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SOCRATIC PROOFS FOR SOME NORMAL MODAL PROPOSITIONAL LOGICS*

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Abstract
Our aim is to adjust the method of Socratic proofs to propositional parts of normal modal logics: K, D, K4, T, KB, S4 and S5. We give a proof of soundness of the method.

0. *Introduction*

In Wiśniewski (2004) the method of Socratic proofs (SP) for Classical Propositional Calculus (CPC) is presented. The method is based on the idea of solving logical problems by pure questioning. The turnstyle $\vdash$, interpreted as referring to CPC-derivability/entailment, is regarded as an expression of a certain object-level language. This language is built upon the language of CPC and comprises, int. al, questions and their answers. A calculus of questions, E^*, is developed. The rules of E^* transform questions into questions. A transformation of this kind starts with a question about CPC-validity of a formula, or about CPC-derivability (CPC-entailment) of a formula. A *Socratic proof* is a successful transformation, that is, a transformation that leads to a question whose affirmative answer is, in a sense, evident. The calculus E^* is sound and complete with respect to CPC. The SP-method provides a decision procedure for CPC. Moreover, any Socratic proof may be transformed into a Gentzen-style proof or an analytic tableau.

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So far the SP-method has been applied to first-order logic (Wiśniewski and Shangin (forthcoming)), to some propositional paraconsistent logics (Wiśniewski, Vanackere and Leszczyńska (2005)) and to intuitionistic logic (Skura (forthcoming)). Some theorem provers based on the SP-method are available on the Web.[1] The aim of this paper is to adjust the SP-method to propositional parts of the modal logics: K, D, K4, T, KB, S4 and S5. Our plan is the following. Section 1 is devoted to a more detailed description of the SP-method for CPC. In Section 2 we characterize an extension of the language M of modal propositional logics. This extension is an object-level language, in which questions about validity of formulas of M can be asked. We call it M^*. Language M^* can be extended further so that questions of derivability would be expressible. We will not do it here, however. In Section 3 we give an intuitive description of the SP-method for modal logics. Then we develop some "erotetic" modal calculi of questions. The problem of soundness is addressed in Section 4.

1. *The Calculus of Questions* E^*

From a proof-theoretical perspective, the SP-method for CPC is a variant of a sequent-calculus method. What is transformed when a rule is applied, is a question containing one or more sequents which are: (a) single-conclusioned (there is always a single formula right of the turnstyle) and (b) both-sided (a non-empty sequence of formulas may occur left of the turnstyle). In the schemas of rules presented below the α, β-notation (after Smullyan (1968)) is used, according to the following table:

α	α_1	α_2		β	β_1	β_2	β_1^*
$A \wedge B$	A	B		$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$\neg(A \vee B)$	$\neg A$	$\neg B$		$A \vee B$	A	B	$\neg A$
$\neg(A \rightarrow B)$	A	$\neg B$		$A \rightarrow B$	$\neg A$	B	A

The rules of calculus E^* are the following:[2]

[1] See `http://logica.ugent.be/albrecht/socratic.html` and `http://logica.ugent.be.albrecht/socratic-modal.htm`.

[2] We present the complete list of rules of E^* without using the α, β-notation in Appendix 1.

L_α: $\dfrac{?(\Phi;S'\alpha'T\vdash C;\Psi)}{?(\Phi;S'\alpha_1{}'\alpha_2{}'T\vdash C;\Psi)}$ $\qquad$ R_α: $\dfrac{?(\Phi;S\vdash\alpha;\Psi)}{?(\Phi;S\vdash\alpha_1;S\vdash\alpha_2;\Psi)}$

L_β: $\dfrac{?(\Phi;S'\beta'T\vdash C;\Psi)}{?(\Phi;S'\beta_1{}'T\vdash C;S'\beta_2{}'T\vdash C;\Psi)}$ $\qquad$ R_β: $\dfrac{?(\Phi;S\vdash\beta;\Psi)}{?(\Phi;S'\beta_1^*\vdash\beta_2;\Psi)}$

$\mathsf{L}_{\neg\neg}$: $\dfrac{?(\Phi;S'\neg\neg A'T\vdash C;\Psi)}{?(\Phi;S'A'T\vdash C;\Psi)}$ $\qquad$ $\mathsf{R}_{\neg\neg}$: $\dfrac{?(\Phi;S\vdash\neg\neg A;\Psi)}{?(\Phi;S\vdash A;\Psi)}$

where Φ, Ψ stand for finite (possibly empty) sequences of sequents, S, T stand for finite (possibly empty) sequences of CPC-formulas, and A represents a single CPC-formula. The semicolon ';' is a concatenation-sign for sequences of sequents, and the sign $'$ is the concatenation-sign for sequences of CPC-formulas. An application of a rule amounts to the elimination of a connective or a double negation. The method proceeds by transforming questions into questions. A transformation terminates when either: (1) each sequent contained in the last question is of one of the following forms:

(a) $T'B'U \vdash B$
(b) $T'B'U'\neg B'W \vdash C$
(c) $T'\neg B'U'B'W \vdash C$

or (2) no rule can be applied to the last question. A *Socratic proof* of sequent $S \vdash A$ *in* E^* is a finite sequence of questions $< Q_1, Q_2, ..., Q_n >$ such that: $Q_1 = ?(S \vdash A)$, each question results from the previous one by a rule of E^*, and each sequent contained in the last question is of the form (a), (b) or (c).

For example, the following is a Socratic proof of $\vdash ((p \to q) \wedge p) \to q$:

1.	$?(\vdash ((p \to q) \wedge p) \to q)$	
2.	$?((p \to q) \wedge p \vdash q)$	by rule $\mathsf{R}_\to$
3.	$?((p \to q), p \vdash q)$	by rule $\mathsf{L}_\wedge$
4.	$?(\neg p, p \vdash q; q, p \vdash q)$	by rule $\mathsf{L}_\to$

It can be shown that a sequent $S \vdash A$ has a proof in E^* if and only if the formula A is CPC-derivable from S (is CPC-entailed by S). Similarly, a sequent of the form $\vdash A$ is provable in E^* if and only if A is CPC-valid. Thus Socratic proofs in E^* are formal devices by means of which both CPC-derivability/entailment and CPC-validity can be established.

In the case of modal logics, however, we are interested mainly in validity. For that reason we shall restrict ourselves to questions based on *right-sided sequents*, that is, sequents of the form $\vdash T$, where T is a finite and non-empty sequence of indexed formulas of the language of a modal propositional logic. These questions are expressions of a certain object-level language M^*, which we will construct in the next section.

2. *The Object-level Language* M^*

2.1. *Syntax of* M^*

Let us designate by M the language of modal propositional calculus with $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (negation), $\Box$ (necessity operator), $\Diamond$ (possibility operator) as primitives. (For simplicity we do not consider equivalence, strict implication and strict equivalence.) We use p, q, r, ... for propositional variables of M, and A, B, C, ... as metavariables for formulas of M. The expression "iff" is an abbreviation of "if and only if". We use the standard set-theoretic terminology and notation.

Language M is our first object-level language. In order to adjust the method of Socratic proofs to modal logics we have to extend the language M to a language M^*, in which questions about logical properties may be asked. The vocabulary of M^* includes the vocabulary of M as well as symbols: $\vdash$, ?, and the numerals 1, 2, ... Sequences of numerals will be called *indices*. First we give some definitions and then we will comment on them.

If A is a formula of M and $< i_1, ..., i_n >$ is a finite, non-empty sequence of numerals (that is an index), then an expression of the form:

(1.1.) $(A)^{i_1,...,i_n}$

is an *indexed formula* of M^*.

By an *indexed literal* we shall mean an indexed variable or an indexed negation of a variable. Indices do not occur inside indexed formulas of M^*. We shall write S, T, ... for finite sequences of indexed formulas. For convenience, we adopt the following convention:

In a metalinguistic expression of the form:

(1.2.) $(A)^{\phi(i_n)}$

symbol $\phi(i_n)$ represents a finite sequence of numerals which has numeral i_n as its last term. Thus the expression $(A)^{\phi(i_n)}$ represents any indexed formula of the form: $(A)^{i_1,...,i_n}$, where $n \geq 1$. For example, indexed formulas: $(p)^3$, $(p \rightarrow q)^{1,2,3}$ and $(\Box\, p)^{1,3}$ are represented by the metalinguistic expression $(A)^{\phi(3)}$.

The role of the indices is twofold. First, we use *numerals* as 'indicators' of possible worlds of a Kripke frame. Second, *the order in which the numerals occur* in an index gives us a partial description of the accessibility relation in the frame.

Now we give a full description of the language M^*. In order to avoid possible misunderstandings, we will be applying the word 'formula' to well-formed expressions of language M, and the abbreviation 'wff' to well-formed expressions of language M^*. The wffs of M^* are: indexed formulas, declarative wffs (d-wffs) and questions.

By a (right-sided) *sequent of* M^* we mean an expression of the form:

(1.3.) $\vdash T$

where T is a finite and non-empty sequence of indexed formulas. We distinguish the class of *atomic sequents of* M^*, that is, expressions of the form:

(1.4.) $\vdash (A)^1$

Atomic d-wffs of M^* are sequents of M^*. *Compound d-wffs of* M^* are sequences of sequents of M^*. We shall use Greek lower-case letters φ, ψ as metalinguistic variables for atomic d-wffs of M^*, and Greek upper-case letters Φ, Ψ as metavariables for compound d-wffs of M^*. For simplicity, we use notions of a one-sequent question and a many-sequent question. A *one-sequent question of* M^* is an expression of the form:

(1.5.) $?(\vdash T)$

where $\vdash T$ is a sequent of M^*. A *many-sequent question of* M^* is an expression of the form:

(1.6.) $?(\vdash S, ..., \vdash T)$

where $\vdash S, ..., \vdash T$ are sequents of M^*. We shall say that sequents $\varphi, ..., \psi$ are *contained in* a question $?(\varphi, ..., \psi)$. *Questions* of M^* are one-sequent questions and many-sequent questions, exclusively. We use $Q, Q', Q_1, ...$ for questions of M^*.

2.2. *Semantics of* M^*

In what follows the letter L will stay for any of the logics: K, D, $\mathsf{K4}$, T, $\mathsf{S4}$, KB, $\mathsf{S5}$. An expression 'L-properties' refers to properties of the accessibility relation that are characteristic to a given logic L. These are listed below.[3]

[3] We use here the terminology of Priest (2001) and Hughes and Cresswell (1996).

Logic L:	L-properties:
K	no properties
D	extendability
K4	transitivity
T	reflexivity
KB	symmetry
S4	transitivity and reflexivity
S5	transitivity, reflexivity and symmetry

In the case of language M we make use of standard notions of Kripke's semantics. By a *frame* we mean an ordered pair $< W, R >$, where W is a non-empty set (intuitively, of *possible worlds*) and R is a dyadic relation defined over W (and called the *accessibility relation*). A *valuation on a frame* $< W, R >$ is a function satisfying the standard conditions. *A model* $< W, R, V >$ is a frame $< W, R >$ together with a valuation V on it.

We shall say that a formula A is *true in a world* w *of a model* $< W, R, V >$ (or that it holds in w) iff $V(A, w) = 1$. A formula A is *valid in a model* $< W, R, V >$ iff for every $w \in W$, $V(A, w) = 1$. A formula A is $\mathsf{K}-valid$ iff A is valid in every model. The notions of D-, K4-, T-, KB-, S4-, S5-validity of a formula of M are defined as usual. Generally, a formula A of M is *L-valid* iff A is valid in every model $< W, R, V >$ in which R has the L-properties.

Before we introduce the notion of validity of a sequent of M^*, we need to 'interpret' sequents in frames. In order to do this we have to define some auxiliary notions.

Let $S = (A_1)^{\phi(i_1)}, ..., (A_n)^{\phi(i_n)}$. The sets $\mathsf{I}\{S\}$ and $\mathsf{I}[S]$ are defined as follows:

- $\mathsf{I}\{S\} = \{j : j$ is a term of some $\phi(i_k)$, where $1 \leq k \leq n\}$
- $\mathsf{I}[S] = \{< j, j' >:< j, j' >$ is a subsequence of some $\phi(i_k)$ (where $1 \leq k \leq n$) and j immediately precedes j' in $\phi(i_k)\}$

Thus, if S is a finite sequence of indexed formulas, then $\mathsf{I}\{S\}$ is the set of all the numerals that occur in indices of terms of S, and $\mathsf{I}[S]$ is the set of all the ordered pairs $< j, j' >$ that are subsequences of indices of terms of S, but with the restriction that j immediately precedes j' in an index. The idea is simple. For a sequent $\vdash S$ and a frame $< W, R >$ we are going to map the set $\mathsf{I}\{S\}$ into W, and, analogously, the set $\mathsf{I}[S]$ — into R. We shall call such a mapping an interpretation of sequent $\vdash S$ in frame $< W, R >$. More formally:

Definition 2.1: *Let* $S = (A_1)^{\phi(i_1)}, ..., (A_n)^{\phi(i_n)}$. *By an* interpretation of *sequent* $\vdash S$ in *a frame* $< W, R >$ *we mean a function* f: $\mathrm{I}\{S\} \mapsto W$ *satisfying the following condition:*

(*) if $< i, j > \in \mathrm{I}[S]$, *then* $< f(i), f(j) > \in R$.

We say that a sequent φ is *interpretable in* a frame $< W, R >$ iff there exists an interpretation of φ in $< W, R >$.

For clarity, let us note a few facts about sequents and their interpretations.

Fact 1: *There usually exists more than one interpretation of a sequent in a given frame. For example, let* $\vdash S$ *be* $\vdash (p)^{1,2}, (p \rightarrow q)^{2,3}$, *and let* $< W, R >$ *be such that:* $W = \{w, w', w'', w'''\}$, *and* $R = \{< w, w' >, < w', w'' >, < w'', w''' >, < w''', w >\}$. *We have* $\mathrm{I}\{S\} = \{1, 2, 3\}$ *and* $\mathrm{I}[S] = \{< 1, 2 >, < 2, 3 >\}$. *There are exactly four interpretations of* $\vdash S$ *in* $< W, R >$, *namely:*

f: $f(1) = w$, $f(2) = w'$, $f(3) = w''$
g: $g(1) = w'$, $g(2) = w''$, $g(3) = w'''$
h: $h(1) = w''$, $h(2) = w'''$, $h(3) = w$
k: $k(1) = w'''$, $k(2) = w$, $k(3) = w'$

Fact 2: *For every sequent* φ *there exists a frame* $< W, R >$ *such that* φ *is interpretable in* $< W, R >$.

As an illustration we give a 'recipe' for constructing, for a given sequent φ: (i) a certain frame $< W, R >$, and (ii) an interpretation of φ in $< W, R >$. Let $\varphi = \vdash S$. We put: (i) $W = \mathrm{I}\{S\}$ and $R = \mathrm{I}[S]$, and (ii) the identity function f: $\mathrm{I}\{S\} \mapsto W$ as the interpretation of $\vdash S$ in $< W, R >$.

A frame constructed for a sequent φ according to (i) will be called a *canonical frame* for φ, and an interpretation of φ in its canonical frame, constructed according to (ii), will be called the *canonical interpretation of* φ *in its canonical frame*.

Fact 3: *Every sequent is interpretable in more than one frame. Indeed, for a given sequent* φ, *it is enough to consider its canonical frame* $< W, R >$ *and any frame* $< W', R' >$ *such that* W *is included in* W', R *is included in* R', *and the inclusion is proper in at least one case. The identity function remains an interpretation of sequent* φ *in any such frame.*

Fact 4: *It may be the case that a sequent is not interpretable in some frames. Here are two examples: sequent* $\varphi = \vdash (p)^{1,2}$ *is not interpretable in any* *frame* $< W, R >$ *such that* R *is empty; sequent* $\varphi =\vdash (p)^{1,2,3}$ *is not interpretable in any frame* $< W, R >$ *such that* $R = \{< w, w' >\}$ *and* $w \neq w'$.

Fact 5: *Every sequent* $\vdash S$ *such that* $\mathsf{I}[S]$ *is empty is interpretable in every frame* $< W, R >$*, as the (*) condition is vacuously satisfied. Indeed, in such a case any function* $f : \mathsf{I}\{S\} \mid\rightarrow W$ *is an interpretation of* $\vdash S$ *in* $< W, R >$.

The notion of validity of a sequent is relativized both to a frame and to an interpretation of the sequent in that frame. In order to define this notion, we shall start with a more elementary notion of satisfaction of a sequent in a model. This notion is also relativized to an interpretation of a sequent. Roughly speaking, a sequent $\varphi =\vdash (A_1)^{\phi(i_1)}, ..., (A_n)^{\phi(i_n)}$ is satisfied in a model $< W, R, V >$ (under an interpretation f of φ in $< W, R >$), if *at least one* formula A_k $(1 \leq k \leq n)$ is true in the world assigned to numeral i_k by interpretation f. More formally:

Definition 2.2: *Let* $< W, R >$ *be a frame and let* V *be a valuation on* $< W, R >$. *A sequent* $\varphi =\vdash (A_1)^{\phi(i_1)}, ..., (A_n)^{\phi(i_n)}$ *is* satisfied in a model $< W, R, V >$ under an interpretation f of φ in frame $< W, R >$ *iff for some* k $(1 \leq k \leq n) : V(A_k, f(i_k)) = 1$.

Definition 2.3: *A sequent* φ *is* valid in a frame $< W, R >$ under an interpretation f of φ in $< W, R >$ *iff for every valuation* V *on frame* $< W, R >$*, the sequent* φ *is satisfied in a model* $< W, R, V >$ *under interpretation* f *of* φ *in frame* $< W, R >$.

Definition 2.4: *A sequent* φ *is* valid in a frame $< W, R >$ *iff* φ *is valid in* $< W, R >$ *under every interpretation* f *of* φ *in* $< W, R >$.

Observe that, according to the above definition, sequent φ is not valid in a frame $< W, R >$ iff there exists an interpretation f of φ in $< W, R >$ such that φ is not valid in $< W, R >$ under f. Therefore we have:

Corollary 2.1: *If a sequent* φ *is not interpretable in a frame* $< W, R >$*, then* φ *is valid in* $< W, R >$.

Definition 2.5: *A sequent* φ *is* K-valid *iff* φ *is valid in every frame.*

The notion of K-validity of a d-wff of M^* may be adjusted easily to any L.

Definition 2.6: *A sequent* φ *is* L-valid *iff* φ *is valid in every frame* $< W, R >$ *such that* R *has the* L*-properties.*

The following corollary immediately follows from the above definitions:

Corollary 2.2: *A sequent φ is not L-valid iff for some model $< W, R, V >$, where R has the L-properties, and for some interpretation f of φ in frame $< W, R >$, the sequent φ is not satisfied in the model $< W, R, V >$ under f.*

Definition 2.7: *A compound d-wff $\Phi = \varphi_1, ..., \varphi_n$ of M^* is L-*valid *iff each term φ_i $(1 \leq i \leq n)$ of Φ is L-valid.*

The notion of validity defined for d-wffs of M^* (*i.e.* for sequences of sequents) generalizes the notion of validity of formulas of the underlying modal language M. In the sequel we will be using two notions of L-validity (*i.e.* L-validity of a formula of M and L-validity of a d-wff of M^*), but the context should prevent any ambiguities. Now we shall prove:

Theorem 2.1: *An atomic sequent $\vdash (A)^1$ is L-valid iff the formula A of language M is L-valid.*

Proof. Let us observe, first, that the set $\mathsf{I}[(A)^1]$ is empty, and thus the sequent $\vdash (A)^1$ is interpretable in every frame (*cf.* Fact 5). We will show that the lack of L-validity of sequent $\vdash (A)^1$ is tantamount to the lack of L-validity of formula A.

1. Suppose that $\vdash (A)^1$ is not L-valid. Then, by Corollary 2.2, for some frame $< W, R >$ (with R having the L-properties) and for some interpretation f of sequent $\vdash (A)^1$ in $< W, R >$, there is a valuation V on $< W, R >$ such that sequent $\vdash (A)^1$ is not satisfied in model $< W, R, V >$ under f. But this means that $V(A, f(1)) = 0$. Therefore formula A is not valid in the model $< W, R, V >$. Hence A is not L-valid.

2. Assume that formula A is not L-valid. Therefore for some frame $< W, R >$ (with the L-properties imposed on R) and some valuation V on it there is $w \in W$ such that $V(A, w) = 0$. We define a function $f : \mathsf{I}\{(A)^1\} \mapsto W$ such that $f(1) = w$. The function f is an interpretation of $\vdash (A)^1$ in $< W, R >$ (*cf.* Fact 5) and, obviously, the sequent $\vdash (A)^1$ is not valid in $< W, R >$ under f, and hence it is not L-valid. □

Theorem 2.1 shows that a one-sequent question of the form $?(\vdash (A)^1)$ may be interpreted as a question about L-validity of the formula A. One may wonder whether this result may be generalized to non-atomic sequents of M^*. Unfortunately, the answer comes to the negative. There is no straightforward correspondence between L-validity of a non-atomic sequent and L-validity of formulas of M that occur in the sequent, as the following examples show. The formula '$p \vee \neg p$' is obviously L-valid. On the other hand, sequent

$\vdash (p)^{\phi(i)}, (\neg p)^{\phi(j)}$ is L-valid iff $i = j$. Sequent $\vdash (\Diamond p)^1, (\neg p)^{1,2}$ is K-valid but formula '$\Diamond p \vee \neg p$' is not K-valid.

However, this lack of correspondence between L-validity of non-atomic sequents and L-validity of formulas that occur in them should not worry us. A Socratic proof, as we shall define it in section 3.2, starts with a question of the form $?(\vdash (A)^1)$. Hence the SP-method allows us to answer questions about validity of formulas of M. As to non-atomic questions, we may say that a non-atomic, one-sequent question asks about L-validity of the sequent contained in it, and that a many-sequent question asks about *joint* L-validity of the sequents contained in it.

3. *Calculi* E^{L}

3.1. *Some examples*

The aim of this section is to give an intuitive account of the SP-method. We do it in a semi-formal way; the appropriate rules will be introduced in the sequel. As we shall see, a transformation of an initial question may be viewed as an attempt to find a Kripke model falsifying the formula considered in our initial question.

Example 1: $\Box(p \to q) \to (\Box p \to \Box q)$

Suppose that one considers whether axiom K is valid in every Kripke model. Alternatively, one may ask if it is the case that in some model there is a world, let us designate it by w_1, in which $\Box(p \to q) \to (\Box p \to \Box q)$ does not hold. The last question may be expressed in M^* in the following way:

(1) $?(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)$

The analyzed formula gets value 0 iff value 1 is assigned to its antecedent (and thus 0 to its negation), and value 0 is assigned to the consequent. The same pertains to formula $\Box p \to \Box q$, so one arrives at:

(2) $?(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (\Box q)^1)$

But formula $\Box q$ does not hold in w_1 iff propositional variable q is assigned value 0 in some world, say w_2, which is accessible from w_1. Thus one may ask the question:

(3) $?(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (q)^{1,2})$

Placing numeral 2 next to 1 indicates that w_2 is accessible from w_1. Obviously, formulas of the form: $\neg\Box A$ and $\Diamond\neg A$ have equal values in any world, so we arrive at:

(4) $?(\vdash (\Diamond\neg(p \to q))^1, (\Diamond\neg p)^1, (q)^{1,2})$

Now observe that a formula of the form $\Diamond A$ is assigned value 0 in world w_1 iff formula A is assigned value 0 in every world accessible from w_1. Thus we may ask:

(5) $?(\vdash (\Diamond\neg(p \to q))^1, (\neg(p \to q))^{1,2}, (\Diamond\neg p)^1, (\neg p)^{1,2}, (q)^{1,2})$

Again, placing numeral 2 next to 1 indicates that w_2 is accessible from w_1. Formula $\neg(p \to q)$ is false at w_2 iff p is false at w_2 or q is true at w_2. Hence we have two possibilities:

(6) $?(\vdash (\Diamond\neg(p \to q))^1, \underline{(p)^{1,2}}, (\Diamond\neg p)^1, \underline{(\neg p)^{1,2}}, (q)^{1,2}$; and
$\vdash (\Diamond\neg(p \to q))^1, \underline{(\neg q)^{1,2}}, (\Diamond\neg p)^1, (\neg p)^{1,2}, \underline{(q)^{1,2}})$

and an evident answer to our initial question.

Example 2: $\Box\Box p \to \Box p$

The transformation starts with the question: is it the case that formula ‘$\Box\Box p \to \Box p$’ does not hold in some world of some Kripke model? We express this question as follows:

(1) $?(\vdash (\Box\Box p \to \Box p)^1)$

As above, we may eliminate the implication:

(2) $?(\vdash (\neg\Box\Box p)^1, (\Box p)^1)$

and the necessity operator:

(3) $?(\vdash (\neg\Box\Box p)^1, (p)^{1,2})$

Next, the negation sign is introduced in the scope of modality:

(4) $?(\vdash (\Diamond\neg\Box p)^1, (p)^{1,2})$

and, as in case of step (5) of the previous example, we arrive at a conclusion that the formula in the scope of possibility operator must be assigned value 0 in w_2, hence:

(5) $?(\vdash (\Diamond\neg\Box p)^1, (\neg\Box p)^{1,2}, (p)^{1,2})$
(6) $?(\vdash (\Diamond\neg\Box p)^1, (\Diamond\neg p)^{1,2}, (p)^{1,2})$

The only transformation that may be carried out at that moment is to repeat the reasoning that lead us to step (5) and to 'introduce' formula $(\neg\Box p)^{1,2}$ once again. In fact, this step may be repeated infinitely many times, but it is obvious that such transformations would be useless. Instead, one may regard that the consideration has been finished and, moreover, one may construct a counter-model using the sequent contained in the last question. What one needs is to consider a canonical frame $< W, R >$ for sequent $\vdash (\Diamond\neg\Box p)^1, (\Diamond\neg p)^{1,2}, (p)^{1,2}$, namely: $W = \{1, 2\}, R = \{< 1, 2 >\}$, its canonical interpretation in this frame and a valuation assigning to literal p value 0 in world $f(2)$. Thus we have: $V(\Box p, 1) = 0$ (as world 2 is accessible from world 1), $V(\Box p, 2) = 1$ (as there is no world accessible from 2 in this frame) and $V(\Box\Box p, 1) = 1$ (as 2 is the only world accessible from 1). Thus $(\Box\Box p \to \Box p)$ does not hold in world 1.

Note that in order to construct a counter-model it was enough to consider only the value of indexed literals — in this example the sole p in 2 — the values of $\Box p$ and $\Box\Box p$ were calculated. It does not have to be the case in general, consider for example the formula '$\Box p \to \Diamond p$'. The transformation will *stop* at a question $?(\vdash (\Diamond\neg p)^1, (\Diamond p)^1)$, but in this case the information that there is only one world under consideration will be sufficient to construct a counter-model.

Example 3: $\Box p \to \Box\Box p$

We shall omit the first three steps and go directly to:

(1) $?(\vdash (\Diamond\neg p)^1, (\Box p)^{1,2})$

Similarly as above, one may regard formula $\neg p$ as false in w_2 and formula p as false in some w_3:

(2) $?(\vdash (\Diamond\neg p)^1, (\neg p)^{1,2}, (p)^{1,2,3})$

The transformation stops at this point. There is *no possibility* to introduce $(p)^{1,3}$, as w_3 is supposed to stand in accessibility relation to w_2, but not necessarily to w_1. The construction of a counter-model is straightforward.

Suppose, however, that one considers the question $?(\vdash (\Box p \rightarrow \Box\Box p)^1)$ once again, but this time one wonders whether formula $\Box p \rightarrow \Box\Box p$ is false in some world of a model $< W, R, V >$, where R is transitive. If R is transitive, then, since world w_2 is accessible from world w_1 and world w_3 is accessible from world w_2, world w_3 is accessible from w_1 as well. If so, from the fact that $\Diamond\neg p$ is false at w_1 one arrives at the conclusion that $\neg p$ is false at w_3. Thus question (2) can be transformed into question $?(\vdash (\Diamond\neg p)^1, (\neg p)^{1,3}, (\neg p)^{1,2}, (p)^{1,2,3})$. We shall generalize this observation in section 3.3.

3.2. *The Calculus* E^{K}

In this section we present rules of calculus of questions pertaining to logic K. We term it calculus E^{K} ("E" after "erotetic"). The calculus has only rules; there are no axioms.

Recall that Φ and Ψ stand for finite (possibly empty) sequences of sequents, and letters S and T represent finite (possibly empty) sequences of indexed formulas. Two concatenation-signs are used below: the sign $'$ is used as a concatenation-sign for sequences of indexed formulas, and the semicolon ';' is used as a concatenation-sign for sequences of sequents. Here are the rules of E^{K}.

$$\mathsf{R}_\alpha: \quad \frac{?(\Phi;\vdash S'(\alpha)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\alpha_1)^{\phi(i)}{}'T;\vdash S'(\alpha_2)^{\phi(i)}{}'T;\Psi)} \qquad \mathsf{R}_\beta: \quad \frac{?(\Phi;\vdash S'(\beta)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\beta_1)^{\phi(i)}{}'(\beta_2)^{\phi(i)}{}'T;\Psi)}$$

$$\mathsf{R}_{\neg\neg}: \quad \frac{?(\Phi;\vdash S'(\neg\neg A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i)}{}'T;\Psi)}$$

$$\mathsf{R}_{\neg\Box}: \quad \frac{?(\Phi;\vdash S'(\neg\Box A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Diamond\neg A)^{\phi(i)}{}'T;\Psi)} \qquad \mathsf{R}_{\neg\Diamond}: \quad \frac{?(\Phi;\vdash S'(\neg\Diamond A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Box\neg A)^{\phi(i)}{}'T;\Psi)}$$

Rules R_α and R_β are, actually, schemas of rules. Instantiations of schemas R_α and R_β are given in Appendix 2. An application of any of the rules: R_α, R_β or $\mathsf{R}_{\neg\neg}$, results in the elimination of a CPC-connective (in case of R_α and R_β) or of a double negation (in case of $\mathsf{R}_{\neg\neg}$). Rules $\mathsf{R}_{\neg\Box}$ and $\mathsf{R}_{\neg\Diamond}$ allow for introducing negation in the scope of modality. Indices of formulas are *not operated on* in case of rules: R_α, R_β, $\mathsf{R}_{\neg\neg}$, $\mathsf{R}_{\neg\Box}$ and $\mathsf{R}_{\neg\Diamond}$. Any modification of indices during the transformation is due to an application of rule $\mathsf{R}_\Box$ or of rule $\mathsf{R}_\Diamond$. The schemas of these rules are the following:

$$\mathsf{R}_\Box: \quad \frac{?(\Phi;\vdash S'(\Box A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i),j}{}'T;\Psi)} \qquad \mathsf{R}_\Diamond: \quad \frac{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T;\Psi)}$$

with the following provisos of applicability of these rules:

rule $\mathsf{R}_\Box$ may be applied provided that numeral j does not occur in the upper question, that is: $j \notin \mathsf{I}\{S'(\Box A)^{\phi(i)}{}'T\}$

rule $\mathsf{R}_\Diamond$ may be applied provided that the ordered pair $< i, j >$ already occurs in the upper question, that is: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$

Let us now repeat the transformation of question $?(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)$ (example 1 from the previous section). Every question (except for the first one) of the sequence of questions presented below has been obtained from the previous one by an application of a rule of E^K; for transparency, we highlight the indexed formula acted upon.

$?(\vdash (\Box(p \to q) \to (\Box p \to \Box q))^1)$	
$?(\vdash (\neg\Box(p \to q))^1, (\Box p \to \Box q)^1)$	by R_β
$?(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (\Box q)^1)$	by R_β
$?(\vdash (\neg\Box(p \to q))^1, (\neg\Box p)^1, (q)^{1,2})$	by $\mathsf{R}_\Box$
$?(\vdash (\Diamond\neg(p \to q))^1, (\neg\Box p)^1, (q)^{1,2})$	by $\mathsf{R}_{\neg\Box}$
$?(\vdash (\Diamond\neg(p \to q))^1, (\Diamond\neg p)^1, (q)^{1,2})$	by $\mathsf{R}_{\neg\Box}$
$?(\vdash (\Diamond\neg(p \to q))^1, (\neg(p \to q))^{1,2}, (\Diamond\neg p)^1, (q)^{1,2})$	by $\mathsf{R}_\Diamond$
$?(\vdash (\Diamond\neg(p \to q))^1, (\neg(p \to q))^{1,2}, (\Diamond\neg p)^1, (\neg p)^{1,2}, (q)^{1,2})$	by $\mathsf{R}_\Diamond$
$?(\vdash (\Diamond\neg(p \to q))^1, (p)^{1,2}, (\Diamond\neg p)^1, (\neg p)^{1,2}, (q)^{1,2};$ $\vdash (\Diamond\neg(p \to q))^1, (\neg q)^{1,2}, (\Diamond\neg p)^1, (\neg p)^{1,2}, (q)^{1,2})$	by R_α

We introduce the notions of a Socratic transformation of a question and a Socratic proof of an atomic sequent in E^K:

Definition 3.1: A Socratic transformation of a question Q via the rules of E^K *is a sequence* $s = Q_1, Q_2, \ldots$ *(possibly infinite) of questions such that:* $Q_1 = Q$*, and for each* $n > 1$*, question* Q_n *results from question* Q_{n-1} *by an application of one of the rules of* E^K.

Definition 3.2: *Let* $\vdash (A)^1$ *be an atomic sequent of* M^*. A Socratic proof of $\vdash (A)^1$ in E^K *is a finite Socratic transformation* s *of the question* $?(\vdash (A)^1)$ *via the rules of* E^K *such that for each sequent* φ *contained in the last question of* s *the following holds:*

(a) φ *is of the form* $\vdash S'(C)^{\phi(i)}{}'T'(\neg C)^{\phi^*(i)}{}'U$, *or*
(b) φ *is of the form* $\vdash S'(\neg C)^{\phi^*(i)}{}'T'(C)^{\phi(i)}{}'U$.

Let us emphasize that the symbols $\phi(i)$ and $\phi^*(i)$ in (a) or (b) may not stand for occurrences of *the same* index. However, the sequences $\phi(i)$ and $\phi^*(i)$ end with the same numeral, and this is the crucial point. Recall that when we deal with an indexed formula $(A)^{i_1,\dots,i_n}$, it is only the numeral i_n that 'indicates' the world in which the value of A is relevant. The role of the remaining part of an index (if there is any) is to determine conditions that must be fulfilled by the accessibility relation in the purported counter-model.

Before we present the rules of other calculi, let us make one more remark. By an analogy to tableau methods (Priest (2001) or Fitting (1983)), it may seem that the rule for eliminating possibility operator $\Diamond$ should have been:

$$\mathsf{R}_{\Diamond}\colon \frac{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{j}{}'T;\Psi)}$$

with the proviso that it may be applied provided that numeral j is new. But it may be shown easily that such a rule would not warrant transmission of joint L-validity. Moreover, if the old proviso, namely $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$, were put, the transmission of joint L-validity from top to bottom would not hold. The situation is similar in the case of the necessity operator.

This difference of the presented method with respect to tableau methods for modal logics is due to the fact that a 'Socratic proof procedure' remains a direct procedure. It *does not* start with the negation of an initial assumption and, hence, it is not an indirect proof method. However, a Socratic proof may still be interpreted as an unsuccessful attempt to find a counter-model, as the examples in section 3.1. have shown.

3.3. *Calculi* E^{L}

In this section L will vary through the following proper extensions of K: D, $\mathsf{K4}$, T, KB, $\mathsf{S4}$, $\mathsf{S5}$. Each calculus E^{L} has, first, rules R_{α}, R_{β}, $\mathsf{R}_{\neg\neg}$, $\mathsf{R}_{\neg\Box}$, $\mathsf{R}_{\neg\Diamond}$, $\mathsf{R}_{\Box}$ (in the form presented above) and, second, a rule $\mathsf{R}_{\Diamond}$ with a proviso of its applicability varying from a calculus to a calculus. The form of the proviso depends on the properties of the accessibility relation specific to L.

For each modal logic L we give a description of the proviso on rule $\mathsf{R}_{\Diamond}$ and examples of transformations. We shall use 'P^{L}' for proviso of applicability of rule $\mathsf{R}_{\Diamond}$ of calculus E^{L}.

Rule $\mathsf{R}_{\Diamond}$:

$$\frac{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T;\Psi)}$$

Calculus E^{D}, P^{D}: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j \notin \mathsf{I}\{S'(\Diamond A)^{\phi(i)}{}'T\}$

In the case of calculus E^{D} rule $\mathsf{R}_\Diamond$ may be applied provided that numeral i immediately precedes numeral j in an index of some formula or numeral j does not occur in the sequent. The second part of the proviso corresponds to extendability of the accessibility relation in D.

Let us stress that in the case of a single application of rule $\mathsf{R}_\Diamond$ *only one part of the proviso* should be satisfied. (Indeed, it is easy to observe that only one part, not both, of the proviso *can* be satisfied in a given case.)

We have already discussed formula '$\Box p \rightarrow \Diamond p$'. The transformation via the rules of E^{D} is the following:

1.	$?(\vdash (\Box p \rightarrow \Diamond p)^1)$	
2.	$?(\vdash (\neg\Box p)^1, (\Diamond p)^1)$	by R_β
3.	$?(\vdash (\Diamond\neg p)^1, (\Diamond p)^1)$	by $\mathsf{R}_{\neg\Box}$
4.	$?(\vdash (\Diamond\neg p)^1, (\neg p)^{1,2}, (\Diamond p)^1)$	by $\mathsf{R}_\Diamond$
5.	$?(\vdash (\Diamond\neg p)^1, (\neg p)^{1,2}, (\Diamond p)^1, (p)^{1,2})$	by $\mathsf{R}_\Diamond$

Questions (4) and (5) are obtained by $\mathsf{R}_\Diamond$, but on different parts of the proviso. In the case of question (4) numeral 2 satisfies condition P^{D}, as 2 is not an element of $\mathsf{I}\{(\Diamond\neg p)^1, (\Diamond p)^1\}$. In the case of question (5), the ordered pair $< 1, 2 >$ is already an element of $\mathsf{I}[(\Diamond\neg p)^1, (\neg p)^{1,2}, (\Diamond p)^1]$.

Calculus E^{K4}, P^{K4}: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or for some numeral k both $< i, k >$ and $< k, j >$ are the elements of $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$

The second part of the proviso corresponds to transitivity of the accessibility relation. Formula '$\Box p \rightarrow \Box\Box p$' has also been discussed above. We repeat the transformation:

1.	$?(\vdash (\Box p \rightarrow \Box\Box p)^1)$	
2.	$?(\vdash (\neg\Box p)^1, (\Box\Box p)^1)$	by R_β
3.	$?(\vdash (\Diamond\neg p)^1, (\Box\Box p)^1)$	by $\mathsf{R}_{\neg\Box}$
4.	$?(\vdash (\Diamond\neg p)^1, (\Box p)^{1,2})$	by $\mathsf{R}_\Box$
5.	$?(\vdash (\Diamond\neg p)^1, (p)^{1,2,3})$	by $\mathsf{R}_\Box$
6.	$?(\vdash (\Diamond\neg p)^1, (\neg p)^{1,3}, (p)^{1,2,3})$	by $\mathsf{R}_\Diamond$

Question (6) is obtained from question (5) by $\mathsf{R}_\Diamond$, as the second part of the proviso is satisfied. For numeral 3 there is a numeral k, namely 2, such that both: $< 1, k >$ and $< k, 3 >$ are the elements of $\mathsf{I}[(\Diamond\neg p)^1, (p)^{1,2,3}]$.

Calculus E^T, P^T: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j = i$

where the second part of the proviso corresponds to reflexivity.

1. $?(\vdash (\Box p \to p)^1)$
2. $?(\vdash (\neg\Box p)^1, (p)^1)$ by R_β
3. $?(\vdash (\Diamond\neg p)^1, (p)^1)$ by $\mathsf{R}_{\neg\Box}$
4. $?(\vdash (\Diamond\neg p)^1, (\neg p)^{1,1}, (p)^1)$ by $\mathsf{R}_\Diamond$

Question (4) is obtained from the previous one by $\mathsf{R}_\Diamond$, on the second part of the proviso, where $j = i = 1$.

Calculus E^KB, P^KB: $<i, j> \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $<j, i> \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$

where $< j, i > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ corresponds to symmetry.

1. $?(\vdash (p \to \Box\Diamond p)^1)$
2. $?(\vdash (\neg p)^1, (\Box\Diamond p)^1)$ by R_β
3. $?(\vdash (\neg p)^1, (\Diamond p)^{1,2})$ by $\mathsf{R}_\Box$
4. $?(\vdash (\neg p)^1, (\Diamond p)^{1,2}, (p)^{2,1})$ by $\mathsf{R}_\Diamond$

The last question results from the previous one by $\mathsf{R}_\Diamond$, on the second part of the proviso: we put $i = 2$ and $j = 1$.

Calculus E^S4, P^S4: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or for some numeral k: $< i, k >, < k, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j = i$

The transformations of questions $?(\vdash (\Box p \to \Box\Box p)^1)$ and $?(\vdash (\Box p \to p)^1)$ remain as in the calculi E^K4 and E^T. We present another example. Formula '$\Box\Diamond\Diamond p \to \Diamond p$' is a thesis of $\mathsf{S4}$.

1. $?(\vdash (\Box\Diamond\Diamond p \to \Diamond p)^1)$
2. $?(\vdash (\neg\Box\Diamond\Diamond p)^1, (\Diamond p)^1)$ by R_β
3. $?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\Diamond p)^1)$ by $\mathsf{R}_{\neg\Box}$

4.	$?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\neg\Diamond\Diamond p)^{1,1}, (\Diamond p)^1)$	by $\mathsf{R}_\Diamond$
5.	$?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\Box\neg\Diamond p)^{1,1}, (\Diamond p)^1)$	by $\mathsf{R}_{\neg\Diamond}$
6.	$?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\neg\Diamond p)^{1,1,2}, (\Diamond p)^1)$	by $\mathsf{R}_\Box$
7.	$?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\Box\neg p)^{1,1,2}, (\Diamond p)^1)$	by $\mathsf{R}_{\neg\Diamond}$
8.	$?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\neg p)^{1,1,2,3}, (\Diamond p)^1)$	by $\mathsf{R}_\Box$
9.	$?(\vdash (\Diamond\neg\Diamond\Diamond p)^1, (\neg p)^{1,1,2,3}, (\Diamond p)^1, (p)^{1,3})$	by $\mathsf{R}_\Diamond$

Question (4) results from the previous one by rule $\mathsf{R}_\Diamond$, on the third part of the proviso. Question (9) results from question (8) by the same rule, but on the second part of the proviso.

Calculus E^{S5}, P^{S5}: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}T]$ or for some numeral k: $< i, k >, < k, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}T]$ or $< j, i > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}T]$ or $j = i$

The transformation of question $?(\vdash (\Box p \to p)^1)$ is the same as in the calculus E^{T}. Here is another example.

1.	$?(\vdash (\Diamond p \to \Box\Diamond p)^1)$	
2.	$?(\vdash (\neg\Diamond p)^1, (\Box\Diamond p)^1)$	by R_β
3.	$?(\vdash (\Box\neg p)^1, (\Box\Diamond p)^1)$	by $\mathsf{R}_{\neg\Diamond}$
4.	$?(\vdash (\neg p)^{1,2}, (\Box\Diamond p)^1)$	by $\mathsf{R}_\Box$
5.	$?(\vdash (\neg p)^{1,2}, (\Diamond p)^{1,3})$	by $\mathsf{R}_\Box$
6.	$?(\vdash (\neg p)^{1,2}, (\Diamond p)^{1,3}, (p)^{3,1})$	by $\mathsf{R}_\Diamond$
7.	$?(\vdash (\neg p)^{1,2}, (\Diamond p)^{1,3}, (p)^{3,2}, (p)^{3,1})$	by $\mathsf{R}_\Diamond$

In case of the first application of rule $\mathsf{R}_\Diamond$ the third part of the proviso is satisfied. The second part of the proviso holds with respect to the last step.

We generalize the notions of a Socratic transformation of a question and of a Socratic proof of a sequent for the case of any calculus E^{L}:

Definition 3.3: A Socratic transformation of a question Q via the rules of E^{L} *is a sequence* $s = Q_1, Q_2, \ldots$ *(possibly infinite) of questions such that* $Q_1 = Q$ *and for each* $n > 1$, *question* Q_n *results from question* Q_{n-1} *by an application of one of the rules of* E^{L}.

Definition 3.4: *Let* $\vdash (A)^1$ *be an atomic sequent of* M^*. A Socratic proof of $\vdash (A)^1$ in E^L *is a finite Socratic transformation* s *of the question* $?(\vdash (A)^1)$ *via the rules of* E^L *such that for each sequent* φ *contained in the last question of* s *the following holds:*

(a) φ *is of the form* $\vdash S'(C)^{\phi(i)}{}'T'(\neg C)^{\phi^*(i)}{}'U$, *or*
(b) φ *is of the form* $\vdash S'(\neg C)^{\phi^*(i)}{}'T'(C)^{\phi(i)}{}'U$.

4. *Soundness*

In this section we address the problem of soundness of the SP-method. L stands, again, for any of: K, D, K4, T, KB, S4, S5. First, we prove that the rules of E^L warrant transmission of L-validity of sequents in both directions.

Theorem 4.1: *If question* $Q' = ?(\Phi')$ *results from question* $Q = ?(\Phi)$ *by one of the rules:* R_α, R_β, $\mathsf{R}_{\neg\neg}$, $\mathsf{R}_{\neg\Box}$ *or* $\mathsf{R}_{\neg\Diamond}$ *of* E^L, *then each term of* Φ *is* L*-valid iff each term of* Φ' *is* L*-valid.*

Proof. We shall consider case R_α only.

Assume that question $Q' = ?(\Phi')$ results from question $Q = ?(\Phi)$ by rule R_α. Then $\Phi = \Psi; \varphi; \Psi_1$ and $\Phi' = \Psi; \psi; \psi'; \Psi_1$, where $\varphi = \vdash S'(\alpha)^{\phi(i)}{}'T$, $\psi = \vdash S'(\alpha_1)^{\phi(i)}{}'T$ and $\psi' = \vdash S'(\alpha_2)^{\phi(i)}{}'T$. It suffices to show that the lack of L-validity of φ is tantamount to the lack of L-validity of at least one of the sequents: ψ, ψ'.

Observe that when rule R_α is applied, no operation is performed on the indices of formulas in sequent φ (and this pertains to every rule listed). This means that sets $\mathsf{I}\{S'(\alpha)^{\phi(i)}{}'T\}$, $\mathsf{I}\{S'(\alpha_1)^{\phi(i)}{}'T\}$ and $\mathsf{I}\{S'(\alpha_2)^{\phi(i)}{}'T\}$ are equal. Consequently, sets $\mathsf{I}[S'(\alpha)^{\phi(i)}{}'T]$, $\mathsf{I}[S'(\alpha_1)^{\phi(i)}{}'T]$ and $\mathsf{I}[S'(\alpha_2)^{\phi(i)}{}'T]$ are also equal. If this is the case, then any interpretation of one of the sequents: φ, ψ or ψ' in a frame $< W, R >$ is also an interpretation of any of the other two sequents in the same frame. Hence, in what follows, we are allowed to consider one interpretation of sequents φ, ψ and ψ' in a specified frame.

Suppose that sequent $\varphi = \vdash S'(\alpha)^{\phi(i)}{}'T$ is not L-valid. Then, by Corollary 2.2, for some frame $< W, R >$ (with R having the L-properties), for an interpretation f of sequent φ in frame $< W, R >$, and for some valuation V on $< W, R >$, the sequent φ is not satisfied in model $< W, R, V >$ under f. This means that $V(A_n, f(i_n)) = 0$ for each term $(A_n)^{\phi(i_n)}$ of S and T, and also that $V(\alpha, f(i)) = 0$. Hence either $V(\alpha_1, f(i)) = 0$ or $V(\alpha_2, f(i)) = 0$. If the first possibility holds then sequent $\psi = \vdash S'(\alpha_1)^{\phi(i)}{}'T$ is not satisfied in model $< W, R, V >$ under interpretation f of ψ in $< W, R >$. But then, by Corollary 2.2, ψ is not L-valid. If the second possibility holds then, by the same reasoning, ψ' is not L-valid. Hence, if sequent φ is not L-valid, then at least one of the sequents: ψ or ψ', is not L-valid.

Similarly, if one of the sequents: $\vdash S'(\alpha_1)^{\phi(i)\prime}T$ or $\vdash S'(\alpha_2)^{\phi(i)\prime}T$ is not L-valid, then the sequent $\vdash S'(\alpha)^{\phi(i)\prime}T$ is not L-valid.

For other rules the details of the proof concerning interpretation functions remain unchanged. And this is only the definition of a valuation function that really counts. □

Now we shall prove that rules $\mathsf{R}_\Box$ and $\mathsf{R}_\Diamond$ of E^L warrant transmission of joint L-validity of sequents in both directions.

Theorem 4.2: *If question $Q' = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by one of the rules $\mathsf{R}_\Box$, $\mathsf{R}_\Diamond$ of E^L, then each term of Φ is L-valid iff each term of Φ_1 is L-valid.*

Proof. Suppose that question $Q' = ?(\Phi_1)$ results from question $Q = ?(\Phi)$ by one of the rules $\mathsf{R}_\Box$, $\mathsf{R}_\Diamond$ of E^L. Then $\Phi = \Psi; \varphi; \Psi_1$, and $\Phi_1 = \Psi; \varphi_1; \Psi_1$, where either $\varphi = \vdash S'(\Box A)^{\phi(i)\prime}T$ and $\varphi_1 = \vdash S'(A)^{\phi(i),j\prime}T$ (in the case of rule $\mathsf{R}_\Box$), or $\varphi = \vdash S'(\Diamond A)^{\phi(i)\prime}T$ and $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ (in the case of rule $\mathsf{R}_\Diamond$). As the terms of d-wffs Ψ and Ψ_1 (if there are any) remain unchanged, for the proof it is enough to show that L-validity of sequent φ entails L-validity of sequent φ_1 and *vice versa*, or, alternatively, that non-L-validity of any of the two sequents entails non-L-validity of the other one. Moreover, in view of Corollary 2.2, it suffices to prove that there is a model in which one of the two sequents is not satisfied (under some interpretation) iff there is a model in which the other sequent is not satisfied (under some interpretation). This is the line we are going to follow.

As the shape of rule $\mathsf{R}_\Box$ does not depend on the choice of L, we will give only a proof that rule $\mathsf{R}_\Box$ preserves K-validity in both directions. In a case of L other than K appropriate conditions should be imposed on the accessibility relation in models we consider, but it is easy to observe that the modification is not essential for our proof. Things are different, however, in the case of rule $\mathsf{R}_\Diamond$. Thus we will consider the applications of $\mathsf{R}_\Diamond$ separately for each E^L.

E^K, rule $\mathsf{R}_\Box$:

We show that non-K-validity of sequent $\varphi_1 = \vdash S'(A)^{\phi(i),j\prime}T$ entails non-K-validity of sequent $\varphi = \vdash S'(\Box A)^{\phi(i)\prime}T$ and *vice versa*. From the proviso on rule $\mathsf{R}_\Box$ we have: $j \notin \mathsf{I}\{S'(\Box A)^{\phi(i)\prime}T\}$.

1. Suppose that sequent φ_1 is not K-valid. Hence there must be a frame $< W, R >$, an interpretation f of sequent φ_1 in $< W, R >$ and a valuation V on $< W, R >$ such that sequent φ_1 is not satisfied in the model

$< W, R, V >$. In particular, $V(A, f(j)) = 0$. We shall construct an interpretation $f*$ of sequent φ in $< W, R >$ such that φ is not satisfied in model $< W, R, V >$ under $f*$.

First, observe that the set of numerals $\mathsf{I}\{S'(\Box A)^{\phi(i)\prime}T\}$ is a (proper) subset of the set $\mathsf{I}\{S'(A)^{\phi(i),j\prime}T\}$, and analogously for sets $\mathsf{I}[S'(\Box A)^{\phi(i)\prime}T]$ and $\mathsf{I}[S'(A)^{\phi(i),j\prime}T]$. Let $f* : \mathsf{I}\{S'(\Box A)^{\phi(i)\prime}T\} \mapsto W$ be a function such that $f*$ and f assign the same values (possible worlds) to the numerals from the set $\mathsf{I}\{S'(\Box A)^{\phi(i)\prime}T\}$. Assume that $< k, k' >$ is a pair of numerals such that $< k, k' >\in \mathsf{I}[S'(\Box A)^{\phi(i)\prime}T]$. Then we have $< k, k' >\in \mathsf{I}[S'(A)^{\phi(i),j\prime}T]$, and $f*(k) = f(k)$ as well as $f*(k') = f(k')$. Since f is an interpretation of $\vdash S'(A)^{\phi(i),j\prime}T$ in $< W, R >$, it must be the case that $< f(k), f(k') >\in R$, and, thus, $< f*(k), f*(k') >\in R$. We have established that if $< k, k' >\in \mathsf{I}[S'(\Box A)^{\phi(i)\prime}T]$, then $< f*(k), f*(k') >\in R$. Hence $f*$ is an interpretation of the sequent $\vdash S'(\Box A)^{\phi(i)\prime}T$ in $< W, R >$.

Second, observe that numeral i immediately precedes numeral j in the index of the wff $(A)^{\phi(i),j}$. Thus the ordered pair $<i, j>$ is an element of the set $\mathsf{I}[S'(A)^{\phi(i),j\prime}T]$. As f is an interpretation of sequent φ_1 in $<W, R>$, the world $f(j)$ must be accessible from world $f(i)$. Hence, in view of $V(A, f(j)) = 0$, it follows that $V(\Box A, f(i)) = 0$ and also $V(\Box A, f*(i)) = 0$. Therefore the sequent $\varphi = \vdash S'(\Box A)^{\phi(i)\prime}T$ is not satisfied in model $<W, R, V>$ under $f*$.

2. Suppose that the sequent φ is not K-valid. As above, we will consider an arbitrary frame $< W, R >$, an interpretation f of sequent $\varphi = \vdash S'(\Box A)^{\phi(i)\prime}T$ in $< W, R >$, and a valuation V on $< W, R >$ such that φ is not satisfied in the model $< W, R, V >$. Since $V(\Box A, f(i)) = 0$, there must be a world w in W accessible from $f(i)$ and such that $V(A, w) = 0$. Now we shall define an interpretation $f*$ of sequent φ_1 in $< W, R >$ and show that φ_1 is not satisfied in $< W, R, V >$ under $f*$.

Let function $f*$ be an extension of f such that $f*(k) = f(k)$ for each $k \neq j$ and $f*(j) = w$. Assume that $< k, k' >$ is an arbitrary ordered pair such that $< k, k' >\in \mathsf{I}[S'(A)^{\phi(i),j\prime}T]$. If $k' \neq j$, then $< k, k' >\in \mathsf{I}[S'(\Box A)^{\phi(i)\prime}T]$. Hence $< f(k), f(k') >\in R$ (because f is an interpretation of $\vdash S'(\Box A)^{\phi(i)\prime}T$ in $< W, R >$) and thus also $< f*(k), f*(k') >\in R$. If $k' = j$, then $< k, k' >=< i, j >$ (this follows from the proviso on rule $\mathsf{R}_\Box$). Therefore, on the basis of $< f(i), w >\in R$, we have $< f*(i), f*(j) >\in R$. Thus $f*$ is an interpretation of $\vdash S'(A)^{\phi(i),j\prime}T$ in $< W, R >$.

For every term $(B_n)^{\phi(i_n)}$ of S and T: $V(B_n, f*(i_n)) = V(B_n, f(i_n)) = 0$. Since $V(A, w) = 0$ and $w = f*(j)$, we get $V(A, f*(j)) = 0$. Hence the sequent $\vdash S'(A)^{\phi(i),j\prime}T$ is not valid in $< W, R >$ under $f*$.

E^{K}, rule $\mathsf{R}_{\Diamond}$:

We will show that the lack of K-validity of sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ is tantamount to the lack of K-validity of sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)\prime}T$. From the proviso on rule $\mathsf{R}_{\Diamond}$ we have: $< i, j > \in I[S'(\Diamond A)^{\phi(i)\prime}T]$. Observe that since j is *not new* in sequent φ_1, the sets: $\mathsf{I}\{S'(\Diamond A)^{\phi(i)\prime}T\}$ and $\mathsf{I}\{S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T\}$, as well as the sets: $\mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}T]$ and $\mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T]$, are equal. Hence any interpretation of one of the sequents in an arbitrary frame is also an interpretation of the other sequent in the same frame.

1. Assume that sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ is not K-valid. Then there exist: a frame $< W, R >$, an interpretation f of sequent φ_1 in this frame, and a valuation V on $< W, R >$ such that for each term $(B_n)^{\phi(i_n)}$ of S or T: $V(B_n, f(i_n)) = 0$ as well as $V(\Diamond A, f(i)) = 0$. It follows that the sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)\prime}T$ is not satisfied in the model $< W, R, V >$ under f.

2. Suppose that sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)\prime}T$ is not K-valid. Then for some frame $< W, R >$, some interpretation f of $\vdash S'(\Diamond A)^{\phi(i)\prime}T$ in $< W, R >$, and some V on $< W, R >$, the sequent φ is not satisfied in $< W, R, V >$ under f. In particular, $V(\Diamond A, f(i)) = 0$. The interpretation f of sequent φ in $< W, R >$ is, as we have observed, also an interpretation of sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ in $< W, R >$. From the proviso on rule $\mathsf{R}_{\Diamond}$ we get $< i, j > \in I[S'(\Diamond A)^{\phi(i)\prime}T]$. Therefore world $f(j)$ is accessible from $f(i)$ in frame $< W, R >$. Thus $V(A, f(j)) = 0$. Hence the sequent $\vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ is not K-valid. This ends the proof for E^{K}.

E^{D}, rule $\mathsf{R}_{\Diamond}$:

We shall prove that the lack of D-validity of sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ yields the lack of D-validity of sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)\prime}T$ and *vice versa*. From the proviso on rule $\mathsf{R}_{\Diamond}$ of the calculus E^{D} we have: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}T]$ or $j \notin \mathsf{I}\{S'(\Diamond A)^{\phi(i)\prime}T\}$. If $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)\prime}T]$, then the reasoning goes as for K, with the exception that the accessibility relation is now extendable. We assume that $j \notin \mathsf{I}\{S'(\Diamond A)^{\phi(i)\prime}T\}$.

1. Suppose that $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)\prime}(A)^{i,j\prime}T$ is not D-valid. Again, there exist: a frame $< W, R >$ (where R is extendable), an interpretation f of φ_1 in that frame, and a valuation V on $< W, R >$ such that φ_1 is not satisfied in $< W, R, V >$ under f. Now we need to define an interpretation $f*$ of sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)\prime}T$ in $< W, R >$. For that purpose it is enough to assume that $f*$ and f assign the same values (*i.e.* possible worlds) to the numerals from $\mathsf{I}\{S'(\Diamond A)^{\phi(i)\prime}T\}$. Under this assignment, for each term $(B_n)^{\phi(i_n)}$ of S or T we have $V(B_n, f*(i_n)) = V(B_n, f(i_n)) = 0$, and also

$V(\Diamond A, f*(j)) = V(\Diamond A, f(j)) = 0$. Hence sequent $\vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not satisfied in $< W, R, V >$ under $f*$.

2. Suppose that sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not D-valid. Again, for a certain frame $< W, R >$ (where R is extendable), some interpretation f of $\vdash S'(\Diamond A)^{\phi(i)}{}'T$ in $< W, R >$, and some V on $< W, R >$, the sequent φ is not satisfied in $< W, R, V >$ under f. In particular, $V(\Diamond A, f(i)) = 0$. As R is extendable, there must be $w \in W$ such that $< f(i), w > \in R$ and, obviously, $V(A, w) = 0$. Now we need an interpretation $f*$ of the sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ in $< W, R >$ such that φ_1 is not satisfied in $< W, R, V >$ under $f*$. We define $f*$ similarly as in the case E^{K}, rule $\mathsf{R}_{\Box}$, and put $f*(j) = w$.

E^{K4}, rule $\mathsf{R}_{\Diamond}$:

We prove that the lack of $\mathsf{K4}$-validity of sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}' T$ yields the lack of $\mathsf{K4}$-validity of sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)}{}'T$, and conversely. From the proviso on rule $\mathsf{R}_{\Diamond}$ of the calculus E^{K4} we have: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or, for some numeral k, both $< i, k >$ and $< k, j >$ are the elements of $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$. If $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$, then the reasoning goes as for K, with the exception that the accessibility relation in the frames we consider is transitive. Suppose that the second possibility holds, that is for some numeral k both $< i, k >$ and $< k, j >$ are the elements of $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$.

Since no new numeral occurs in sequent φ_1 the sets: $\mathsf{I}\{S'(\Diamond A)^{\phi(i)}{}'T\}$ and $\mathsf{I}\{S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T\}$ are equal, whereas the sets: $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ and $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T]$ differ at most with respect to pair $< i, j >$. It is easy to observe that if sequent φ_1 results from sequent φ on the second part of the proviso (as we have assumed) then any interpretation of one of the sequents in a transitive frame is also an interpretation of the other sequent in the same frame.

1. Suppose that sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is not $\mathsf{K4}$-valid. Again, there exist: a frame $< W, R >$ (where R is transitive), an interpretation f of sequent φ_1 in this frame, and a valuation V on $< W, R >$ such that for each term $(B_n)^{\phi(i_n)}$ of S and T: $V(B_n, f(i_n)) = 0$, as well as $V(\Diamond A, f(i)) = 0$. It follows that the sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not satisfied in model $< W, R, V >$ under f.

2. Suppose that sequent $\varphi = \vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not $\mathsf{K4}$-valid. Hence for some frame $< W, R >$ such that R is transitive, for some interpretation f of $\vdash S'(\Diamond A)^{\phi(i)}{}'T$ in $< W, R >$, and for some V on $< W, R >$, the sequent φ is not satisfied in $< W, R, V >$ under f. In particular, $V(\Diamond A, f(i)) = 0$. By the proviso on rule $\mathsf{R}_{\Diamond}$, there is a numeral k such that both $< i, k >$ and

$< k, j >$ are the elements of $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$. Thus, in view of transitivity of R, we have $V(A, f(j)) = 0$. Therefore the sequent $\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is not valid in $< W, R >$ under f.

E^{T}, rule $\mathsf{R}_\Diamond$:

We shall prove that the lack of T-validity of sequent $\varphi_1 =\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is tantamount to the lack of T-validity of sequent $\varphi =\vdash S'(\Diamond A)^{\phi(i)}{}'T$. From the proviso on rule $\mathsf{R}_\Diamond$ of calculus E^{T} we have: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j = i$. We suppose that the second possibility holds and observe, as in the previous case, that under this assumption any interpretation of one of the sequents in a reflexive frame is also an interpretation of the other sequent in the same frame.

1. Suppose that sequent $\varphi_1 =\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is not T-valid. Then there exist: a frame $< W, R >$ (where R is reflexive), an interpretation f of φ_1 in $< W, R >$, and a valuation V on $< W, R >$ such that for each term $(B_n)^{\phi(i_n)}$ of S or T, $V(B_n, f(i_n)) = 0$ as well as $V(\Diamond A, f(i)) = 0$. Hence sequent $\varphi =\vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not satisfied in $< W, R, V >$ under f.

2. Suppose that sequent $\varphi =\vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not T-valid. Then there exist: a frame $< W, R >$, where R is reflexive, an interpretation f of φ in $< W, R >$, and a valuation V on $< W, R >$ such that φ is not satisfied in model $< W, R, V >$ under f. In particular, $V(\Diamond A, f(i)) = 0$. Since $j = i$ and R is reflexive, we have $V(A, f(j)) = 0$. Thus the sequent $\varphi_1 =\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is not satisfied in $< W, R, V >$ under f.

E^{KB}, rule $\mathsf{R}_\Diamond$:

Again, we shall prove that non-KB-validity of sequent $\varphi_1 =\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is equivalent to non-KB-validity of sequent $\varphi =\vdash S'(\Diamond A)^{\phi(i)}{}'T$. From the proviso on rule $\mathsf{R}_\Diamond$ of the calculus E^{KB} we have: $< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $< j, i > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$. We suppose that the second possibility holds. As in previous cases, we may observe that each interpretation of one of the sequents in a symmetric frame is an interpretation of the second sequent in the same frame.

1. For the first implication the reasoning is the same as that for rule $\mathsf{R}_\Diamond$ of calculus E^{K4} or E^{T} (with the exception that R is symmetric).

2. For the second one suppose that sequent $\varphi =\vdash S'(\Diamond A)^{\phi(i)}{}'T$ is not KB-valid. It follows that there are: a frame $< W, R >$, where R is symmetric, an interpretation f of φ in $< W, R >$, and a valuation V on $< W, R >$ such that, first, B_n is false in $f(i_n)$ for each term $(B_n)^{\phi(i_n)}$ of S or T, and second, $V(\Diamond A, f(i)) = 0$. Since the ordered pair $< j, i >$ is an element of $\mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ and R is symmetric, formula A is false in $f(j)$. Therefore

the sequent $\varphi_1 = \vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T$ is not satisfied in $< W, R, V >$ under f.

E^{S4}, rule $\mathsf{R}_{\Diamond}$:

The reasoning goes as for E^{K} if $< i, j > \in I[S'(\Diamond A)^{\phi(i)}{}'T]$; as for E^{K4} if for some numeral k: $< i, k >, < k, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$; and as for E^{T} if $j = i$.

E^{S5}, rule $\mathsf{R}_{\Diamond}$:

The reasoning is a combination of those for calculi E^{K}, E^{K4}, E^{T} and E^{KB}. □

It is clear that from Theorem 4.1 and Theorem 4.2 we get:

Corollary 4.1: *If question $Q' = ?(\Phi')$ results from question $Q = ?(\Phi)$ by a rule of E^{L}, then each term of Φ is L-valid iff each term of Φ' is L-valid.*

Lemma 4.1: *If φ is a sequent of the form: $\vdash S'(C)^{\phi(i)}{}'T'(\neg C)^{\phi^*(i)}{}'U$, or of the form: $\vdash S'(\neg C)^{\phi^*(i)}{}'T'(C)^{\phi(i)}{}'U$, then φ is L-valid.*

Proof. Let $< W, R >$ stand for an arbitrary frame. Suppose that φ is of one of the forms specified above. If φ is not interpretable in frame $< W, R >$ then, by Corollary 2.1, φ is valid in $< W, R >$. Suppose that there exists an interpretation f of φ in $< W, R >$. Let V be a valuation on $< W, R >$. Then either $V(C, f(i)) = 1$ or $V(\neg C, f(i)) = 1$. As V was an arbitrary valuation, the sequent φ is valid in frame $< W, R >$ under f. But, as $< W, R >$ was an arbitrary frame and f was an arbitrary interpretation of φ in $< W, R >$, the sequent φ is valid in every frame, independently of the properties of R. □

Theorem 4.3: (soundness) *If there exists a Socratic proof of a sequent $\vdash (A)^1$ in E^{L}, then the formula A is L-valid.*

Proof. Let $s = s_1, s_2, ..., s_n$ be a Socratic proof of $\vdash (A)^1$ in E^{L}. Thus $s_1 = ?(\vdash (A)^1)$ and $s_n = ?(\Phi)$, where each term of Φ is of the form $\vdash S'(C)^{\phi(i)}{}'T'(\neg C)^{\phi^*(i)}{}'U$, or of the form $\vdash S'(\neg C)^{\phi^*(i)}{}'T'(C)^{\phi(i)}{}'U$. By Lemma 4.1 each term of Φ is L-valid. By Corollary 4.1, if question $s_{i+1} = ?(\Phi_{i+1})$ results from question $s_i = ?(\Phi_i)$ (where $1 \leq i < n$) and each term of Φ_{i+1} is L-valid, then each term of Φ_i is L-valid. Hence (by induction), the sequent $\vdash (A)^1$ is L-valid. Therefore, in view of Theorem 2.1, the formula A of language M is L-valid. □

APPENDIX 1

The list of inferential rules of the calculus E^* written down without using the α, β-notation:

$\mathsf{L}_\wedge$: $\frac{?(\Phi;S'A\wedge B'T\vdash C;\Psi)}{?(\Phi;S'A'B'T\vdash C;\Psi)}$ $\mathsf{R}_\wedge$: $\frac{?(\Phi;S\vdash A\wedge B;\Psi)}{?(\Phi;S\vdash A;S\vdash B;\Psi)}$

$\mathsf{L}_{\neg\vee}$: $\frac{?(\Phi;S'\neg(A\vee B)'T\vdash C;\Psi)}{?(\Phi;S'\neg A'\neg B'T\vdash C;\Psi)}$ $\mathsf{R}_{\neg\vee}$: $\frac{?(\Phi;S\vdash\neg(A\vee B);\Psi)}{?(\Phi;S\vdash\neg A;S\vdash\neg B;\Psi)}$

$\mathsf{L}_\rightarrow$: $\frac{?(\Phi;S'A\rightarrow B'T\vdash C;\Psi)}{?(\Phi;S'\neg A'T\vdash C;S'B'T\vdash C;\Psi)}$ $\mathsf{R}_\rightarrow$: $\frac{?(\Phi;S\vdash A\rightarrow B;\Psi)}{?(\Phi;S'A\vdash B;\Psi)}$

$\mathsf{L}_{\neg\rightarrow}$: $\frac{?(\Phi;S'\neg(A\rightarrow B)'T\vdash C;\Psi)}{?(\Phi;S'A'\neg B'T\vdash C;\Psi)}$ $\mathsf{R}_{\neg\rightarrow}$: $\frac{?(\Phi;S\vdash\neg(A\rightarrow B);\Psi)}{?(\Phi;S\vdash A;S\vdash\neg B;\Psi)}$

$\mathsf{L}_{\neg\wedge}$: $\frac{?(\Phi;S'\neg(A\wedge B)'T\vdash C;\Psi)}{?(\Phi;S'\neg A'T\vdash C;S'\neg B'T\vdash C;\Psi)}$ $\mathsf{R}_{\neg\wedge}$: $\frac{?(\Phi;S\vdash\neg(A\wedge B);\Psi)}{?(\Phi;S'A\vdash\neg B;\Psi)}$

$\mathsf{L}_\vee$: $\frac{?(\Phi;S'A\vee B'T\vdash C;\Psi)}{?(\Phi;S'A'T\vdash C;S'B'T\vdash C;\Psi)}$ $\mathsf{R}_\vee$: $\frac{?(\Phi;S\vdash A\vee B;\Psi)}{?(\Phi;S'\neg A\vdash B;\Psi)}$

$\mathsf{L}_{\neg\neg}$: $\frac{?(\Phi;S'\neg\neg A'T\vdash C;\Psi)}{?(\Phi;SA'T\vdash C;\Psi)}$ $\mathsf{R}_{\neg\neg}$: $\frac{?(\Phi;S\vdash\neg\neg A;\Psi)}{?(\Phi;S\vdash A;\Psi)}$

APPENDIX 2

The list of inferential rules of calculi E^{L} written down without using the α, β-notation:

$\mathsf{R}_\wedge$: $\frac{?(\Phi;\vdash S'(A\wedge B)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i)}{}'T;\vdash S'(B)^{\phi(i)}{}'T;\Psi)}$ $\mathsf{R}_{\neg\wedge}$: $\frac{?(\Phi;\vdash S'(\neg(A\wedge B))^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\neg A)^{\phi(i)}{}'(\neg B)^{\phi(i)}{}'T;\Psi)}$

$\mathsf{R}_{\neg\vee}$: $\frac{?(\Phi;\vdash S'(\neg(A\vee B))^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\neg A)^{\phi(i)}{}'T;\vdash S'(\neg B)^{\phi(i)}{}'T;\Psi)}$ $\mathsf{R}_\vee$: $\frac{?(\Phi;\vdash S'(A\vee B)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i)}{}'(B)^{\phi(i)}{}'T;\Psi)}$

$\mathsf{R}_{\neg\rightarrow}$: $\frac{?(\Phi;\vdash S'(\neg(A\rightarrow B))^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i)}{}'T;\vdash S'(\neg B)^{\phi(i)}{}'T;\Psi)}$ $\mathsf{R}_\rightarrow$: $\frac{?(\Phi;\vdash S'(A\rightarrow B)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\neg A)^{\phi(i)}{}'(B)^{\phi(i)}{}'T;\Psi)}$

$\mathsf{R}_{\neg\neg}$: $\frac{?(\Phi;\vdash S'(\neg\neg A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i)}{}'T;\Psi)}$

$\mathsf{R}_{\neg\Box}$: $\frac{?(\Phi;\vdash S'(\neg\Box A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Diamond\neg A)^{\phi(i)}{}'T;\Psi)}$ $\mathsf{R}_{\neg\Diamond}$: $\frac{?(\Phi;\vdash S'(\neg\Diamond A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Box\neg A)^{\phi(i)}{}'T;\Psi)}$

$\mathsf{R}_\Box$: $\frac{?(\Phi;\vdash S'(\Box A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(A)^{\phi(i),j}{}'T;\Psi)}$ $\mathsf{R}_\Diamond$: $\frac{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'T;\Psi)}{?(\Phi;\vdash S'(\Diamond A)^{\phi(i)}{}'(A)^{i,j}{}'T;\Psi)}$

proviso of applicability of rule $\mathsf{R}_{\Box}$ in each E^{L}: $j \notin \mathsf{I}\{S'(\Box A)^{\phi(i)}{}'T\}$

provisos of applicability of rule $\mathsf{R}_{\Diamond}$:

calculus:	proviso:
E^{K}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$
E^{D}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j \notin \mathsf{I}\{S'(\Diamond A)^{\phi(i)}{}'T\}$
E^{K4}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or for some $k :< i, k >, < k, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$
E^{T}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j = i$
E^{KB}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $< j, i > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$
E^{S4}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or for some $k :< i, k >, < k, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j = i$
E^{S5}	$< i, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or for some $k :< i, k >, < k, j > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $< j, i > \in \mathsf{I}[S'(\Diamond A)^{\phi(i)}{}'T]$ or $j = i$

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