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PARACONSISTENT LOGICS WITH SIMPLE SEMANTICS

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Paraconsistent logics are designed to isolate the effect of contradictions in reasoning so that not all propositions are derviable from an inconsistency. In this paper I will show how contradictions can be isolated in terms of their semantic content.

Subject matter relatedness logic S, first presented in 1979, is designed to model the idea that the subject matter of propositions should be taken into account in reasoning, where the notion of the subject matter of a proposition is taken as primitive. By modifying the semantic consequence relation of S to take account of content as well as truth-values of propositions, we can create a paraconsistent logic in which from a contradiction only propositions related in subject matter to the contradiction can be derived. These modified semantics bring into question Tarski's axioms for consequence operators.

The same method can be used to create a paraconsistent logic based on dependence logic, D, in which from a contradiction only propositions whose referential content is contained in that of the contradiction can be derived. Paraconsistent versions of two similar logics are also investigated, with a comparison of how they model intuitions about content and contradictions.

Relatedness Logic

The propositional logic S was first presented in *Epstein, 1979*. It was further developed with a strongly complete axiomatization and clearer motivation in *Epstein, 1990*, from which this brief description is culled.

A *subject matter relatedness model* is:

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$$\mathrm{L}(\neg, \rightarrow, \mathrm{p}_0, \mathrm{p}_1, \ldots)$$

$$\Big\downarrow \quad \text{realization}$$

$$\{p_0, p_1, \ldots, \text{complex propositions formed from these using } \neg, \rightarrow\}$$

$$\Big\downarrow \ \mathfrak{v}, \mathfrak{s}, \text{ and truth-tables}$$

$$\{\mathsf{T}, \mathsf{F}\}$$

Here p_0, p_1, ... are the realizations of p_0, p_1, ..., which we take to be atomic. They are English language sentences; from a realist perspective these would be understood to correspond to or represent abstract propositions. And here $\mathfrak{v}$ is an assignment of truth-values to the atomic propositions. We also have a *subject matter assignment*:

a set of topics $\mathfrak{S} \neq \varnothing$;
an assignment $\mathfrak{s}$ that for every atomic proposition p,

$$\mathfrak{s}(\mathrm{p}) \subseteq \mathfrak{S} \text{ and } \mathfrak{s}(\mathrm{p}) \neq \varnothing.$$

The extension of $\mathfrak{s}$ to compound propositions is given by taking the subject matter of a proposition to be the sum of the subject matters of its parts:

$$\mathfrak{s}(\mathrm{A}) = \bigcup\{\mathfrak{s}(\mathrm{p}) : \text{ p appears in A}\}$$

The subject matter assignment determines a (*subject matter*) *relatedness relation* on the propositions of the model, meant to model the notion that A has subject matter overlap with B:

$$\mathfrak{R}(\mathrm{A}, \mathrm{B}) \text{ iff } \mathfrak{s}(\mathrm{A}) \cap \mathfrak{s}(\mathrm{B}) \neq \varnothing$$

The truth-value assignment $\mathfrak{v}$ is extended inductively to all propositions by using the classical table for $\neg$ and the table for the related conditional for $\rightarrow$:

A	B	$\mathfrak{R}(\mathrm{A}, \mathrm{B})$	$\mathrm{A} \rightarrow \mathrm{B}$
any	value	fails	F
T	T		T
T	F	holds	F
F	T		T
F	F		T

A proposition A of the semi-formal language is *true* in the model if $\mathfrak{v}(A) = \mathsf{T}$, *false* if $\mathfrak{v}(A) = \mathsf{F}$. We refer to a model as $\mathfrak{M} =< \mathfrak{v}, \mathfrak{s} >$, and write $\mathfrak{M} \vDash A$ for $\mathfrak{v}(A) = \mathsf{T}$.

In *Epstein, 1990* it is shown that the class of relatedness relations is characterized as those that satisfy:

R1. $\mathfrak{R}(A, A)$

R2. $\mathfrak{R}(A, B)$ iff $\mathfrak{R}(\neg A, B)$

R3. $\mathfrak{R}(A, B)$ iff $\mathfrak{R}(B, A)$

R4. $\mathfrak{R}(A, B \to C)$ iff $\mathfrak{R}(A, B)$ or $\mathfrak{R}(A, C)$

It is also shown that this is equivalent to $\mathfrak{R}$ being a reflexive, symmetric relation on the propositional variables (propositions) such that $\mathfrak{R}(A, B)$ iff for some p in A, some q in B, $\mathfrak{R}(p, q)$. From R1–R4 we have that the only transitive relatedness relation is the universal relation on wffs, since for any A, B, both $\mathfrak{R}(A, A \to B)$ and $\mathfrak{R}(A \to B, B)$.

Letting capital Greek letters stand for collections of propositions (atomic or compound) we have the following usual definition of semantic consequence.

Semantic consequence in S

$\Gamma \vDash_{\mathsf{S}} A$ iff for any model $\mathfrak{M}$, if for every B in Γ, $\mathfrak{M} \vDash B$, then $\mathfrak{M} \vDash A$.

(*Subject matter*) *relatedness logic* S is then the collection of these models using this semantic consequence relation.

Since the universal relation is a relatedness relation, the logic S is a sublogic of classical propositional logic, that is: If $\Gamma \vDash_{\mathsf{S}} A$, then $\Gamma \vDash_{\text{classical}} A$.

We make the following definitions in S:

$$A \wedge B \quad \equiv_{\text{Def}} \quad \neg(A \to (B \to \neg((A \to B \to (A \to B)))))$$
$$\mathfrak{R}(A, B) \quad \equiv_{\text{Def}} \quad A \to (B \to B)$$

Then, as you can check:

$$< \mathfrak{v}, \mathfrak{s} > \vDash A \wedge B \quad \text{iff} \quad < \mathfrak{v}, \mathfrak{s} > \vDash A \text{ and } < \mathfrak{v}, \mathfrak{s} > \vDash B$$
$$< \mathfrak{v}, \mathfrak{s} > \vDash \mathfrak{R}(A, B) \quad \text{iff} \quad \mathfrak{s}(A) \cap \mathfrak{s}(B) \neq \varnothing$$
$$\text{iff} \quad < \mathfrak{v}, \mathfrak{s} > \vDash (A \wedge \neg A) \to B.$$

The connective $\vee$ can be defined from $\neg$ and $\rightarrow$ as either classical disjunction or as relatedness disjunction. Since that decision is inessential and distracting, I will not consider disjunction in the discussions that follow.

In *Epstein, 1990* a strongly complete axiomatization of S is given using the definition of R plus R1–R4: A finite number of axiom schema with the single rule *modus ponens* suffice such that for any collection of wffs Γ, $\Gamma \vdash_{\mathsf{S}}$ A iff $\Gamma \models_{\mathsf{S}}$ A.

In *Epstein and Krajewski, 2004* the semantics of S are extended to the language of predicate logic with relations, equality, and names. An axiomatization for that language is described, with a proof that it is strongly complete to appear in *Epstein, 200?*.

Paraconsistent Relatedness Logic

The Deduction Theorem fails for the logic S. That is, we do not have: A $\models_{\mathsf{S}}$ B iff $\models_{\mathsf{S}}$ A $\rightarrow$ B. In particular, for any distinct propositional variables p and q, p $\models$ q $\rightarrow$ q, but $\not\models$ p $\rightarrow$ (q $\rightarrow$ q). An implication is not equivalent to the validity of a conditional. Thus we have two notions of "follows from" in our logic. Compare:

$\not\models_{\mathsf{S}}$ (The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\rightarrow 2+2=4$

(The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\models_{\mathsf{S}} 2+2=4$

The Deduction Theorem fails because the conditional takes into account both content and truth-values, while semantic consequence in relatedness logic takes into account only truth-values. In 1985 Newton da Costa suggested modifying the semantic consequence relation to incorporate relatedness (see *Epstein, 1990*, p. 123). A variation of his idea is the following definition.

Related semantic consequence

$\models_{\mathsf{PS}}$ A iff $\models_{\mathsf{S}}$ A.

For $\Gamma \neq \varnothing$, $\Gamma \models_{\mathsf{PS}}$ A iff there are $C_1, \ldots, C_n$ in Γ such that:

(i) if for every B in Γ, $\mathfrak{M} \models$ B, then $\mathfrak{M} \models$ A, and

(ii) for any model $\mathfrak{M} = < \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(C_1 \wedge \ldots \wedge C_n) \cap \mathfrak{s}(A) \neq \varnothing$.

That is, A follows from a collection of propositions iff it is impossible for all those propositions to be true and A to be false, and A is related in subject matter to some finite collection of those.

Paraconsistent relatedness logic

PS is the logic of the semantics of relatedness logic except that semantic consequence is taken to be related semantic consequence.

In PS, then, negation is classical, and there is only one negation. Hence, in formalizations of ordinary language propositions the word "not" is modeled as in classical logic. There is no recourse to any defined connective meant to be interpreted as "is consistent" as in logics of formal inconsistency (see the survey of paraconsistent logics *Carnielli and Marcos, 2002*). It is the conditional and the consequence relation that are not classical.

I leave the proof of the following to you.

Theorem 1 Semantic consequence in PS

a. *The Deduction Theorem*: $A \vDash_{\mathsf{PS}} B$ iff $\vDash_{\mathsf{PS}} A \rightarrow B$.

b. *Variable sharing criteria*: If $\Gamma \vDash_{\mathsf{PS}} B$, then some propositional variable (atomic proposition) that appears in B appears also in some wff (proposition) in Γ.

c. $\Gamma \vDash_{\mathsf{PS}} A$ iff there is some C in Γ such that for any model $\mathfrak{M} = <\mathfrak{v}, \mathfrak{s}>$, $\mathfrak{s}(C) \cap \mathfrak{s}(A) \neq \varnothing$ and if for every B in Γ, $\mathfrak{M} \vDash B$, then $\mathfrak{M} \vDash A$.

The semantics of PS are clear and simple. In terms of formalizations of ordinary language propositions the discussions for S in *Epstein, 1990* and *Epstein and Krajewski, 2004* apply. And the semantics are paraconsistent.

Theorem 2 The paraconsistent nature of PS

a. $(A \wedge \neg A) \vDash_{\mathsf{PS}} B$ iff in every model $<\mathfrak{v}, \mathfrak{s}>$, $\mathfrak{s}(A) \cap \mathfrak{s}(B) \neq \varnothing$.

b. $(A \wedge \neg A) \vDash_{\mathsf{PS}} B$ iff there is some propositional variable (proposition) that appears in both A and B.

c. There is no formula A such that for all B, $(A \wedge \neg A) \vDash_{\mathsf{PS}} B$.

d. *Strong non-explosiveness of* PS *(finite non-trivializability)*
There is no finite collection of wffs Γ such that for all B, $\Gamma \vDash_{\mathsf{PS}} B$.

One can thus trace the effect of a contradiction in a system of propositions by considering its subject matter. Since we have considerable freedom in establishing the relatedness relation, it should be possible to apply this system to databases, isolating the effect of contradictions via their subject matter or place in a relatedness relation.

From Theorem 2 we see that not every wff follows from a contradiction, but only those wffs that must have some subject matter in common with the contradiction. Similarly, not every wff entails a valid wff, but only those that must have some subject matter in common with the valid wff. Valid wffs are isolated in terms of what they are consequences of just as contradictions are isolated in terms of what consequences they have.

Theorem 3 Valid wffs as consequences in PS

a. $\mathrm{A} \models_{\mathsf{PS}} \mathrm{B} \rightarrow \mathrm{B}$ iff in every model $< \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(\mathrm{A}) \cap \mathfrak{s}(\mathrm{B}) \neq \varnothing$.

b. $\mathrm{A} \models_{\mathsf{PS}} \mathrm{B} \rightarrow \mathrm{B}$ iff there is some propositional variable (proposition) that appears in both A and B.

c. There is no formula A such that for all B, $\mathrm{A} \models_{\mathsf{PS}} \mathrm{B} \rightarrow \mathrm{B}$.

We can utilize the proof theory of S to define a proof theory for PS.

Axiomatization of PS

$\vdash_{\mathsf{PS}} \mathrm{A}$ iff $\vdash_{\mathsf{S}} \mathrm{A}$.

For $\Gamma \neq \varnothing$, $\Gamma \vdash_{\mathsf{PS}} \mathrm{A}$ iff there is a finite collection of wffs $\mathrm{C}_1, \ldots, \mathrm{C}_n$ in Γ such that $\vdash_{\mathsf{S}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \rightarrow \mathrm{A}$.

Theorem 4 The axiomatization of PS is *strongly complete*. That is, for any Γ, $\Gamma \vdash_{\mathsf{PS}} \mathrm{A}$ iff $\Gamma \models_{\mathsf{PS}} \mathrm{A}$.

Proof. The theorem is true by definition if Γ is empty. So suppose $\Gamma \neq \varnothing$.

If $\Gamma \vdash_{\mathsf{PS}} \mathrm{A}$, then for some $\{\mathrm{C}_1, \ldots, \mathrm{C}_n\} \subseteq \Gamma$, $\vdash_{\mathsf{S}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \rightarrow \mathrm{A}$. So by the completeness of the axiomatization of S, $\models_{\mathsf{S}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \rightarrow \mathrm{A}$. Hence, in every model $\mathfrak{M} =< \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \cap \mathfrak{s}(\mathrm{A}) \neq \varnothing$, and if $\mathfrak{M} \models \mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n$, then $\mathfrak{M} \models \mathrm{A}$. That is, $\Gamma \models_{\mathsf{PS}} \mathrm{A}$.

Now suppose $\Gamma \models_{\mathsf{PS}} \mathrm{A}$. Then for some $\{\mathrm{C}_1, \ldots, \mathrm{C}_n\} \subseteq \Gamma$, we have (i) and (ii) as in the definition of semantic consequence. By (i), $\Gamma \models_{\mathsf{S}} \mathrm{A}$. Hence by the completeness of the axiomatization of S, we have $\Gamma \vdash_{\mathsf{S}} \mathrm{A}$. Since the syntactic consequence relation of S, defined in the usual way in terms of axioms and rules, is compact, for some $\{\mathrm{D}_1, \ldots, \mathrm{D}_m\} \subseteq \Gamma$, $\{\mathrm{D}_1, \ldots, \mathrm{D}_m\} \vdash_{\mathsf{S}} \mathrm{A}$. Hence, $\{\mathrm{C}_1, \ldots, \mathrm{C}_n\} \cup \{\mathrm{D}_1, \ldots, \mathrm{D}_m\} \vdash_{\mathsf{S}} \mathrm{A}$, so

$$\{\mathrm{C}_1, \ldots, \mathrm{C}_n\} \cup \{\mathrm{D}_1 \wedge \ldots \wedge \mathrm{D}_m\} \models_{\mathsf{S}} \mathrm{A}.$$

By (i) we then have $\models_{\mathsf{S}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n \wedge \mathrm{D}_1 \wedge \ldots \wedge \mathrm{D}_m) \rightarrow \mathrm{A}$, and so by the completeness of the axiomatization of S, $\vdash_{\mathsf{S}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n \wedge \mathrm{D}_1 \wedge \ldots \wedge \mathrm{D}_m) \rightarrow \mathrm{A}$. Hence, $\Gamma \vdash_{\mathsf{PS}} \mathrm{A}$. □

By Theorem 4, we have that the conditions for $\Gamma \models_{\mathsf{PS}} \mathrm{A}$ are equivalent to:

(‡) There are $C_1, ..., C_n$ in Γ such that $\models_{\mathsf{S}} (C_1 \wedge ... \wedge C_n) \rightarrow A$.

For an extension of PS to the language of predicate logic with relations, equality, and names, we can modify the semantics of the extension of S to that language and use instead related semantic consequence. In *Epstein and Krajewski, 2004* a strongly complete axiomatization of the extension of S to the language of predicate logic is described, and a strongly complete axiomatization of PS in the language of predicate logic can then be given in the same manner as for the propositional logic.

Returning to the propositional case, we find some anomalies in this logic.

Theorem 5

a. It is not the case that Γ, A $\models_{\mathsf{PS}}$ B iff $\Gamma \models_{\mathsf{PS}} A \rightarrow B$.

b. The semantic consequence relation of PS is not transitive.

c. It is not the case that if $\models_{\mathsf{PS}}$ A then for any Γ, $\Gamma \models_{\mathsf{PS}}$ A.

Proof. For part (a), letting p, q be distinct propositional variables:

$$\{p, q\} \models_{\mathsf{PS}} q \text{ but } \{p\} \not\models_{\mathsf{PS}} q \rightarrow q$$

$$\{p \wedge \neg p\} \models_{\mathsf{PS}} p \rightarrow q, \text{ but } \{p \wedge \neg p, p\} \not\models_{\mathsf{PS}} q$$

For part (b), letting p, q, r to be distinct propositional variables:

$$\{p \wedge \neg p\} \models_{\mathsf{PS}} (p \wedge \neg p) \wedge (q \wedge \neg q)$$

$$\{(p \wedge \neg p) \wedge (q \wedge \neg q)\} \models_{\mathsf{PS}} (q \wedge \neg q) \wedge (r \wedge \neg r)$$

$$\{p \wedge \neg p\} \not\models_{\mathsf{PS}} (q \wedge \neg q) \wedge (r \wedge \neg r)$$

Part (c) follows by Theorem 3.c. □

In 1930, Alfred Tarski presented an axiomatic analysis of consequence operations. Besides assuming that the set of sentences is denumerable, his three axioms were:

$$\Gamma \subseteq \mathrm{Cn}(\Gamma)$$

$$\mathrm{Cn}(\Gamma) = \bigcup\{\mathrm{Cn}(\Sigma) : \Sigma \text{ is finite and } \Sigma \subseteq \Gamma\}$$

$$\mathrm{Cn}(\mathrm{Cn}(\Gamma)) = \mathrm{Cn}(\Gamma)$$

Tarski did not present his theory as prescriptive, that is, every deductive system should satisfy the axioms, but only descriptive.

> We will now state four axioms which express certain elementary properties of the primitive concepts and are satisfied in all known formalized disciplines. p. 63

With PS we have a semantically well-motivated example of a consequence operation that satisfies only the first axiom of Tarski's theory: the second axiom fails due to Theorem 5.c, and the third by Theorem 5.b*.

However, a simple modification of the semantics of PS yields a different paraconsistent logic whose consequence relation is transitive. It is based on the propositional logic D, which is very similar to S.

Dependence Logic

The propositional logic D was first presented in *Epstein, 1990* to model the idea that in a true conditional the consequent must be contained in the antecedent.

To define the logic D we modify the notion of a model for S. Set-assignments are defined as before, except that we understand them in terms of

* T.J. Smiley, 1959 also proposed a consequence relation that is not transitive. He modified the classical syntactic consequence relation by defining:

> $A_1, ..., A_n \vdash B$ iff $(A_1 \wedge ... \wedge A_n) \supset B$ is a substitution instance of a tautology of classical logic $(A'_1 \wedge ... \wedge A'_n) \supset B'$ such that neither $\neg(A'_1 \wedge ... \wedge A'_n)$ nor B' is a classical tautology.

Smiley comments on the issue of non-transitivity by saying:

> The need for an unrestrictedly transitive entailment-relation for serious logical work is no reason at all against accepting a relation which is not unrestrictedly transitive as being a satisfactory reconstruction of an intuitive idea of entailment. But the need itself is undeniable: the whole point of logic as an instrument, and the way in which it brings us new knowledge, lies in the contrast between the transitivity of "entails" and the non-transitivity of "obviously entails," and all this is lost if transitivity cannot be relied on. Of course if there is an effective way of predicting when transitivity will hold then most of the objection vanishes; there is such a way where the present system is concerned, for it is decidable. p. 242

His logic, too, is paraconsistent, since for distinct p and q, $(p \wedge \neg p) \nvdash q$.

referential content. In a model, we define a *dependence relation* $\mathfrak{D}$ to replace the relatedness relation $\mathfrak{R}$ governing the truth-table for the conditional:

$$\mathfrak{D}(\mathrm{A},\mathrm{B}) \text{ iff } \mathfrak{s}(\mathrm{A}) \supseteq \mathfrak{s}(\mathrm{B})$$

The evaluation of $\neg$ in a model is classical, and the evaluation of the conditional is just as for S except reading $\mathfrak{D}$ for $\mathfrak{R}$. Though classical conjunction, $\wedge$, can be defined in this logic, in this presentation I will take $\wedge$ to be an additional primitive connective that we will interpret as classical conjunction. *Dependence logic* D is the collection of these models using the semantic consequence relation defined in the usual way.

Semantic consequence in D
$\Gamma \vDash_{\mathsf{D}}$ A iff for any model $\mathfrak{M}$, if for every B in Γ, $\mathfrak{M} \vDash$ B, then $\mathfrak{M} \vDash$ A.

We define in D:

$$\mathrm{D}(\mathrm{A},\mathrm{B}) \equiv_{\mathrm{Def}} \mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{B})$$

Then for any model $< \mathfrak{v}, \mathfrak{s} >$ of dependence logic:

$$\begin{aligned} < \mathfrak{v}, \mathfrak{s} > \vDash \mathrm{D}(\mathrm{A},\mathrm{B}) \quad & \text{iff} \quad \mathfrak{s}(\mathrm{A}) \supseteq \mathfrak{s}(\mathrm{B}) \\ & \text{iff} \quad < \mathfrak{v}, \mathfrak{s} > \vDash (\mathrm{A} \wedge \neg \mathrm{A}) \rightarrow \mathrm{B} \end{aligned}$$

In *Epstein, 1990* it is shown that dependence relations can be characterized as those satisfying:

$\mathfrak{D}$ is reflexive

$\mathfrak{D}$ is transitive

$\mathfrak{D}(\neg\mathrm{A}, \mathrm{A})$

$\mathfrak{D}(\mathrm{A}, \neg\mathrm{A})$

$\mathfrak{D}(\mathrm{A}, \mathrm{B} \wedge \mathrm{C})$ iff $\mathfrak{D}(\mathrm{A}, \mathrm{B})$ and $\mathfrak{D}(\mathrm{A}, \mathrm{C})$

$\mathfrak{D}(\mathrm{A}, \mathrm{B} \rightarrow \mathrm{C})$ iff $\mathfrak{D}(\mathrm{A}, \mathrm{B})$ and $\mathfrak{D}(\mathrm{A}, \mathrm{C})$

Using this characterization and the definition of D, a strongly complete axiomatization of D is given *in Epstein, 1990*: $\Gamma \vdash_{\mathsf{D}}$ A iff $\Gamma \vDash_{\mathsf{D}}$ A.

Paraconsistent Dependence Logic

Paraconsistent dependence logic can be built from D just as paraconsistent relatedness logic is built from S.

Dependence semantic consequence

$\vDash_{\mathsf{PD}}$ A iff $\vDash_{\mathsf{D}}$ A.

For $\Gamma \neq \varnothing$, $\Gamma \vDash_{\mathsf{PD}}$ A iff there are $C_1, \ldots, C_n$ in Γ such that

(i) if for every B in Γ, $\mathfrak{M} \vDash$ B, then $\mathfrak{M} \vDash$ A, and

(ii) for any model $\mathfrak{M} = < \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(C_1 \wedge \ldots \wedge C_n) \supseteq \mathfrak{s}(A)$.

Paraconsistent dependence logic

PD is the logic of the semantics of dependence logic but with dependence semantic consequence.

The semantic consequence relation of paraconsistent dependence logic satisfies the Deduction Theorem and is transitive, as summarized in the following theorem, whose proof I leave to you.

Theorem 6 Semantic consequence in PD

a. The Deduction Theorem A $\vDash_{\mathsf{PD}}$ B iff $\vDash_{\mathsf{PD}}$ A $\rightarrow$ B.

b. Transitivity of semantic consequence If $\Gamma \vDash_{\mathsf{PD}}$ A and A $\vDash_{\mathsf{PD}}$ B, then $\Gamma \vDash_{\mathsf{PD}}$ B.

c. The cut rule If $\Gamma \vDash_{\mathsf{PD}}$ A and Δ, A $\vDash_{\mathsf{PD}}$ B, then $\Gamma \cup \Delta \vDash_{\mathsf{PD}}$ B.

d. Variable sharing criteria If $\Gamma \vDash_{\mathsf{PD}}$ B, then every propositional variable (atomic proposition) that appears in B also appears in some wff (proposition) in Γ.

Unlike paraconsistent relatedness logic, we cannot reduce the containment clause in semantic consequence to using only one proposition in Γ (Theorem 1.c). For example, $\{p, q\} \vDash_{\mathsf{PD}} p \wedge q$, but $p \nvDash_{\mathsf{PD}} p \wedge q$ and $q \nvDash_{\mathsf{PD}} p \wedge q$.

I leave the proof of the next two theorems to you.

Theorem 7 The paraconsistent nature of PD

a. $(A \wedge \neg A) \vDash_{\mathsf{PD}}$ B iff in every model $< \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(A) \supseteq \mathfrak{s}(B)$.

b. $(A \wedge \neg A) \vDash_{\mathsf{PD}}$ B iff every propositional variable (proposition) that appears in B appears in A.

c. There is no formula A such that for all B, $(A \wedge \neg A) \vDash_{\mathsf{PD}}$ B.

d. Strong non-explosiveness of PD
There is no finite collection of wffs Γ such that for all B, $\Gamma \vDash_{\mathsf{PD}}$ B.

e. If for all B, $\Gamma \models_{\mathsf{PD}}$ B, then for each propositional variable in B there is a formula in Γ in which that variable appears.

Theorem 8 Valid wffs as consequences in PD

a. A $\models_{\mathsf{PD}}$ B $\rightarrow$ B iff in every model $< \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(\mathrm{A}) \supseteq \mathfrak{s}(\mathrm{B})$.

b. A $\models_{\mathsf{PD}}$ B $\rightarrow$ B iff every propositional variable (proposition) that appears in B appears also in A.

c. There is no formula A such that for all B, A $\models_{\mathsf{PD}}$ B $\rightarrow$ B.

Theorem 9

a. If $\Gamma \models_{\mathsf{PD}}$ A $\rightarrow$ B, then Γ, A $\models_{\mathsf{PD}}$ B.

b. It is not the case that if Γ, A $\models_{\mathsf{PD}}$ B then $\Gamma \models_{\mathsf{PD}}$ A $\rightarrow$ B.

c. It is not the case that if $\models_{\mathsf{PD}}$ A then for any Γ, $\Gamma \models_{\mathsf{PD}}$ A.

Proof. Part (a) follows because if there are $\mathrm{C}_1, \ldots, \mathrm{C}_n$ in Γ such that in $< \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \supseteq \mathfrak{s}(\mathrm{A}) \cup \mathfrak{s}(\mathrm{B})$, then $\mathfrak{s}(\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n \wedge \mathrm{A}) \supseteq \mathfrak{s}(\mathrm{B})$.

For part (b), $\{\mathrm{p}, \mathrm{q}\} \models_{\mathsf{PD}}$ q but $\{\mathrm{p}\} \not\models_{\mathsf{PD}}$ q $\rightarrow$ q.

Part (c) follows by Theorem 8.b. □

Axiomatization of PD

$\vdash_{\mathsf{PD}}$ A iff $\vdash_{\mathsf{D}}$ A.

For $\Gamma \neq \varnothing$, $\Gamma \vdash_{\mathsf{PD}}$ A iff there is a finite collection of wffs $\mathrm{C}_1, \ldots, \mathrm{C}_n$ in Γ such that $\vdash_{\mathsf{D}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \rightarrow \mathrm{A}$.

The proof of the following is similar to that for Theorem 4.

Theorem 10 The axiomatization of PD is strongly complete. That is, for any Γ, $\Gamma \vdash_{\mathsf{PD}}$ A iff $\Gamma \models_{\mathsf{PD}}$ A.

By Theorem 10 we have that the conditions for $\Gamma \models_{\mathsf{PD}}$ A are equivalent to:

There are $\mathrm{C}_1, \ldots, \mathrm{C}_n$ in Γ such that $\models_{\mathsf{D}} (\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \rightarrow \mathrm{A}$.

There is an infinitistic aspect to dependence that is not found in relatedness. In a model we may have an infinite collection Γ and some A such that $\bigcup\{\mathfrak{s}(\mathrm{C}) : \mathrm{C} \text{ is in } \Gamma\} \supseteq \mathfrak{s}(\mathrm{A})$, but for no finite collection $\mathrm{C}_1, \ldots, \mathrm{C}_n$ in Γ is it the case that $\mathfrak{s}(\mathrm{C}_1 \wedge \ldots \wedge \mathrm{C}_n) \supseteq \mathfrak{s}(\mathrm{A})$. Thus, if one considers infinite collections of propositions on a model-by-model basis, dependence relations are considerably more complicated than relatedness relations. It might seem, then, that we should adopt a different notion of semantic consequence:

$\Gamma \vDash$ A iff if for every B in Γ, $\mathfrak{M} \vDash$ B, then $\mathfrak{M} \vDash$ A, and
for any model $\mathfrak{M} = <\mathfrak{v}, \mathfrak{s}>$, $\bigcup\{\mathfrak{s}(\mathrm{C}) : \mathrm{C} \text{ is in } \Gamma\} \supseteq \mathfrak{s}(\mathrm{A})$.

But this would make no difference for the consequence relation. Any A contains only finitely many propositional variables (atomic propositions), say q_1, ..., q_n. Hence, if in every model, $\bigcup\{\mathfrak{s}(\mathrm{C}) : \mathrm{C} \text{ is in } \Gamma\} \supseteq \mathfrak{s}(\mathrm{A})$, it must be that every propositional variable appearing in A must appear in some wff in C. Let C_i, $i \leq n$, be wffs in Γ such that q_i appears in C_i. Then in every model, $\mathfrak{s}(C_1 \wedge ... \wedge C_n) \supseteq \mathfrak{s}(\mathrm{A})$.

The consequence operation of PD satisfies the first and third axioms of Tarski's theory of consequence relations. We could modify the consequence relation of PD to satisfy the second axiom by setting:

$\Gamma \vDash *_{\mathsf{PD}}$A iff $\Gamma \vDash_{\mathsf{PD}}$ A or $\vDash_{\mathsf{PD}}$ A.

But in this case, we would no longer be able to isolate contradictions and tautologies completely, since for any A and B, A $\vDash *_{\mathsf{PD}}$B $\rightarrow$ B. Still, we could isolate the effect of contradictions on non-tautologous formulas, as Theorem 7 would hold for any B that is not a tautology. Hence, the logic of $\vDash *_{\mathsf{PD}}$ is paraconsistent, too. Whether it is a better tool for reasoning with inconsistencies is a question for further research.

Two Further Logics and a Comparison

The logic DualD, presented in *Epstein, 1990*, is defined exactly as D except that the relation governing the truth-table for the conditional A $\rightarrow$ B is $\mathfrak{s}(\mathrm{A}) \subseteq \mathfrak{s}(\mathrm{B})$. The following definitions establish a paraconsistent version of DualD.

Dual dependence semantic consequence
$\vDash_{\mathsf{PDualD}}$ A iff $\vDash_{\mathsf{DualD}}$ A.
For $\Gamma \neq \varnothing$, $\Gamma \vDash_{\mathsf{PDualD}}$ A iff there are C_1, ... , C_n in Γ such that
(i) if for every B in Γ, $\mathfrak{M} \vDash$ B, then $\mathfrak{M} \vDash$ A, and
(ii) for any model $\mathfrak{M} = <\mathfrak{v}, \mathfrak{s}>$, $\mathfrak{s}(C_1 \wedge ... \wedge C_n) \subseteq \mathfrak{s}(\mathrm{A})$.

Paraconsistent dual dependence logic
PDualD is the logic of the semantics of dual dependence logic but with dual dependence semantic consequence.

The logic Eq of equality of contents is also defined in *Epstein, 1990* in the same manner as D except that the relation governing the truth-table for the conditional is equality, $\mathfrak{s}(\text{A}) = \mathfrak{s}(\text{B})$. The following definitions establish a paraconsistent version of Eq.

Equality content semantic consequence

$\vDash_{\mathsf{PEq}}$ A iff $\vDash_{\mathsf{E}}$ A.

For $\Gamma \neq \varnothing$, $\Gamma \vDash_{\mathsf{PEq}}$ A iff there are $\text{C}_1, \ldots, \text{C}_n$ in Γ such that

(i) if for every B in Γ, $\mathfrak{M} \vDash$ B, then $\mathfrak{M} \vDash$ A, and

(ii) for any model $\mathfrak{M} =< \mathfrak{v}, \mathfrak{s} >$, $\mathfrak{s}(\text{C}_1 \wedge \ldots \wedge \text{C}_n) = \mathfrak{s}(\text{A})$.

Paraconsistent equality content logic

PEq is the logic of the semantics of Eq but with equality content semantic consequence.

These logics can be analyzed in the same manner as PD, and I leave to you to formulate the corresponding theorems. Their interest lies in comparing how contradictions are handled in formalizing reasoning.

Consider the example we looked at earlier:

(The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\rightarrow 2 + 2 = 4$

For each of the logics discussed above, in any model in which there is no content overlap between "The moon is made of green cheese" and "$2 + 2 = 4$", this proposition is false. And so for each of the logics above, we also have:

(The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\nvDash 2 + 2 = 4$

But on the assumption that in every model "$4 \cdot 3 = 12$" has the same content as "$2 + 2 = 4$", we have:

(The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\wedge$ $(4 \cdot 3 = 12) \vDash_{\mathsf{PS}} 2 + 2 = 4$

(The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\wedge$ $(4 \cdot 3 = 12) \vDash_{\mathsf{PD}} 2 + 2 = 4$

This may seem anomalous, since the contradiction itself does not have any content overlap with the proposition that is derived from it. In this respect paraconsistent dual dependence fares better:

(The moon is made of green cheese $\wedge$ $\neg$(the moon is not made of green cheese)) $\wedge$ $(4 \cdot 3 = 12) \nvDash_{\mathsf{PDualD}} 2 + 2 = 4$

However, paraconsistent dual dependence logic has a similar anomaly in the derivation of tautologies:

$(2 + 2 = 4) \vDash_{\mathsf{PDualD}} \neg(\neg$(The moon is made of green cheese $\wedge\neg$(The moon is made of green cheese)) $\wedge \neg(4 \cdot 3 = 12))$

If we wish to fully isolate contradictions and tautologies according to their content, it would seem that paraconsistent Eq is more suitable. We have:

(The moon is made of green cheese $\wedge\neg$(the moon is not made of green cheese)) $\wedge\, 4 \cdot 3 = 12 \nvDash_{\mathsf{PEq}} 2 + 2 = 4$

$(2 + 2 = 4) \nvDash_{\mathsf{PEq}} \neg(\neg$(The moon is made of green cheese $\wedge\neg$(The moon is made of green cheese)) $\wedge \neg(4 \cdot 3 = 12))$

Further Research

1. It should be possible to extend each of dependence logic D, dual dependence logic DualD, and the logic of equality of contents Eq to a predicate logic in the same manner as subject matter relatedness logic S is extended, with both clear semantics and a strongly complete axiomatization. This would then yield a predicate logic versions of paraconsistent D, DualD, and Eq.

2. A comparison of the paraconsistent logics presented here with other paraconsistent logics would clarify the underlying semantic assumptions of all of them.

3. Using one of the paraconsistent logics discussed here as the method of reasoning about an inconsistent database would illuminate whether these logics are indeed useful in reasoning about inconsistencies.

4. An investigation of the algebras of paraconsistent dependence logic would give a different perspective on its semantic assumptions and may yield further counterexamples to long-held assumptions about the nature of algebras of logics, as begun with the investigation in *Epstein, 1987*.

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