particularly, to "extend the 'Russell notion' of truth... to cover the undecidable sentence P"? The *only* reason there could be, and the only reason Steiner offers, is a *semantic* reason: If PM is consistent, then 'P' is true because there does not exist a natural number with a particular property and that is precisely what 'P' *says or means*. This, of course, is the standard interpretation of Gödel's theorem; as Steiner says, "[e]ach substitution instance of the Gödel theorem... is provable in Arithmetic," and "the truth of each substitution instance implies the truth of the generalization, *i.e.*, the Gödel sentence, G." Virtually everyone accepts this line of reasoning, but Wittgenstein does not. For Wittgenstein, standard number-theoretic propositions *do not say anything about* "the set of natural numbers" because they do not say or mean anything in this sense. Thus, a universally quantified formula is not true in PM on condition that every substitution instance is provable — it is true *iff* it is proved in PM. If, therefore, we asked Wittgenstein, following Steiner, to choose to call 'P' either true or false, he would ask us which system it has been proved or refuted in. Steiner will tell him 'none,' but then point out that if 'P' is added to PM, we must call it true, otherwise we "will violate one of the fundamental laws of arithmetic truth — which is that a generalization of ϕ is true if and only if ϕ itself is satisfied by every natural number." Wittgenstein will ask *why* we want to add 'P' to PM. Steiner seems to have two answers at this juncture: first, 'P' has a strong organic (i.e., syntactic) relation to the propositions of PM (e.g., it is a "well-formed proposition" in the language of PM); second, as a number-theoretic proposition, 'P' must be either true or false.

Wittgenstein's response to the first point is given in (§7): a proposition that is true or false and "written in [Russell's] symbolism," but not provable or refutable in PM, must be proved/provable or refuted/refutable "in another [existent] system." But this simply is not the case with 'P.' At the moment that Gödel proves his theorem, 'P' is homeless — it is "written in [Russell's] symbolism," but it is neither provable or refutable in any existent system. For Wittgenstein, the fact that we have a "well-formed" concatenation of symbols in Russell's symbolism cannot, in and of itself, tell us whether that concatenation is a mathematical proposition; the fact that 'P'

has a strong organic (i.e., syntactic) relation to the propositions of PM is *irrelevant* to Wittgenstein. Since Steiner's entire argument rests on the claim that Wittgenstein has a good *Wittgensteinian* reason to make 'P' a home, in PM_2, the only reason left is that 'P' must be either true or false, which is clearly a *semantic reason*. "[T]hough calling the Gödel sentence for *PM* true may involve an extension of the concept of truth," Steiner says, "the extension is not arbitrary (except in that we are not compelled to extend it), but mathematically fixed in advance by the rules of *PM*" (Steiner, 2001, 278).[23] We are *not* compelled to call 'P' true or false, but *if* we want to do so, we *must* call 'P' true, we must extend PM, and we must extend out concept of truth. If, that is, we accept the standard semantics for number theory or "Gödel's theorem in its semantic version," or we grant that 'P' is a well-formed number-theoretic proposition, we *are* compelled to grant that 'P' is either true or false, which, in turn, compels us to grant that it is true, to extend PM and our concept of truth (i.e., to "true in PM_2"), and, because this forcing can be reiterated *ad infinitum*, to adopt Tarski truth for mathematics.

The problem for Steiner is that Wittgenstein does not and cannot accept these *semantic* presuppositions. As I have just said, the fact that an expression has a particular syntactical form does not, of itself, ensure that it is a mathematical proposition. Furthermore, a *central*, *distinguishing* feature of Wittgenstein's Philosophy of Mathematics, from 1929 through 1944, is that expressions that (semantically) quantify over (i.e., speak about) "an infinite mathematical domain" are either non-mathematical expressions (e.g., *PR* §§126 & 174; *PG* 451–52) or must be understood in a non-standard, piecemeal way. In his intermediate period (1929–34), Wittgenstein maintains that an expression is a meaningful *mathematical* proposition *only* within a given calculus (*PG* 376, *PR* §152) and *iff* we *knowingly* have in hand an applicable and effective decision procedure by means of which we can decide it [(*PR* §§148–152), (*PG* 366, 387, 451–52, 468)]. There cannot be "undecidable propositions," Wittgenstein argues (*PR* §173), because such 'propositions' would have no 'sense,' "and the consequence of this is precisely that the propositions of logic lose their validity for [them]" — the Law of the Excluded Middle does *not* apply and, hence, "we aren't dealing with propositions of mathematics" [(*PR* §§121 & 151); cf. (*PG* 367–368)]. In Wittgenstein's later period, where these positions are somewhat softened, Wittgenstein says as late as 1942 that the 'proposition' "[T]he pattern ϕ (...e.g. '770') will occur in the infinite expansion of π" is 'queer' (*RFM* V, §9) and that if you say "the rule of expansion *determine*[s] the series completely" and hence that "it must implicitly determine *all* questions

[23] As noted above in Note #21, Steiner similarly says (p. 268): "The only way to avoid adopting Tarski's notion of truth (*other than, of course, refusing to extend the notion at all*) is to attack the theorem..." (italics mine).

about the structure of the series," "you are thinking of *finite* series" (*RFM* V, §11; italics on 'finite' mine). At (*RFM* V, §9), Wittgenstein reaffirms his intermediate position on algorithmic decidability, saying that "[t]he question... changes its status, when it becomes decidable," "[f]or a connexion is made then, which formerly *was not there*." And, at (*RFM* V, §§9–13, 17), Wittgenstein repeatedly stresses that the invocation of the Law of the Excluded Middle in such cases begs the question: "When someone hammers away at us with the law of excluded middle as something which cannot be gainsaid, it is clear that there is something wrong with his question" (*RFM* V, §10). This is an unorthodox view, yes, and it may be an indefensible view, but it is *Wittgenstein's* view, not an aberration from Wittgenstein's view.

The foregoing considerations show that Wittgenstein's (§8) 'riposte' does *not* "ignore[] Wittgenstein's own views about 'family resemblance' concepts and about mathematical proof." Wittgenstein has no Wittgensteinian reason to "want[] to extend the 'Russell notion' of truth... to cover the undecidable sentence P." The only Wittgensteinian reason that Wittgenstein could have for extending *PM* to include 'P' would be if 'P' *were needed* in an application of *PM* to the real world (i.e., in *mufti*), for "[i]t is the use outside mathematics, and so the *meaning* ['*Bedeutung*'] of the signs, that makes the sign-game into mathematics" (*RFM* V, §2; MS 126, October 28, 1942). But there is no such reason *for Wittgenstein*, since he doubts and/or denies that 'P' has any useful application. "Which extra-mathematical application can we give to Gödel's theorem?" (MS 163, 42r–42v; July 11, 1941), Wittgenstein rhetorically asks; "What application do we have for a proposition that mathematically asserts its own unprovability?" (42v). At (*RFM* App. III, §19) Wittgenstein explicitly says that we "can make no use of ['P']," and at (MS 121, 80r; December 31, 1938) he states that "[m]ost importantly, you do not have the slightest use for such a sentence."[24] Indeed, just as Steiner says "that the Gödel sentence P has no mathematical interest" (Steiner, 2001, 272), Wittgenstein says *in 1944* (MS 124, 115r; March 10): "What's unphilosophical in Gödel's essay is that he doesn't understand the relationship

[24] Cf. (MS 121; 34v–35r, June 9, 1938): "Someone once said mathematics is the handmaiden of the sciences; and whether or not it is now, its whole behaviour derives from the fact that it once *was*. In everything it imitates its earlier mistress." For Wittgenstein, this last fact has two faces. On the one face, a formal calculus is only a mathematical calculus (language-game) if it has an application to the real world in a realm of contingent/empirical propositions; on the other face, mathematics has, almost from the very beginning, borrowed the viewpoint and terminology of contingent/empirical propositions and the modern scientific approach, which has created the erroneous "conception of mathematics as the physics of 'mathematical objects"' (MS 163, 46r–46v; July 11, 1941). See note #30, below.

between mathematics and its application. In this, he maintains the slippery concepts of most mathematicians."[25]

6. *Steiner's Two Later Wittgensteins*

There is a very strong tension permeating Steiner's paper, which rears its head at least twice, and which reflects, I think, another tension, namely Steiner's strong appreciation for Wittgenstein's views and his erroneous belief that Wittgenstein's conceptions of "family resemblance," "organic relation," etc., should compel him to grant the forced extension of "true in PM." In effect, Steiner has two *later* Wittgensteins (i.e., 1937–44) at war with one another.

To see this, first note Steiner's claim (268) that "[t]he only way to avoid adopting Tarski's notion of truth (other than, of course, [1] refusing to extend the notion at all) is to [2] attack the theorem, so we return to my original conclusion: Wittgenstein's 'prose' is *in effect* ([...]) an attack on Gödel's theorem..." Here Steiner says that *of* the two possible ways "to avoid adopting Tarski's notion of truth" — i.e., [1] "refusing to extend the notion at all" *or* [2] "attack[ing] [Gödel's] theorem" — Wittgenstein chooses *not* to [1] refuse to extend the notion, but rather to [2] attack Gödel's theorem. In Footnote #41, Steiner quotes Stuart Shanker (1988) and makes two very interesting comments:

> Thus, when Shanker ([1988], p. 229), says, on behalf of Wittgenstein, "... the meaning of a proposition is strictly determined by the rules governing its use in a specific system. If dealing with autonomous calculi then no matter how similar the rules of the two systems might be, as long as they differ — as long as we are dealing with distinct mathematical systems — it makes no sense to speak of the *same* proposition occurring in each. The most that can be concluded is that *parallel* propositions occur in the two systems which can easily be mapped onto one another..." he is making the same mistake Wittgenstein made. (*For this reason, however, Shanker's passage probably does reflect Wittgenstein's point of view when writing his critique of Gödel!*) [Italics in parentheses mine; Shanker actually writes "the meaning of a mathematical proposition."]

[25] It is worth noting that these two sentences occur in MS 124 between the passages that were used for the 2nd and 3rd paragraphs of (*RFM* VII, §31) and the three paragraphs constitutive of (*RFM* VII, §31) are otherwise continuous in MS 124 (March 10, 1944). (Wittgenstein's "schleimigen Begriffe" could perhaps be translated as "slimy concepts," but this makes, I think, less sense of the passage than does "slippery concepts." In my (2002) I used "murky notions," though now I think that "slippery concepts" is preferable.)

In contrast with the parenthetical remark, Steiner immediately adds:

> The entire point is that the system in which the Gödel sentence is undecidable and that in which it is provable are *not* 'autonomous', given general properties of the universal quantifier and of truth which Wittgenstein certainly accepted in *RFM*. Since the extension of the incomplete Russell system is not arbitrary, it is reasonable to say that the Gödel sentence, though not decided in 'Russell's system' is still true, because it is decided in a forced extension.

Here Steiner says that Shanker *correctly* presents Wittgenstein's view of these matters as Wittgenstein viewed things when he wrote Appendix III (1937–38), namely, as [1] a flat-out *refusal* to accept the aforementioned extension because it is a contradiction-in-terms to extend *PM* to PM_2 on the grounds that *one and the same proposition*, namely 'P,' is "true but unprovable in *PM*" and "true and provable in PM_2." In other words, Steiner says that the Wittgenstein of (*RFM* App. III) [2] attacks Gödel's Theorem by [1] claiming that "a true but unprovable mathematical proposition of *PM*" is a contradiction-in-terms from one (i.e., *Wittgenstein's*) standpoint, but in saying this, Steiner grants that Wittgenstein [2] attacks Gödel's Theorem simply by [1] "refusing to extend the notion at all." Immediately after granting this, however, Steiner reverts to his view that Wittgenstein has good *Wittgensteinian* reasons to endorse the (full) extension, for "given general properties of the universal quantifier and of truth *which Wittgenstein certainly accepted in RFM*" (italics mine), Wittgenstein has good reasons to say that the two systems "are *not* 'autonomous'," and thus the best possible reasons to grant "that the Gödel sentence, though not decided in 'Russell's system' is still true, because it is decided in a forced extension."

There is a tension here and it is not easily resolved. On the one hand, Wittgenstein, according to Steiner, rejects the very notion of a true but unprovable proposition of *PM* for the very reasons I have detailed in this paper. On the other hand, Wittgenstein's *RFM* views on the "general properties of the universal quantifier and of truth", with, apparently, the single exception of (*RFM* App. III), *force* the endorsement of the extension all the way to Tarski-truth for mathematics. But whence the latter claim, which, evidently, immediately yields Steiner's conclusion? Indeed, if Wittgenstein in *RFM* 'certainly' held certain views on mathematical truth, universal quantification, and the non-autonomy of calculi such as *PM* and PM_2, why not straight-away quote the relevant passages and be done with it?

Two facts are salient in this connection. First, it is *anything but* clear or 'certain' that the later Wittgenstein of *RFM* — with the sole exception of (*RFM* App. III) — held these views. Steiner is making the *very strong* and *very contentious* claim that Wittgenstein's views in (*RFM* App. III), which may well be (and indeed *are*) consonant with his intermediate (1929–34) views (i.e., *PR* and *PG*), are *anomalous* with respect to the rest of the material in *RFM*. This means that (*RFM* App. III), written in 1937 and revised in 1938, is inconsistent with other *RFM* material written *earlier* (e.g., at least some of MS 117 used for *RFM* I), other *RFM* material written *later* (e.g., most of *RFM* II–VII; taken from, e.g., MSS 121 [1938–39], 122 [1939–40], 117 [1940], 125 [1942], 124 [1941, 1944], 126 [1942], 127 [1944], 164 [1941, 1944]), *and* other *RFM* material written at precisely the same time (MS 117, 1937; for TS 222 & 223, for *RFM* I).[26] Second, and most importantly, this highly contentious claim is in serious need of a supporting argument, but Steiner provides none.

To argue cogently for this/these new claim(s), Steiner would have to do three things: (1) provide proof text for these new claims; (2) demonstrate that Wittgenstein's (*RFM* VII, §§21–22) remarks on Gödel disagree with (*RFM* App. III) and agree with Steiner's new claims; and (3) demonstrate that the positions taken and arguments made in (*RFM* App. III) are truly anomalous in Wittgenstein's lectures and copious writings on mathematics from 1937–1944 (since, as Steiner well knows, *RFM* consists of selections from various MSS and TSS). As regards (1), I am not aware of *any* proof text for it and it is *certainly* not the case that "Wittgenstein *certainly* accepted in *RFM*" (last italics mine) the views here attributed to him by Steiner.[27] As for (2), at (*RFM* VII, §§21–22) Wittgenstein demonstrates a partially improved

[26] Most of (*RFM* App. III) was *written* during the three-day period of Sept. 22–24, 1937 (MS 118, 105v–114r), but Wittgenstein had it typed as part of TS 221 (which G.H. von Wright dates as 1938), *adding* (§20) from (MS 159; 24r–24v), which von Wright also dates as 1938 [G.H. von Wright, "The Wittgenstein Papers," in (1982), p. 47 (TS 221), p. 44 (MS 118), and p. 45 (MS 159)]. Since (*RFM* App. III) is taken from TS 223, which, according to von Wright (1982, 48), is "composed of cuttings from 221," and given especially that Wittgenstein made several hand-written changes to the relevant part of TS 221, which collectively *constitutes* TS 223, (*RFM* App. III) is best dated to the period 1937–1938. See my (2003, Note #23) for more detailed information about TS 223.

[27] Though an argument against Steiner's claim is well beyond the scope of this paper, I have argued in various papers that, in *RFM*, Wittgenstein maintains his intermediate finitism, with, seemingly, certain less-than-clear softenings. *At a minimum*, the matter is *not* clear-cut, contra Steiner. For example, it is hard to see that Wittgenstein "certainly accepted [general properties of the universal quantifier and of truth] in *RFM*" when one reads, e.g., (*RFM* V, §§9–25). "The symbols "$(x).\phi x$" and "$(\exists x).\phi x$," says Wittgenstein (*RFM* V, §13), "are certainly useful in mathematics so long as one is acquainted with the technique of the proofs of the existence or non-existence to which the Russellian signs *here* refer. If however this is left open, then these concepts of the old logic are extremely misleading." One of Wittgenstein's principal points here certainly seems to be that a putative ('queer') 'proposition' such as "The pattern '770' does/does not occur in the infinite decimal expansion of π" (*RFM* V, §9), which is *not* connected with a decision procedure or "technique of proof," "makes sense only" "where it is in the *rule* for this series, not to contain the pattern" (*RFM* V, §11) (i.e., when it is algorithmically decidable; cf. (*RFM* V, §13): "The general proposition that that pattern does not occur in the expansion can only be a *commandment*")). At (*RFM* V, §12),

understanding of Gödel's proof, but, as I have argued elsewhere,[28] Wittgenstein still erroneously thinks that Gödel's proof requires a self-referential proposition. And, finally, as regards (3), as I have argued here and elsewhere, the positions adopted and the arguments made in (*RFM* App. III) are a piece with Wittgenstein's lectures and writings on mathematics, including numerous passages not contained in *RFM*, where he continues to both reject the existence of "true but unprovable propositions of calculus Γ" and misunderstand Gödel's Theorem; they are anything but anomalous.[29]

There is, however, one line of argument hinted at by Steiner's reference to Wittgenstein's *certain* acceptance of views about mathematical truth and universal quantification, which bears mentioning here. Despite Wittgenstein's articulations of strongly formalist views from 1929 through 1944, one *can* find passages throughout that period which, one might argue, indicate an acceptance (i.e., in spite of himself) of semantics for mathematical calculi, including, perhaps, an intensional (if not referential) theory of meaning for mathematical terms and propositions. Though one can find, I believe, an intensional element in Wittgenstein's Philosophy of Mathematics, anyone who looks for strong evidence of a standard semantics for mathematical calculi will come up wanting.[30] To consider just one example, one might

Wittgenstein argues that the Law of the Excluded Middle cannot prove that this is a mathematical proposition since its invocation begs the question; at (*RFM* V, §25) he stresses that "[g]enerality in mathematics does not stand to particularity in mathematics in the same way as the general to the particular elsewhere" because "one can form a mathematical proposition in a grammatically correct way without understanding its meaning." This last statement is strongly reminiscent of Wittgenstein's intermediate position, especially when he says that we delude ourselves when we claim (*WVC* 37) that we can lay down principles or rules for well-formedness "(among which are 'all' and 'there is')" which enable us "to say whether the axioms are relevant to this proposition or not" (*PR* §149) and that (*PR* §126) "you can't say '$(n)\varphi n$', precisely because 'all natural numbers' isn't a bounded concept." At (*RFM* V, §11) Wittgenstein similarly says that if you say 'It must either reside in the rule for this series that the pattern occurs, or the opposite', "you are thinking of *finite* series" (italics mine). In the midst of these *RFM* remarks, when Wittgenstein (in)famously speaks of (V, §24) "'[t]he disastrous invasion" of mathematics by logic,' he adds that "logical technique is only an auxiliary technique in mathematics," which is a piece with his (*RFM* V, §25) claim that "one can form a mathematical proposition in a grammatically correct way without understanding its meaning." These passages are obviously very close in *RFM*; though there are some unpublished passages in between §24 and §25 in MS 126, §24 was written on November 28, 1942, and §25 was written on December 8, 1942.

[28] See my (2002) and (2003).

[29] The details are contained in my (2002) and (2003).

[30] In MS 163, where Wittgenstein wrestles with Gödel, he also wrestles with the idea that a mathematical proposition is or can be "contentually interpreted" (i.e., it allegedly has or can have a semantic *meaning*). If anything, Wittgenstein's ruminations indicate that this semantic

investigate Wittgenstein's views on universal quantification in mathematics by examining his various ruminations, claims, and arguments on mathematical induction. But here, however, the news is bad for Steiner's aims, for (at least the intermediate) Wittgenstein rejects the so-called 'conclusion' of a proof by mathematical induction, claiming that it is not even a mathematical proposition, but rather a pseudo-proposition standing proxy for a proved inductive base and inductive step.[31] Indeed, as I have argued, Wittgenstein's radical position on mathematical induction and his intermediate finitism are inextricably connected with his rejection of *undecidable* (yet meaningful) mathematical propositions. Wittgenstein, admittedly, does say if we know a "recursive proof" "with endless possibility" (*PR* §164), beginning "with '$A(1)$' and continu[ing] through '$A(2)$' etc. etc," the proof "spares me the trouble of proving each proposition of the form '$A(7)$'," he does say that the direct proof of, say, "$A(311)$" (i.e., without 310 iterations of *modus ponens*) "cannot have a still better proof, say, by my carrying out the derivation as far as this proposition itself" (*PR* §165), and he does say as late as 1939 (*LFM* 266) that it "is the queerest thing in the world... that one should have a short cut through logic," "[f]or if the proof of the proposition is the step-by-step proof, how can anything else be a proof of it?" "This is most important," Wittgenstein concludes, "[i]t's puzzled me more than I can say."[32] If one is looking for unequivocal acceptance of universal quantification in mathematics in the later Wittgenstein's work and writings (e.g., around 1938–39), the best that can be said for Steiner's aims is that it is not there to be found (or, much worse, that Wittgenstein's finitism is still radical in nature). Indeed, it is worth noting, as I and others (e.g., Floyd) have noted, that Wittgenstein's position on Gödel in *LFM* (1939), 1–2 years *after* he wrote (*RFM* App. III) is virtually identical to his articulation in (*RFM* App. III).

viewpoint is misguided and mistaken, for he says that "[t]he phrase "contentually interpreted" is a wretched, shoddy effort... aris[ing] from a false idea of the nature of the application of mathematics" (MS 163, 43r–43v; July 11, 1941), and that "[t]he whole idea of contentual interpretation rests on the conception of mathematics as the physics of 'mathematical objects"' (MS 163, 46r–46v; July 11, 1941). It should be noted that the (*RFM* VII, §§21–22) passages on Gödel were first written in (MS 163, 19rff; July 2–3, 1941) and then re-written, with some changes, in (MS 124, 87–95, July 2–3, 1941) on the very same two days. Thus, Wittgenstein's anti-semantical views of the time are *not* in opposition to his continued opposition to, and misunderstanding of, Gödel's Theorem.

[31] See (*WVC* 135), (*PR* §§129, 163–165), and (*PG* 406–407).

[32] Cf. (MS 121, 33r–33v; June 4, 1938): "If one understands the inductive proof as a short cut, then it is a short cut that leads as it were through a new space; as if one were shortening the way from here to Vienna by traveling *through* the earth instead of over its surface."

> One often hears statements about "true" and "false" — for example, that there are true mathematical statements which can't be proved in *Principia Mathematica*, etc. In such cases the thing is to avoid the words "true" and "false" altogether, and to get clear that to say that p is true is simply to assert p; and to say that p is false is simply to deny p or to assert $\sim p$. It is not a question of whether p is "true in a different sense". It is a question of whether we assert p. (*LFM* 188)

In sum, this demonstrates, viz. (3), that the positions taken and arguments made in (*RFM* App. III) are not anomalous in Wittgenstein's writings in 1938 and his lectures after 1938. I refer the reader to my previous papers, which show that those positions are *not* anomalous relative to Wittgenstein's numerous MSS devoted to mathematics.

It is perhaps worth asking: "Who would be convinced by Steiner's arguments to extend her/his concept of mathematical truth?" If, as I have argued here, Wittgenstein is not his own worst enemy, is there a notable someone contemporaneous with Wittgenstein, who, if not Wittgenstein's worst enemy, might be moved by the very arguments that Steiner thinks should move Wittgenstein? I propose David Hilbert, circa 1929–1930, as just such a person. In his (1929), Hilbert posed the problem of a completeness proof for elementary number theory. He thought, it seems, that such a proof was possible. Let us imagine, therefore, that, in 1930, Hilbert accepted that "P is true in PA *iff* P is provable in PA." Faced with Gödel's 1931 proof, Hilbert might well have extended his conception of "true in PA" feeling *forced* as Steiner describes. But Hilbert, a life-long Kantian, had a very good and natural reason to respond in this fashion: he claimed that $(\forall n)$F(n) is true *iff* F(1), F(2), F(3),... are true — if one can prove that F(1), F(2), F(3),... are provable, one has proved that $(\forall n)$F(n) is true, even if, as it turned out, one cannot always prove $(\forall n)$F(n) within PA. Naturally, Hilbert would have been compelled by Steiner's forced extension, but Hilbert would have been so compelled because he (then) viewed F(1), F(2), F(3), and $(\forall n)$F(n) as meaningful, 'contentual' propositions. (Indeed, Hilbert's 1931 introduction of the ω–rule is ample evidence of this.)[33]

But Hilbert was never a pure formalist. 'Contentual' mathematical propositions have meaning, he claimed, and they are true or false because of their respective meanings, which enables one to ask, as he did, whether all truths

[33] See (Hilbert 1931, 271). Indeed, as Daniel Isaacson says ("Some Considerations on Arithmetical Truth and the ω–Rule," in M. Detlefsen (ed), *Proof, Logic and Formalization* (London: Routledge, 1992), p. 101), "Given the relative timing of this work of Hilbert and Gödel's discovery of incompleteness in formal systems for arithmetic (1931a), it is natural to wonder whether Hilbert's idea to use the ω–rule in his (1931a) paper was in response to the Gödel incompleteness theorems." "Feferman considers this question carefully (1986: 208) and concludes that the available evidence offers no clear-cut answer."

of a certain kind (e.g., number-theoretic truths) are provable in a particular system. For Wittgenstein *qua* formalist, on the other hand, one cannot ask this question, for "true in PM" just means "proved/provable in PM," and this is so even if and when he made extra-systemic application a necessary condition of a mathematical calculus (versus a mere "sign-game"). Thus, while Hilbert, in 1929–1930, would have been compelled by the considerations described by Steiner, Wittgenstein, in 1929–30 (*or* in 1937–44), would certainly *not* have been moved, as is evidenced by his numerous intermediate statements *against* meaningful and undecidable *mathematical propositions*.[34]

Conclusion

If the arguments of this paper are sound, Steiner fails to show that Wittgenstein's "remarks on Gödel's theorem are an aberration" simply because they are *not*. Steiner's enhanced compatibility argument fails because it reduces, rather than enhances, the compatibility of Wittgenstein's views. Wittgenstein, *qua* Radical Constructivist, always identifies "true in calculus Γ" and "proved/provable in calculus Γ" and in (*RFM* App. III) he *explicitly* uses this identification to argue against the Gödelian. Moreover, Wittgenstein's formalism and his conception of intra-systemic, syntactical "mathematical truth" are *far more central* to his Philosophy of Mathematics than his looser comments on mathematics and family resemblance, which, as we have seen, simply cannot support an inference to the claim that Wittgenstein views truth and/or mathematical truth as (a) family resemblance concept(s).[35] It follows that when discussing Gödel's theorem, (directly and indirectly) from 1937 through 1944, Wittgenstein is not his own worst enemy, violating central tenets of his Philosophy of Mathematics; rather he wrestles with Gödel's

[34] See, e.g., (*PR* §173–174) and (*PG* 376).

[35] Another way to put this point is to ask: *Could* Wittgenstein claim that truth and/or mathematical truth is/are (a) family resemblance concept(s)? *Could* Wittgenstein make this or these claims *without* giving up central theses that he held (for, perhaps, his entire philosophical life)? The answer is 'No.' As I have said, Wittgenstein *always* strongly contrasts contingent/empirical propositions and *truth* with mathematical 'propositions' and "mathematical truth." It, therefore, would be utterly shocking if Wittgenstein makes this or these claims anywhere in the *Nachlaß*. Thus, although Steiner's Wittgenstein must make this or these claims since Steiner's Wittgenstein says "mathematical truth is multicolored," the real Wittgenstein cannot possibly make such claims (as is borne out by the textual evidence).

theorem from a point of view he consistently holds from 1929 through at least 1944.[36]

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