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YET ANOTHER "CHOICE OF PRIMITIVES" WARNING: NORMAL MODAL LOGICS

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1. *Introduction*

The sensitivity of some claims about modal logics – in particular about the structure of the lattice of all such logics – to the choice of boolean primitives was pointed out in [Ma], while the sensitivity of some such claims – in particular about the completeness of particular axiomatizations – to the choice of modal primitives was stressed in [Mi]. In the latter case, the logic S0.5 was at issue, while in the former, the example depends again on the inclusion, in the range of modal logics considered, of non-normal (or more generally, non-congruential) logics. Here we point out that the choice of modal primitives does not cease to matter when it is specifically, claims about *normal* modal logics that are at issue, by noting the inadequacy of a certain style of (putative) axiomatization.

It should be emphasized that the cases alluded to above of alternative sets of primitives, what is involved is a choice between pairs of sets of primitives which are interdefinable: we are not considering the effect of weakening, to take the boolean case, the stock of primitives in such a way as to make negation or implication, for instance, not definable in terms of the primitives chosen (as in [Hu2], [Du], for instance). Similarly, in the modal case, the distinction in [Mi] is between taking necessity ($\Box$) as primitive and taking possibility ($\Diamond$) as primitive, rather than using some weaker basis definable in terms of either of these and the boolean connectives, but from which $\Box$ and $\Diamond$ cannot in turn be defined, such as contingency (or noncontingency), on which see [Hu5], [Ku]. (One might think necessity could be defined on this basis by saying that what is necessary is what is both noncontingent and also true: but we do not wish to restrict ourselves to logics in which necessity implies truth, so such a definition is not generally available. Of course this shows that "necessary", "possible", "noncontingent" etc., and so on, are not really appropriate terms, but we take them simply as convenient verbalizations of $\Box$, $\Diamond$, and – to use some obvious notation – $\Box\_\_ \vee \Box\neg\_\_$.)

The prototype for discussions such as that of [Mi], as well as the inspiration for the title of [Ma], is the classic discussion [Hi] in which the incompleteness of an axiomatization for conjunction and negation is shown, this axiomatization being the result of translating the axioms and rules of a complete system for (classical) disjunction and negation, using the obvious De Morgan style translation. The topic has been taken up further in [Fr] and [Sh]; the point is of course not specific to classical logic, and it was raised for a putative axiomatization of intuitionistic propositional logic in [Hu6]. The theme of [Ma] was further discussed in §4.6 of [Se2] and §3 of [Hu3].[1] The simplest illustration of Hiż's phenomenon known to the present author concerns the axiomatization of classical propositional logic with $\rightarrow$ and $\bot$ as primitive, with $\neg A$ defined as $A \rightarrow \bot$. Using axioms (1) and (2) from Section 4 below, and the further axiom $\neg\neg p \rightarrow p$, together with *Modus Ponens* and Uniform Substitution as rules – or alternatively, using schematic formulations of these axioms and just the rule *Modus Ponens* – we have a complete axiomatization of classical propositional logic, though if we take as our primitives $\rightarrow$ and $\neg$ (bearing in mind the definability of $\bot$ in terms of these primitives) the above axiomatization is incomplete, with, for example $\neg\neg(p \rightarrow p)$ being unprovable therefrom, as interpreting $\neg A$ as false for all A reveals.[2]

2. *A Failed Axiomatization of* K

A common axiomatization of K, smallest normal modal logic, supplies a stock of boolean primitives, a set of axioms which together with the rule *Modus Ponens* provides a complete axiomatization of classical (non-modal) propositional logic, and adds a new axiom – or, since we shall work here without the rule of uniform substitution, an infinite set of axioms, namely all instances of the axiom-schema

$$(\mathsf{K}) \quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

[1] Readers curious about the authorship of [Mi] are referred to [Hu4].

[2] That is, we keep the usual truth-functional interpretation of $\rightarrow$ but say that $\neg$ is to be assigned the 1-ary constant false truth-function; in the case in which $\neg$ was defined we could not arrange for this, since there is no way for __ $\rightarrow \bot$ to be associated with this function (since whatever fills the __ blank may have the same truth-value as $\bot$). This example first came to my attention as a result of correspondence with Rob Goldblatt in about 1980; a variant on the example arises for intuitionistic and for (Johansson's) Minimal logic, and is given for the latter case in Proposition 2.7.4 on p. 137 of [Wó]. (The axioms there mentioned by number are identified on p. 108.) Wójcicki suggests the example illustrates the point of [Ma], rather than, as claimed here, that of [Hi].

and a new modal rule, Necessitation, licensing the passage from A to $\Box A$. This is well and good provided that $\Box$ is taken as a primitive connective. Given such a choice, one may introduce a dual operator $\Diamond$, by defining $\Diamond A$ as $\neg\Box\neg A$. One may, noting the K-provability of all formulas of the form $\Box A \leftrightarrow \neg\Diamond\neg A$ be tempted to take $\Diamond$ rather than $\Box$ as the primitive modal operator, treating $\Box A$ as defined to be $\neg\Diamond\neg A$, and then stick with the above axiomatization in $\Box$, now *alias* $\neg\Diamond\neg$, and think one has a complete axiomatization of K on one's hands. (Some who have succumbed to this temptation will be mentioned in Section 4, along with an earlier observation of its failure – from [Hu1] – and some remarks as to how the present section's approach differs from the argument given there.) It will help to have the proposed axiomatization before us explicitly, before we go on to note its incompleteness. The truth-functional axioms and rule need not be listed – though we note that the axioms are to include all instances in the present language of any chosen set of schemata complete (with *Modus Ponens*) for truth-functional logic – since the changes occur at the modal level, with what we have written as (K) becoming, given the change of primitives, the following schema once the occurrences of the defined "$\Box$" are spelt out in primitive notation:

$$(\mathsf{K})_{\Diamond} \quad \neg\Diamond\neg(A \to B) \to (\neg\Diamond\neg A \to \neg\Diamond\neg B)$$

and with the rule of necessitation becoming

$$(\text{Nec.})_{\Diamond} \quad \textit{From} \vdash A \textit{ to} \vdash \neg\Diamond\neg A.$$

To show that not every theorem of traditional $\Box$-based K (with $\Diamond$ understood as $\neg\Box\neg$) is forthcoming on this basis (with $\Box$ understood as $\neg\Diamond\neg$), it will help to have a modification of the Kripke semantics for normal modal logics on hand. We will dispense with accessibility relations, since this makes for a simpler semantics relative to which all we need is the soundness of the current $\Diamond$-based axiomatization – not its completeness, but make use of a device familiar in a different connection (entering into the clause for the modal primitive in the definition of truth) from [Kr], the idea of a subset of "normal worlds", to be used here for the treatment of negation. (Compare [Cr], p. 453*ff.*) For present purposes, then, a *model* is a triple $\langle W, N, V\rangle$ in which W is a set with $\emptyset \neq N \subseteq W$ and V is a function assigning to each propositional variable (sentence letter) a subset of W. Here we take as the boolean primitives, $\to$ and $\neg$ with (of course) $\Diamond$ as our modal primitive. Given a model $\mathcal{M} = \langle W, N, V\rangle$ we define the truth of a formula A at a point $u \in W$ – notated thus: "$\mathcal{M} \models_u A$" – by induction on the complexity of A, with the basis clause

- $\mathcal{M} \models_u A \Leftrightarrow u \in V(A)$ for A a propositional variable and the inductive clauses for A of the forms $B \to C$, $\neg B$, $\Diamond B$:

- $\mathcal{M} \vDash_u B \to C \Leftrightarrow \mathcal{M} \nvDash_u B$ or $\mathcal{M} \vDash_u C$,
- $\mathcal{M} \vDash_u \neg B \Leftrightarrow u \in N$ and $\mathcal{M} \nvDash_u B$,
- $\mathcal{M} \vDash_u \Diamond B \Leftrightarrow$ for some $v \in W$, $\mathcal{M} \vDash_v B$.

The formula A is *true in* the model $\mathcal{M} = \langle W, N, V\rangle$, written as $\mathcal{M} \vDash A$, just in case for all $u \in N$, we have $\mathcal{M} \vDash_u A$, and *valid* if it is true in every model.

Theorem 2.1: *Every formula provable in the above extension of truth-functional logic by means of* $(\mathsf{K})_\Diamond$ *and* $(Nec.)_\Diamond$*, alongside Modus Ponens, is valid.*

Proof. It suffices to show that every substitution-instance of a classical tautology (in $\to$ and $\neg$) is true in every model, and likewise for any instance of the schema $(\mathsf{K})_\Diamond$, and that the rules $(\text{Nec.})_\Diamond$ and *Modus Ponens* preserve, for an arbitrary model $\mathcal{M}$, the property of being true in $\mathcal{M}$. For the truth-functional aspects of this axiomatization, it suffices to note that since truth in a model is a matter of truth throughout the set N of 'normal' points, at which the above clause for $\neg$ reduces to the usual truth-table stipulation. This leaves us with $(\mathsf{K})_\Diamond$ and $(\text{Nec.})_\Diamond$ to check. In the case of the former, suppose for a contradiction that we have for some model $\mathcal{M} = \langle W, N, V\rangle$ and some $u \in N$, $\mathcal{M} \nvDash_u \neg\Diamond\neg(A \to B) \to (\neg\Diamond\neg A \to \neg\Diamond\neg B)$, for some formulas A and B, i.e., (1), (2), and (3):

$$(1)\mathcal{M} \vDash_u \neg\Diamond\neg(A \to B) \qquad (2)\mathcal{M} \vDash_u \neg\Diamond\neg A \qquad (3)\mathcal{M} \nvDash_u \neg\Diamond\neg B.$$

Since $u \in N$, these mean respectively that

$$(1)'\mathcal{M} \nvDash_u \Diamond\neg(A \to B) \qquad (2)'\mathcal{M} \nvDash_u \Diamond\neg A \qquad (3)'\mathcal{M} \vDash_u \Diamond\neg B.$$

By $(3)'$ there exists $v \in W$ for which $\mathcal{M} \vDash_v \neg B$, and by $(1)'$ and $(2)'$, $\mathcal{M} \nvDash_v \neg(A \to B)$ and $\mathcal{M} \nvDash_v \neg A$. Since we have $\mathcal{M} \vDash_v \neg B$, $v \in N$, so these last two claims are equivalent to claiming that $\mathcal{M} \vDash_v A \to B$ and $\mathcal{M} \vDash_v A$, which is not consistent with $\mathcal{M} \vDash_v \neg B$, itself equivalent to $\mathcal{M} \nvDash_v B$. Finally, we check that $(\text{Nec.})_\Diamond$, preserves truth in a model. Suppose that for $\mathcal{M} = \langle W, N, V\rangle$ we have $\mathcal{M} \vDash A$ but not $\mathcal{M} \vDash \neg\Diamond\neg A$. The latter means that for some $u \in N$, $\mathcal{M} \nvDash_u \neg\Diamond\neg A$ and thus that $\mathcal{M} \vDash_u \Diamond\neg A$ (since $u \in N$), so for some $v \in W$, $\mathcal{M} \vDash_v \neg A$. Since it verifies a negated formula, this v also belongs to N, and $\mathcal{M} \nvDash_v A$, which facts together contradict our supposition that $\mathcal{M} \vDash A$. $\square$

Corollary 2.2: *Not every theorem of* K *is provable from the axiomatization described in Theorem 2.1.*

Proof. Consider the K-provable (with $\Diamond$ abbreviating $\neg\Box\neg$, p a propositional variable) formula $\Diamond p \to \Diamond\neg\neg p$, which, considered as a formula of the language with $\Diamond$ primitive is not derivable from the axiomatization using $(\mathsf{K})_\Diamond$ and $(\text{Nec.})_\Diamond$, because by Theorem 2.1 it is not valid in the sense of that Theorem. We can see this with the aid of a two-element model $\mathcal{M}_0 = \langle W, N, V\rangle$ with $W = \{u, v\}(u \neq v)$, $N = \{u\}$, and $V(p) = \{v\}$. (What V assigns to other variables is immaterial.) $\mathcal{M}_0 \models_u \Diamond p$, since there is an element of W, viz. v, at which p is true, while $\mathcal{M}_0 \not\models_u \Diamond\neg\neg p$, since there is no element of W at which $\neg\neg p$ is true: not u itself, since $u \in N$ and $\mathcal{M}_0 \not\models_u p$, and not v since although $\mathcal{M}_0 \not\models_v \neg p$, $v \notin N$. □

"For the benefit," as Krister Segerberg so memorably once put it,[3] "of matrix minded readers", we note that the 'frame' $\langle W, N\rangle$ of the above model $\mathcal{M}_0$ can be converted into a (logical) matrix, with V corresponding to a particular assignment of matrix elements to the formulas, by setting $1 = \{u, v\}$, $2 = \{u\}$, $3 = \{v\}$ and $4 = \emptyset$. Since we defined validity as truth at every 'normal' point and u is the only normal point, the designated elements of the matrix are 1 and 2 – as indicated by the asterisks in Figure 1 – corresponding to the sets containing u.

$\to$	1	2	3	4	$\neg$		$\Diamond$		$\Box$	
*1	1	2	3	4	4	1	1	1	2	1
*2	1	1	3	3	4	2	1	2	2	2
3	1	2	1	2	2	3	1	3	4	3
4	1	1	1	1	2	4	4	4	4	4

Figure 1

The example presented in the proof of Coro. 2.2 then takes the following form: while every theorem of the logic there considered is valid in the matrix of Figure 1, in the sense of assuming a designated value whatever values are assigned to the propositional variables, the same does not hold for the formula $\Diamond p \to \Diamond\neg\neg p$, since when p takes the value 3, this formula assumes the value $\Diamond 3 \to \Diamond\neg\neg 3 = 1 \to \Diamond\neg 2 = 1 \to \Diamond 4 = 1 \to 4 = 4$, an undesignated value.

Of course, while the axiomatization discussed in this section fails to yield K, there are simple axiomatic bases to be found in the literature which take $\Diamond$

[3] At p. 196 of [Se1].

rather than $\Box$ as primitive and are complete. (See, e.g., [Ch], Thms. 4.4, 4.5, and [BRV], pp. 9 and 31: these authors take $\Diamond$ as primitive, supplementing (K) (= $(\mathsf{K})_{\Diamond}$) with $\Diamond p \leftrightarrow \neg\Box\neg p$ as an additional axiom to avoid the current difficulty.) We should also note that while our title is intended to echo the terminology of [Ma] and – more pertinently here – [Mi], it would have been equally apt to speak of a 'choice of axioms' problem, echoing that of [Hi], rather than a 'choice of primitives' problem. The point is simply that if one is wedded antecedently to a particular axiomatization, then one has to be careful about making a choice of primitives, given which that axiomatization does what is claimed for it, while if one is antecedently wedded to a particular choice of primitives, then one has to be careful when choosing axioms and rules, to make sure, again, that a suitable combination has been selected for the purpose at hand. What we have seen in this section is the unsuitability of combining the choice of $\Diamond$ as primitive with the choice of $(\mathsf{K})_{\Diamond}$ and $(\text{Nec.})_{\Diamond}$, i.e. naively using the translation of a successful axiomatic basis when $\Box$ is primitive.

3. *Some Issues Arising*

In drawing up Figure 1, we included also the table for the defined connective $\Box$, computed by taking $\Box$ as $\neg\Diamond\neg$. In the model-theoretic semantics the corresponding clause would be (relative to a model $\mathcal{M} = \langle W, N, V\rangle$, for any $u \in W$):

- $\mathcal{M} \models_u \Box B \Leftrightarrow u \in N$ and for all $v \in N$, $\mathcal{M} \models_v B$.

There is something of a philosophical disagreement over the status of defined expressions such as $\Box$ here. (See the discussion and references in [Hu6].) On one view, that underpinning our own discussion, such expressions are not part of the object language: rather, when we use an expression such as "$\Box p$" we are simply helping ourselves to an abbreviation in the metalanguage, the actual formula referred to by this abbreviation being the formula $\neg\Diamond\neg p$. On the alternative view, a definition like that with which we are currently concerned adds a new symbol to the object language itself, one with which it is interreplaceable in any context provided by any theorem of the logic under consideration by the corresponding *definiens*. On this approach $\Box p$ and $\neg\Diamond\neg p$ are distinct formulas which are 'synonymous', as [Sm] puts it, in the logic. It is clear that taking this second position makes no difference to our discussion, and in particular to Theorem 2.1 and Coro. 2.2, though a development along those lines would more naturally place the above clause for $\Box$ alongside those given for $\rightarrow$, $\neg$ and $\Diamond$ before the formulation of Theorem 2.1.

In a normal modal logic the provability of $A \leftrightarrow B$ (or, given the current choice of primitives, the provability of both $A \rightarrow B$ and $B \rightarrow A$: though of course we could use the definition of $\leftrightarrow$ from $\rightarrow$ and $\neg$ to provide a single formula) suffices for the synonymy of A and B, and in fact this so for any merely *congruential* modal logic (one in which the provable equivalence of A with B suffices for the provable equivalence of OA with OB, where O is whichever of $\Box$ and $\Diamond$ is primitive[4]). The logic of Theorem 2.1 is clearly not congruential, since its theorems include, amongst all other truth-functional tautologies, $p \rightarrow \neg\neg p$ and the converse implication, while the example of Coro. 2.2 shows that p and $\neg\neg p$ are not interreplaceable in the context $\Diamond(\cdot)$. In this setting, therefore, there is no guarantee that we can, while taking the second of the two views on definition distinguished in the preceding paragraph, secure the interreplaceability of *definiens* and *definiendum* by stipulating that they are to be provably equivalent. For the logic axiomatized in Section 2, it would be enough (for obtaining a complete axiomatization of K with $\Diamond$ as primitive) to add the schema $\Diamond A \leftrightarrow \Diamond\neg\neg A$, which recalls the advice given by Hiż ([Hi], p. 614): "A translation of a complete set of axioms to another set of primitives would be complete only if from the resulting axioms the definitions of the first set of primitives followed."[5] In the present case, this means that we need to be able to prove (for any A) $\neg\Box\neg A \leftrightarrow \Diamond A$, i.e., in primitive notation, $\neg\neg\Diamond\neg\neg A \leftrightarrow \Diamond A$, which is an easy consequence of the schema just mentioned ($\Diamond A \leftrightarrow \Diamond\neg\neg A$). However, this advice is not in general the last word, since there may not be a set of formulas constituting "the definitions of the first set of primitives". (For example, in the logic S0.5 mentioned in our opening paragraph there is no binary connective playing the role here played by $\leftrightarrow$, with the provability of the resulting compound securing the synonymy of the two components: See [Po], Theorem 3.1.)

In the proof of Coro. 2.2, only one half of the a representative instance of the schema considered in the preceding paragraph figured, $\Diamond p \leftrightarrow \Diamond\neg\neg p$, namely the $\rightarrow$ direction. It would be interesting to find an equally simple argument, perhaps a variation on that given there, establishing the unprovability of the converse implication from the failed axiomatization of K. The author's attempts in this direction have not been successful. An idea with some initial promise for finding a notion of validity on which $\Diamond\neg\neg p \rightarrow \Diamond p$

[4] This notion of congruentiality is adapted from [Ma], where it is remarked that the problem of sensitivity to a choice of primitives there considered does not arise for congruential modal logics.

[5] This point about being able to recover the definitions from the new axiomatization is familiar in the setting of first-order theories as what distinguishes definitional equivalence from (mere) mutual interpretability: see [Co1], [Co2], [Co3], for discussion and references.

is invalid but with respect to which that axiomatization is sound is the following. One retains models of the $\langle W, N, V\rangle$ form, with validity as truth throughout N in each such model, and the following change to the clause for $\neg$:

- $\mathcal{M} \models_u \neg B \Leftrightarrow u \notin N$ or $\mathcal{M} \not\models_u B$.

Whereas the earlier clause ruled out the joint truth, but allowed for the simultaneous falsity at a point of a formula and its negation, this new clause rules out the simultaneous falsity but allows for the joint truth of B and $\neg B$. The possibilities allowed for in both cases are realized only at points outside of N, however, so all (substitution instances of) classical tautologies are again valid. Every instance of $(\mathsf{K})_\Diamond$ also turns out valid, so we are almost home, in view of the fact that $\Diamond\neg\neg p \to \Diamond p$ is invalid. (Simply adjust the specification of $\mathcal{M}_0$ in the proof of Coro. 2.2 so that $V(p) = \emptyset$.) Almost, but not quite: the rule $(\text{Nec.})_\Diamond$ does not preserve truth in a model, or even validity: $\neg\Diamond\neg(p \to p)$ is invalid, for example, since whenever $W \setminus N \neq \emptyset$, $\langle W, N, V\rangle \models \Diamond\neg(p \to p)$ regardless of V, any negated formula being true throughout $W \setminus N$. So we must leave open the problem of supplying a variation on the earlier semantics which demonstrates the unprovability of $\Diamond\neg\neg p \to \Diamond p$.

Allen Hazen has pointed out to the author that the semantics of the preceding paragraph can be used instead to demonstrate the unprovability of $\Box\neg\neg p \to \Box p$, from an axiomatization in which $\Box$ is primitive but which is an injudicious translation of a complete $\Diamond$-based axiomatization (such as that of [Ch], mentioned in Section 2); the idea is that the present semantics makes all negated formulas true at non-normal worlds, so if $V(p)$ comprises precisely the normal worlds in a model in which there are also non-normal worlds, all the normal worlds will verify $\Box\neg\neg p$ and falsify $\Box p$. It would be interesting to know whether any such translation has in fact ever been mistakenly proposed as complete.

In view of the significance ([Ma] and the supplementary discussion cited in Section 1) of the choice, amongst boolean primitives, between taking $\to$, $\neg$ as our discussion has, on the one hand, and $\to$, $\bot$ on the other, it is of some interest to see what becomes of Section 2 with a change to the latter pair, with $\Box A$ now defined to be $\Diamond(A \to \bot) \to \bot$. If in the model-theoretic semantics we retain the previous clause for $\to$, there will be no way of having $A \to \bot$ amount to $\neg A$ as previously treated, since the latter could only be true at normal worlds, while whatever is said about $\bot$, the former will be true at any world at which A fails to be true. For the analogue of the axiomatization in the primitives $\to$, $\bot$, $\Diamond$ – and because of its length we will not write out (as we did with $(\mathsf{K})_\Diamond$ what (K) looks like in the new primitive notation – we can still show that the current incarnation of $\Diamond p \to \Diamond\neg\neg p$, namely $\Diamond p \to \Diamond((p \to \bot) \to \bot)$ is not derivable, by the following modification of the earlier semantics. We stipulate that, relative to a model $\langle W, N, V\rangle$, $\bot$ is false at every point, and an implication $B \to C$ is true at an arbitrary $u \in W$ if and only if $u \in N$ and either B is false at u or C is true at u. Validity continues to be defined as truth throughout N in any $\langle W, N, V\rangle$, and we leave the reader to check that every theorem derivable from the current axiomatization is valid, while $\Diamond p \to \Diamond((p \to \bot) \to \bot)$ is not. The point has nothing in particular to do with $\bot$: putting q (another propositional variable) or p itself, for $\bot$ in this formula again gives something that ought to be provable but, as we see from its invalidity, isn't.

4. *Examples From Life*

Thomason, in [Th] – a paper justly celebrated for its early recognition that not every (bi)modal logic was determined by a class of Kripke frames – presents what purports to be an axiomatization of the minimal tense logic, often called K_t (though referred to by Thomason as T_0). The language has primitive connectives $\to$, $\neg$, F and P, the latter two being the tense logical $\Diamond$-operators ("future" and "past" respectively), G and H abbreviating $\neg F \neg$ and $\neg P \neg$. Thomason uses the rules *Modus Ponens*, Uniform Substitution, and the (Nec.)-like rules which pass from A to GA and from A to HA, and the following axioms (numbered as in [Th], p. 150):

(1) $p \to (q \to p)$
(2) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
(3) $(\neg p \to \neg q) \to (q \to p)$
(4) $G(p \to q) \to (Gp \to Gq)$
(5) $H(p \to q) \to (Hp \to Hq)$
(6) $p \to HFp$
(7) $p \to GPp$

The role of the axioms (1)–(3) is, with the aid of *Modus Ponens* and Uniform Substitution, to yield proofs of all formulas which are substitution-instances of truth-functional tautologies. The intended role of (4) and (5), together with the H and G Necessitation rules, is to secure that we have a normal bimodal logic (i.e., each of the operators H, G, is normal in the resulting logic) – an intention we do not expect to succeed in view of Section 2 above. There remains the possibility that further inclusion of (6) and (7) – the bridging axioms which reflect the fact that the accessibility relations for H and G are each other's converses – safeguards the present axiomatization from the objection raised for the monomodal case in Section 2. This possibility, however, is not realized. We find our old example from Coro. 2.2, $\Diamond p \to \Diamond\neg\neg p$, turns out to be unprovable in either the form $Fp \to F\neg\neg p$ or the form $Pp \to P\neg\neg p$ (both of which are provable in K_t). For one readily sees that all instances of the schema (sometimes called B) $A \to \Box\Diamond A$ are valid in the sense of Section 2, so we can interpret P and F in the same way as the semantics there provided interprets $\Diamond$, and all axioms of Thomason's axiomatization will be valid, the various rules preserving validity, while the formulas just cited are invalid and therefore unprovable. Note that because we are now using Uniform Substitution, we can no longer say that each rule preserves, for any model, the property of being true in that model. However, we still have the claimed preservation property for this rule for a reason which is familiar from the literature but which we spell out here because of its bearing on the topic of the following paragraph. If a formula $A(B)$ results from a formula $A(p_i)$ by uniform substitution of B for p_i, then for any model $\mathcal{M} = \langle W, N, V\rangle$ with, for $u \in W$, $\mathcal{M} \nvDash_u A(B)$, setting V' as like V except that $V'(p_i) = \{v \in W \mid \mathcal{M} \vDash_v B\}$, for the model $\mathcal{M}' = \langle W, N, V'\rangle$ we have $\mathcal{M}' \nvDash_u A(p_i)$.[6] So from the invalidity of $A(B)$ the invalidity of $A(p_i)$ follows.

This brings us to our second example, which again concerns the axiomatization of K_t with F and P taken as primitive. McArthur, at p. 18 of [Mc], offers a variant on the axiomatization with (1)–(7) which replaces these axioms by corresponding axiom-schemata, distinct propositional variables being replaced by distinct schematic letters, thereby avoiding the need to include Uniform Substitution amongst the rules.[7] The same definitions are given of G and H in terms of F and P as in the previous example. The observation that this does not provide a complete axiomatization of K_t was made in note 13 of [Hu1], with a justification using only bivalent valuations (truth-value assignments), as opposed to the model-theoretic apparatus brought to bear in Section 2 above (or its many-valued incarnation, such as Figure 1 provides for a two-element model). By a *boolean* valuation we understand a valuation v satisfying for all formulas A, B, the conditions (i) $v(\neg A) = T$ iff $v(A) = F$ and (ii) $v(A \to B) = T$ iff $v(A) = F$ or $v(B) = T$. (If other boolean primitives are employed, the corresponding truth-table conditions should be imposed.) For the language of tense logic,

[6] One may draw this conclusion from the fact, proved by induction on the construction of $A(p_i)$, that for $\mathcal{M}$ and $\mathcal{M}'$ as described here, we have $\mathcal{M} \vDash_v A(B)$ iff $\mathcal{M}' \vDash_v A(p_i)$, for all $v \in W$.

[7] In fact McArthur also avoids the need for the necessitation rules with G and H by the device of counting not only any instance of these schemata, but also the result of prefixing such an instance with G or H, as an axiom. He also collapses the effect of axioms (1)–(3) into a single schema counting any (substitution-instance of a) truth-functional tautology as an axiom. These differences do not affect our discussion.

with additional operators F and P, what in the *ad hoc* terminology of [Hu1] is called *McValuation* is a boolean valuation v satisfying the further conditions that $v(FA) = v(PA) = T$ iff A is not of the form $\neg B$. Thus for any formula A and any McValuation v, $v(GA) = v(HA) = T$. (Here we are assuming the first – 'metalinguistic abbreviation' – view of definitions distinguished in Section 3.) This observation helps us to see that every instance of the McArthur axiom schemes is assigned the value T by every McValuation, and that the rules *Modus Ponens* and G- and H-necessitation preserve the property of being assigned this value by every McValuation, whereas the formula $Fp \to F\neg\neg p$ (as well as the corresponding formula with P) lacks this property: in fact no McValuation verifies this formula, since all such valuations verify its antecedent while falsifying its consequent. We conclude (as in note 13 of [Hu1]) that the given formula, while K_t-provable, is not forthcoming on the basis of McArthur's axiomatization.

The argument just sketched would have worked just as well as that given in Coro. 2.2 to show that the axiomatization discussed in Section 2 failed as an axiomatization of K, by giving $\Diamond$ the same treatment just accorded to F and P. Is there any advantage to be gained, then, from the route taken in Section 2, over this simpler route to the same result? There certainly is, and we have just seen, in the case of Thomason's would-be axiomatization of K_t, one instance of the greater power of the methods of Section 2. The key difference between Thomason's and McArthur's axiomatizations lies in the former's using and the latter's eschewing the rule of Uniform Substitution. For brevity, let us call a formula true on every McValuation *McValid*. Whereas the model-theoretically defined notion of validity deployed in Section 2 is preserved by Uniform Substitution, as explained two paragraphs back, McValidity is not so preserved, so the simple argument given – that all provable formulas are McValid which proceeded by showing that the axioms had this property and the rules preserved it – would fail if Uniform Substitution were amongst the rules. As an illustration of the failure of this rule to preserve McValidity, consider the formula Fp and its substitution instance $F\neg p$: the first is McValid while the second is not. Now it may be said that we can supplement the McValuations argument, when we wish to use it *à propos* of an axiomatization using Uniform Substitution, with the observation that the same theorems are evidently provable from the axiomatization dropping this rule in favour of schemata. Certainly: but no such supplementary considerations are needed if instead the method of Section 2 is employed.

It would probably be worthwhile to make a more detailed study of the differences (in respect of range of application) of the $\langle W, N, V\rangle$ models and the method of McValuations – or of the 'dualized' version of the former (attributed to Hazen in Section 3) and a corresponding adaptation of the latter – but we shall not undertake any such further investigations here.

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