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PREFERENCE SEMANTICS FOR DEONTIC LOGIC PART I — SIMPLE MODELS

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Abstract
This paper presents determination results for some deontic logics with respect to a simple preference-based semantics, in which possible worlds are ranked by comparative value but none need be supposed to be best or top-ranked among alternatives. This kind of semantics is useful for defining deontic logics that allow for conflicts of obligation. Monadic standard deontic logic (SDL) is determined by the class of frames in which the preference ranking is reflexive, transitive and connected. The weak deontic logic P, which allows for normative conflicts, is determined by the class of all preference frames. These results are extended to corresponding dyadic deontic logics that formalize the logic of conditional obligation and the logic of preference itself.

The preference semantics for deontic logic contrasts with the more familiar Kripke-style relational semantics derived from normal modal logic. In that framework deontic formulas OA are interpreted to be true just in case A is true at all the deontically perfect, or ideal, or normatively best, alternative possible worlds, where it is always presupposed that there are such worlds (cf., e.g., [6], p. 163). In the preference semantics no worlds need to be supposed to be ideal or best in any respect. Instead, they are merely compared, so that some may be deontically better than, or normatively preferable to others. Then a formula OA is said to be true just in case there is a world where A is true that marks a threshold, as it were, for A in that there is no world as good as it is where A does not obtain.

It is widely known that a standard preference semantics is adequate for standard deontic logic (SDL). Curiously, however, no direct proof of that result seems to have been published.[1] To fill that gap I present such a proof

[1] This surprising fact came to light in the aftermath of the conference DEON'98, the 4th International Workshop on Deontic Logic in Computer Science, held in Bologna, Italy, January, 1998, when Paul McNamara and Henry Prakken were writing their introduction to the

here; that determination theorem is the chief result of Section 1 below. The argument is quite straight-forward, and I do not claim any special originality for it. Nevertheless, it helps to introduce the framework of the preference semantics. The subsequent sections explore two kinds of generalization from the original, standard framework. First, conditions on the preference relation that make it adequate for SDL can be relaxed. This leads to the deontic logic I call P that is valuable for analyzing normative systems in which there might be conflicts of obligation. In Section 2, I demonstrate the soundness and completeness of P; the argument for this result, which is new, is considerably more complex than that for SDL. The second sort of generalization is to introduce dyadic deontic connectives for conditional obligation and preference; it was for these that the pattern of the preference semantics was originally developed. In Sections 3 and 4 below the results of the earlier sections are extended to apply to logics with such dyadic connectives. Section 3 presents a dyadic counterpart of SDL and also a standard logic of preference itself. Section 4 gives a dyadic counterpart of the weaker logic P, as well as a corresponding weaker logic of preference; these too are new. All of these logics are demonstrated to be both sound and complete with respect to their preference semantics. Establishing these several results is the primary purpose of this paper. In the proof of completeness for P and its dyadic counterparts an extension of the preference semantics is briefly introduced. I call this 'multiple preference semantics'. That is investigated further in Part II of this work [4].

The results given here are chiefly formal; I do not try to motivate the logics discussed, or the pattern of the preference semantics, from a philosophical point of view. (See my paper [3] for more of that discussion.)[2] Furthermore, this work is meant to be foundational. It is generally agreed that for deontic logic to do the philosophical work expected of it, it must be more elaborate

workshop's published proceedings [9]. I suspect it is due to the preference semantics originally being developed for dyadic deontic logics, logics of conditional obligation, whence its application to monadic SDL is effected through the postulated equivalence between OA and $O(A/\top)$. Then the completeness of SDL may be extracted from completeness results for systems of dyadic deontic logic. This can be found, for example, in David Lewis [7], esp. §6.3, but there it is embedded in far more general concerns. Lewis was one of the pioneers of this sort of interpretation for deontic logic, especially dyadic deontic logic (see also Lewis [8]), but even so, his principle proofs were given in terms of other semantic structures, notably systems of spheres, and only derivatively for preference rankings. Other pioneers in using preference models in this way for the interpretation of dyadic deontic operators, though with variations, were Sven Danielson [2] and Bas van Fraassen [12], and, with a somewhat different approach, Bengt Hansson [5] (see note 14 below). We examine dyadic deontic logics in Sections 3 and 4.

[2] In [3] I also sketched some of the present results; here I provide alternative, more direct and simpler proofs, and more in the way of details.

than we see here. It probably needs to take into account aspects of agency and action, not to mention temporality and alethic modality, and it needs to reach beyond the propositional level to include at least first-order quantification. None of that is included here; rather the systems that are examined provide a potential platform for such further sophistication. Also, it should be said, I present here *one* basic way to apply the notion of preference in deontic logic. There are others; I do not try to survey them at all.

1. *Standard Deontic Logic (SDL)*

All of the logics discussed here are propositional deontic logics, and so their language consists, as usual, of a vocabulary of atomic formulas, $p, q, r \ldots$ with the connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$, understood classically, and, in this section and the next, the single monadic deontic operator O, with formation rules as usual. Call this language L_o. Letters 'A', 'B', 'C', etc. are used as variables for well-formed formulas in L_o. Models for this language are based on *simple preference frames* $F = \langle W, \mathrm{P} \rangle$ where W is a non-empty set of points, or so-called 'possible worlds', and P is a function assigning each $a \in W$ a binary preference relation P_a that ranks points in W as being better or worse, or more or less acceptable, more or less desirable, more or less valuable, etc., than others. Thus, '$b\mathrm{P}_a c$' says that, according to a's standard, b is at least as good (acceptable, desirable, valuable, etc.) as c.[3]

In all that follows, for every preference frame $F = \langle W, \mathrm{P} \rangle$, all relations P_a assigned to points $a \in W$ are required to be non-empty; i.e. there must be at least one pair of points b and c such that $b\mathrm{P}_a c$. Where $\mathcal{F}\mathrm{P}_a$ is the field of P_a, i.e., $\{b : \exists c(b\mathrm{P}_a c \text{ or } c\mathrm{P}_a b)\}$, it is thus required that for any $a \in W$, $\mathcal{F}\mathrm{P}_a$ not be empty. This simply says that every point must be able to see, or care about, some world, and compare it, for better or worse, to some world. Relations P_a that are (i) reflexive, (ii) transitive, and (iii) connected on their fields — i.e., that for all $b, c, d \in \mathcal{F}\mathrm{P}_a$, (i) $b\mathrm{P}_a b$, (ii) if $b\mathrm{P}_a c$ and $c\mathrm{P}_a d$ then $b\mathrm{P}_a d$, (iii) $b\mathrm{P}_a c$ or $c\mathrm{P}_a b$ — will be called *standard*, and frames and models all of whose assigned relations P_a are reflexive, transitive or standard will

[3] I use the notation 'P_a' here, rather than, say, '$\geq_a$', in order to prevent some of the later proofs becoming an optical nightmare. Notice that each $a \in W$ has its own ranking relation, P_a. There could, of course, be frames in which the ranking relations are universal, in the sense that for every $a, b \in W, \mathrm{P}_a = \mathrm{P}_b$. Equivalently, one could define a preference frame as a pair $\langle W, \mathrm{P} \rangle$ where P is now itself a binary relation on W with the appropriate properties. Models on such frames would, however, validate principles of the iteration of deontic modalities, such as the S4 principle, $OA \rightarrow OOA$, and the S5 principle, $\neg OA \rightarrow O\neg OA$, which go beyond standard deontic logic. Hence, I do not consider this option further.

likewise be called by the same terms.[4] In this section we look at standard models; in the next we consider others.

Given a frame $F = \langle W, \mathrm{P}\rangle$, a *model* M based on that frame is a pair $\langle F, v\rangle$ where v is an assignment function for atomic formulas, p, such that $v(p) \subseteq W$. Formulas are then evaluated in the usual way, so that[5]

(p)	$M, a \models_P p$ iff $a \in v(p)$
$(\neg)$	$M, a \models_P \neg A$ iff not-($M, a \models_P A$)
$(\wedge)$	$M, a \models_P A \wedge B$ iff $M, a \models_P A$ and $M, a \models_P B$
$(\vee)$	$M, a \models_P A \vee B$ iff $M, a \models_P A$ or $M, a \models_P B$
$(\rightarrow)$	$M, a \models_P A \rightarrow B$ iff not-($M, a \models_P A$) or $M, a \models_P B$

and in particular, for deontic formulas OA,

(P-O) $M, a \models_P OA$ iff there is a $b \in \mathcal{F}\mathrm{P}_a$ such that $M, b \models_P A$ and for any c such that $c\mathrm{P}_a b$, $M, c \models_P A$

That is to say, OA is true just when there is a point to which a's standard applies where A is true and there is no world that is as good as it (according to that standard) where A is not true.

For all the types of models to follow, we say that a formula A *holds at* a point a *in* a model M just in case $M, a \models A$. A *holds on* a model M just when A holds at every point in W in that model. A is *valid on* a frame F just when A holds on every model based on F. A is *valid for* a class of frames $\mathcal{F}$ just when A is valid on every $F \in \mathcal{F}$. When every member of a set of formulas S is valid on a frame F, then F is a *frame for* S. When every member of S is valid for a class of frames $\mathcal{F}$, then S is *sound* with respect to $\mathcal{F}$, and when only members of S are valid for $\mathcal{F}$, S is *complete* with respect to $\mathcal{F}$. When S is both sound and complete with respect to $\mathcal{F}$, $\mathcal{F}$ *characterizes* or *determines* S. Similarly, a set S is sound or complete with respect to a class of models $\mathcal{M}$ just in case all, or only, members of S hold on every $M \in \mathcal{M}$. In what follows, we equate a logic, like SDL, with the set of its theorems.

Standard deontic logic, SDL, is the normal modal logic D (aka KD) for the operator O; i.e., it is the class of formulas axiomatized by

[4] Of course, reflexivity follows from connectednes; it is mentioned separately because it will have a separate role to play in what follows.

[5] Here, and below, I index the sign for the modelling relation, as with the present subscript 'P', to indicate what kind of model M is, here a model on a simple preference frame, and hence the evaluation rule for statements 'OA', since later on some different model structures will be introduced, and this reduces ambiguity.

(PC)	All classical tautologies in L_o
(MP)	If $\vdash A \rightarrow B$ and $\vdash A$ then $\vdash B$
(K)	$O(A \rightarrow B) \rightarrow (OA \rightarrow OB)$
(D)	$OA \rightarrow \neg O \neg A$
(RN)	If $\vdash A$ then $\vdash OA$

Other axiomatizations are also common. Thus (K) might be replaced by the combination of the aggregation principle

(C) $(OA \wedge OB) \rightarrow O(A \wedge B)$

and the inheritance rule

(RM) If $\vdash A \rightarrow B$ then $\vdash OA \rightarrow OB$

Likewise, given (RM), (D) could be replaced by

(P) $\neg O \bot$

(sometimes called (OD)),[6] and (RN) could be replaced by

(N) $O\top$

where $\top$ is an arbitrary tautology, and $\bot$ abbreviates $\neg\top$. All of these are derivable from the first formulation. They will be useful to have in mind later on.

We now establish that the preference semantics is adequate for SDL, or more precisely,

Theorem 1: *SDL is sound and complete with respect to the class of all standard preference frames.*

Proof: Soundness, as usual, is easy to demonstrate, and may be left to the reader. For completeness, we apply familiar Henkin-style techniques to define a canonical model. Let $F = \langle W, \mathrm{P} \rangle$, where W is the set of all maximal consistent extensions of SDL, and for each $a \in W$, a binary relation P_a is defined on W so that

$$\mathrm{P}_a = \{\langle b, c \rangle : b, c \in W \text{ and either } O^{-1}a \subseteq b \text{ or not-}(O^{-1}a \subseteq c)\}$$

where $O^{-1}a = \{A : OA \in a\}$. P assigns P_a to a. Let $M = \langle F, v \rangle$ where

$$v(p) = \{a : a \in W \text{ and } p \in a\}$$

Lemma 2: *M is a model on a standard preference frame.*

[6] Both names are found in Chellas [1], e.g., p. 133, p. 191.

It is easy enough to show that the relations P_a are reflexive, transitive and connected on W that this can be left to the reader.

Before introducing the key lemma that leads to the completeness theorem, it is helpful to have this lemma, which is familiar from modal logic.

Lemma 3: *(i) $O^{-1}a$ is consistent; (ii) if $OA \notin a$, then $O^{-1}a \cup \{\neg A\}$ is consistent.*

Proof: For part (ii), suppose $OA \notin a$ but that $O^{-1}a \cup \{\neg A\}$ is not consistent. $O^{-1}a \neq \emptyset$ since $\top \in O^{-1}a$ because $O\top \in a$ by (N). Hence, there are $C_1, \ldots, C_n \in O^{-1}a$ such that $\vdash (C_1 \wedge \ldots \wedge C_n) \rightarrow A$. For each C_i, $OC_i \in a$; hence $OC_1 \wedge \ldots \wedge OC_n \in a$. By (C), $O(C_1 \wedge \ldots \wedge C_n) \in a$. Since $\vdash (C_1 \wedge \ldots \wedge C_n) \rightarrow A$, $\vdash O(C_1 \wedge \ldots \wedge C_n) \rightarrow OA$ by (RM). Consequently, $OA \in a$, contrary to the opening hypothesis. Hence, if $OA \notin a$, then $O^{-1}a \cup \{\neg A\}$ must be consistent. The argument for part (i) is the same given that $O\bot \notin a$ by virtue of (P).

Lemma 4: *For all formulas A and all $a \in W$, $A \in a$ iff $M, a \models_{P} A$.*

Proof: By induction on A; we show only the inductive case when $A = OB$, supposing the lemma to hold for B. (a) Suppose $OB \in a$. By the preceding lemma we know that $O^{-1}a$ is consistent; hence it has a maximal consistent extension, b. Since P_a is reflexive on W, $b \in \mathcal{F}\mathrm{P}_a$. Since $B \in O^{-1}a$, $B \in b$, and so by the inductive hypothesis, $M, b \models_{P} B$. Now consider any c such that $c\mathrm{P}_a b$. By definition of P_a, since $O^{-1}a \subseteq b$, $O^{-1}a \subseteq c$, and so $B \in c$. Thus, by the inductive hypothesis, $M, c \models_{P} B$. These suffice for $M, a \models_{P} OB$. (b) Suppose, for the converse, that $M, a \models_{P} OB$, so that there is a $b \in \mathcal{F}\mathrm{P}_a$ such that $M, b \models_{P} B$ and for any c such that $c\mathrm{P}_a b$, $M, c \models_{P} B$. Suppose $OB \notin a$. Then, by the preceding lemma, $O^{-1}a \cup \{\neg B\}$ is consistent, and so has a maximal consistent extension, c. $O^{-1}a \subseteq c$, and so automatically $c\mathrm{P}_a b$. Hence $M, c \models_{P} B$, and so, by the inductive hypothesis, $B \in c$, contrary to the consistency of c. Hence, if $M, a \models_{P} OB$, $OB \in a$, as required.

Theorem 1 now follows from Lemmas 2 and 4 in the usual way. If A is not provable in SDL, then $\{\neg A\}$ is consistent, and so has a maximal consistent extension, a, in W. By Lemma 4, $\neg A$ holds at a on a standard preference model, and so A does not hold at a on that model. In other words, if A is valid in all standard preference frames, it must be provable in SDL.

The preference based semantics offers the opportunity to interpret deontic formulas over worlds where none are maximally preferred, and instead there may be infinitely ascending chains of worlds. But this is not required. Indeed, SDL is equally well characterized by the class of frames in which there are no such chains. This is a spin-off from the proof of Theorem 1. Given a frame $F = \langle W, \mathrm{P}\rangle$, for each preference ranking P_a, define its strict counterpart S_a so that $b\mathrm{S}_a c$ iff $b\mathrm{P}_a c$ and not-($c\mathrm{P}_a b$). (This will be asymmetric and transitive when P_a is transitive.) Then, for any $X \subseteq W$, say that $b \in X$ is *maximal*$_a$ *in* X iff there is no $c \in X$ such that $c\mathrm{S}_a b$. Let us say that X is *limited*$_a$ iff for every $b \in X$, b is either maximal$_a$ in X or there is a $c \in X$ such that c is maximal$_a$ in X and $c\mathrm{S}_a b$. A frame F is *limited*$_a$ iff for all $X \subseteq W$, X is limited$_a$, and F is limited *per se* iff F is limited$_a$ for every $a \in W$.

Corollary 5: SDL is characterized by the class of limited standard preference frames.

This follows from the fact that the canonical frame defined for Theorem 1 is limited. (For any $a \in W$, consider any $X \subseteq W$. For any $b \in X$, if $O^{-1}a \subseteq b$, then b is maximal$_a$ in X. If $O^{-1}a \nsubseteq b$, then if there is a $c \in X$ such that $O^{-1}a \subseteq c$, then c is maximal$_a$ in X and $c\mathrm{S}_a b$. If there is no member of X containing $O^{-1}a$ then all members of X are maximal$_a$ in X.) Indeed, we can say more, for the canonical frame is merely 'two-tiered' in the sense that, for any $a \in W$, all worlds are either maximally ideal or minimally subideal. Thus, call $\mathrm{MAX}_a = \{b : \neg\exists c \in W(c\mathrm{S}_a b)\}$ and $\mathrm{MIN}_a = \{b : \neg\exists c \in W(b\mathrm{S}_a c)\}$. (Given that P_a is connected over W, every member of MAX_a will be maximal in the sense that if $b \in \mathrm{MAX}_a$ then for all $c \in W$, $b\mathrm{P}_a c$, and therefore too all members of MAX_a will be equal valued, i.e., for $b, c \in \mathrm{MAX}_a$, $b\mathrm{P}_a c$ and $c\mathrm{P}_a b$. Similarly for MIN_a. Without connectedness, this need not be so.) Call a frame two-tiered if, for all $a \in W$, $\mathrm{MAX}_a \cup \mathrm{MIN}_a = W$. (This allows that, for some $a \in W$, $\mathrm{MAX}_a = \mathrm{MIN}_a = W$.)

Corollary 6: SDL is characterized by the class of two-tiered standard preference frames.

This too follows from the proof of Theorem 1 since the canonical frame there is two-tiered. ($\mathrm{MAX}_a = \{b : O^{-1}a \subseteq b\}$ and if there are any points b such that $O^{-1}a \nsubseteq b$ then $\mathrm{MIN}_a = \{b : O^{-1}a \nsubseteq b\}$, otherwise $\mathrm{MIN}_a = \mathrm{MAX}_a = W$.) Thus putting these additional constraints of limitedness or two-tieredness on frames adds no new theorems to the logic.[7]

[7] Like Theorem 1 itself, these corollaries are well-known. The first is related to what Lewis called the Limit Assumption; cf. [7] §1.4 and [8]. This will be described more in Section 3 below. The second is little more than the standard completeness result for SDL with respect to serial binary relational frames in the familiar Kripke semantics. There MAX_a is the set of points b such that aRb, for R the deontic alternativeness relation.

2. *The Logic P*

The pattern of interpretation for deontic formulas in the preference semantics is ungainly compared to the pattern in Kripke-style relational semantics, and the preference semantics does not have the kind of flexibility that makes the Kripke semantics such a powerful tool in general modal logic. Nevertheless, it offers a different dimension of variation that is not available in the customary Kripke-style relational semantics, and which lets it apply to weak deontic systems that have no Kripke semantics. This is important for deontic logic for it allows the development of systems that accept the possibility of conflicts of obligation, cases where both OA and $O\neg A$ may be true. In SDL this is explicitly excluded by the principle (D), but even without (D), so long as the logic contains the aggregation principle (C) and the rule (RM), such a conflict of obligation would generate the collapse of the normative structure since it would entail OB for every B. To avoid such consequences, a deontic logic that allows for conflicts of obligation must therefore reject both (D) and (C) of SDL (if it preserves (RM) and all of classical logic). A preference-based deontic logic can do that, while one based on a standard Kripke-style relational semantics cannot.[8]

As demonstrated, standard deontic logic is characterized by the class of all standard preference frames. There the requirement that relations P_a be connected on their fields is essential to validating both (D) and (C). If such connectedness is not required, we come to the logic I call P, i.e., the logic characterized by the class of all preference frames, without conditions on the assigned preference relations P_a. As it turns out, P is also characterized by the class of frames in which all relations P_a are reflexive or transitive or both, but perhaps not connected.

The logic P is the set of formulas axiomatized by (PC) with *modus ponens* together with just the rule (RM): If $\vdash A \to B$ then $\vdash OA \to OB$, and the axioms (N), $O\top$, and (P), $\neg O\bot$, that were mentioned above as alternative postulates for SDL.

P does not contain either (C) or (D). The rule (RN): If $\vdash A$ then $\vdash OA$, is, however, derivable for P, as are the rules (RMP): If $\vdash A \to B$ then $\vdash \neg O\neg A \to \neg O\neg B$, and (RP): If $\vdash A$ then $\vdash \neg O\neg A$. Also, as noted in Section 1, if the aggregation principle (C) of SDL is added to P, the result is

[8] As remarked below, the neighborhood models of Segerberg [11], also called 'minimal models' by Chellas [1], Ch. 7–9, provide another way to interpret the language of a system that allows for conflicts of obligation. Another approach is to extend the standard Kripke-semantics to include multiple deontic accessibility relations. This is the approach taken by Schotch and Jennings [10]; cf. also my [3]. In Part II of this work [4] we will see logics that have a preference semantics within the present framework but neither a neighborhood semantics nor a multiple relational semantics.

equivalent to SDL, and since (K) suffices for (C), given (RN), adding (K) to P also yields SDL.[9]

P is a non-normal, classical modal logic in the sense of Segerberg [11]. As such, it does not have a simple Kripke-style relational semantics. As a classical modal logic, it is usually considered from the viewpoint of neighborhood semantics (cf. Segerberg [11] or Chellas [1], Ch. 7–9). Here I will show the adequacy of the framework of preference models for P. Soundness is easy enough, but the proof of completeness, Theorem 12 below, is complicated and seems to require a detour through a generalization of the semantics so far presented. This generalization, which introduces multiple preference relations, offers opportunities for further extensions of P itself. Those are the subject of Part II of this work [4].[10] For the present, we might wish for a simpler proof, but I do not know of any.

Let a *multiple preference frame* F be a structure $\langle W, \mathcal{P}\rangle$ where, as before, W is a non-empty set of points or possible worlds, and $\mathcal{P}$ now assigns to each $a \in W$ a non-empty *set* $\mathcal{P}_a$ of binary relations $\mathrm{P} \subseteq W \times W$. We assume that every relation in each $\mathcal{P}_a$ is non-empty.[11] A multiple preference model, $M = \langle F, v\rangle$, interprets deontic formulas, OA, according to the rule:

(MP-O) $\quad M, a \models_{MP} OA$ iff there is a $\mathrm{P} \in \mathcal{P}_a$ such that $M, \mathrm{P} \models_{MP} A$

where the notation '$M, \mathrm{P} \models_{MP} A$' abbreviates

> there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A$ and, for every c, if $c\mathrm{P}b$ then $M, c \models_{MP} A$

corresponding, for a given relation P, to the original evaluation condition in the simple, non-multiple preference semantics.

So far, no restrictions are put on the multiple relations $\mathrm{P} \in \mathcal{P}_a$, except that they not be empty. If all the relations in $\mathcal{P}_a$ are standard, then the frame will be called standard, similarly if they are all reflexive on their fields or

[9] P, or a very similar system, has been recommended by others as a way to accommodate normative conflicts; e.g., Schotch and Jennings [10]. Van Fraassen, [13], p. 16 and Chellas, [1], p. 202, propose a variant that lacks (N) and (RN) (though van Fraassen, p. 18, backs away from this system as being too weak).

[10] These multiple preference frames extend the simple preference frames previously described in much the way that the multiple relational frames of Schotch and Jennings mentioned in footnote 8 extend the simple frames of the Kripke semantics for modal logics; see also my [3].

[11] For present purposes it would be enough to require merely that at least one relation in $\mathcal{P}_a$ not be empty. In Part II of this work [4], however, we will want the stronger condition, and so it is convenient to incorporate it from the beginning.

transitive. But these properties are not required. The logic P is characterized by any of these classes of multiple preference frames.

Theorem 7: P *is sound and complete with respect to (a) the class of all multiple preference frames, (b) the class of all reflexive or transitive multiple preference frames, and (c) the class of all standard multiple preference frames.*

Proof: Soundness, as usual, may be left to the reader. For completeness we adapt the argument for Theorem 1. Let $F = \langle W, \mathcal{P}\rangle$ where W is the set of all maximal consistent extensions of P, and $\mathcal{P}$ is defined thus: For each $a \in W$ and each formula $A \in L_o$, define a binary relation P_a^A such that

$$\mathrm{P}_a^A = \{\langle b, c\rangle : b, c \in W \text{ and either } OA \notin a \text{ or } A \in b \text{ or } A \notin c\}.$$

Let

$$\mathcal{P}_a = \{\mathrm{P} : \exists A(\mathrm{P} = \mathrm{P}_a^A)\}.$$

$\mathcal{P}$ assigns $\mathcal{P}_a$ to a. Let $M = \langle F, v\rangle$ where, as usual,

$$v(p) = \{a : a \in W \text{ and } p \in a\}.$$

Lemma 8: M *is a model on a standard multiple preference frame.*

Proof: Since there are formulas obviously $\mathcal{P}_a$ is non-empty. Obviously too, every relation in $\mathcal{P}_a$ is reflexive, hence non-empty, and transitive and connected on the whole of W.

Lemma 3 no longer holds for P since the virtue of this system is that it allows for inconsistent obligations. In its place it will be helpful to have the next small lemma. First, though, some notation. I will write ‘$[A]$’ for $\{a : a \in W \text{ and } A \in a\}$; then the key Lemma 10 below will say, in effect, that $M, a \models_{MP} A$ iff $a \in [A]$, and thus that the semantics and syntax for this model are equivalent. To have a similar syntactical counterpart for the notation $M, \mathrm{P} \models_{MP} A$, defined above, let us write ‘$\mathrm{P}\ \varepsilon\ [A]$’ to abbreviate

there is a $b \in \mathcal{F}\mathrm{P}$ such that $A \in b$ and, for every c, if $c\mathrm{P}b$ then $A \in c$

Then Lemma 10 will imply that $M, \mathrm{P} \models A$ iff $\mathrm{P}\ \varepsilon\ [A]$ to reflect the desired semantical and syntactical equivalence.

Lemma 9: *For all formulas A and B, (i) If $\mathrm{P}_a^A\ \varepsilon\ [B]$ then $\vdash A \to B$; (ii) if $\mathrm{P}_a^A\ \varepsilon\ [B]$ and $OA \notin a$ then $\vdash B$; (iii) $\mathrm{P}_a^A\ \varepsilon\ [A]$ iff $a \in [OA]$.*

Proof: For (i), suppose $\mathrm{P}_a^A \; \varepsilon \; [B]$, so that there is a $b \in \mathcal{F}\mathrm{P}_a^A$ and $B \in b$ and for all c such that $c\mathrm{P}_a^A b$, $B \in c$. Suppose that $\nvdash A \rightarrow B$. Then $\{A, \neg B\}$ is consistent and has a maximal consistent extension, c. Since $A \in c$, $c\mathrm{P}_a^A b$, and so $B \in c$, contrary to its consistency. For (ii), suppose $\mathrm{P}_a^A \; \varepsilon \; [B]$, so that again there is a $b \in \mathcal{F}\mathrm{P}_a^A$ such that $B \in b$ and, for all c, if $c\mathrm{P}_a^A b$ then $B \in c$; suppose also that $OA \notin a$, but that $\nvdash B$. So $\{\neg B\}$ is consistent and has a maximal consistent extension, c. Since $OA \notin a$, automatically $c\mathrm{P}_a^A b$. So $B \in c$, contrary to its consistency. For (iii), (a) Suppose $\mathrm{P}_a^A \; \varepsilon \; [A]$, but that $a \notin [OA]$, i.e., $OA \notin a$. Then, by (ii), $\vdash A$, so $\vdash OA$ by (RN) and $OA \in a$, a contradiction. (b) Suppose $a \in [OA]$, i.e., $OA \in a$. $\{A\}$ is consistent (else $\vdash A \rightarrow \bot$, and then $\vdash OA \rightarrow O\bot$, and so $O\bot \in a$, but, by (P), $\vdash \neg O\bot$, so $\neg O\bot \in a$, contrary to its consistency). Thus $\{A\}$ has a maximal consistent extension, b. $A \in b$, hence $b\mathrm{P}_a^A b$ and $b \in \mathcal{F}\mathrm{P}_a^A$. Obviously, $b \in [A]$. Suppose any c such that $c\mathrm{P}_a^A b$. Either $OA \notin a$ or $A \notin b$ or $A \in c$. The first two are ruled out, leaving the third, which suffices for $\mathrm{P}_a^A \; \varepsilon \; [A]$.

Lemma 10: *For all formulas A and all $a \in W$, $A \in a$ iff $M, a \models_{MP} A$.*

Proof: By induction on A; I consider only the case where $A = OB$, supposing the lemma to hold for B. (a) Suppose $OB \in a$, i.e., $a \in [OB]$. Then by Lemma 9.iii, $\mathrm{P}_a^B \; \varepsilon \; [B]$, i.e., there is a $b \in \mathcal{F}\mathrm{P}_a^B$ such that $B \in b$ and for every c if $c\mathrm{P}_a^B b$ then $B \in c$, but, with the inductive hypothesis, this yields $M, a \models_{MP} OB$ directly since $\mathrm{P}_a^B \in \mathcal{P}_a$. (b) Suppose $M, a \models_{MP} OB$, i.e., there is a $\mathrm{P} \in \mathcal{P}_a$ and $M, \mathrm{P} \models_{MP} B$. $\mathrm{P} = \mathrm{P}_a^C$ for some C. Since $M, \mathrm{P}_a^C \models_{MP} B$, the inductive hypothesis yields $\mathrm{P}_a^C \; \varepsilon \; [B]$, and so by Lemma 9.i, $\vdash C \rightarrow B$, whence, by (RM) $\vdash OC \rightarrow OB$. Suppose, however, $OB \notin a$. Then $OC \notin a$. But then, by Lemma 9.ii, $\vdash B$, in which case $\vdash OB$ and $OB \in a$, a contradiction. Therefore, if $M, a \models_{MP} OB$, then $OB \in a$.

Theorem 7 now follows in the usual way. Suppose A is not provable in P. Then $\{\neg A\}$ is consistent, and so has a maximal consistent extension, a. By Lemma 10, A does not hold at a on the canonical model M. Hence, A is not valid for any class of models containing M. Since, by Lemma 8, M is a model on a standard multiple preference frame, that means that A is not valid for the class of such standard frames (part (c) of the theorem). But, of course, M is also a model on a reflexive frame, and on a transitive frame, and on a frame *simpliciter*; hence A is not valid with respect to those wider classes of multiple preference frames as well (parts (a) and (b) of the theorem). By contraposition, if A is valid with respect to any of those classes of frames, it must be provable in P.

At the end of Section 1, we noted that SDL is characterized by the class of limited preference frames. A similar result obtains for P. In the framework of multiple preference frames, let each relation $\mathrm{P} \in \mathcal{P}_a$ have its strict counterpart be given by $b\mathrm{S}^{\mathrm{P}}c$ iff $b\mathrm{P}c$ and not-($c\mathrm{P}b$). For any $X \subseteq W$, say that $b \in X$ is P-maximal$_a$ in X iff there is no $c \in X$ such that $c\mathrm{S}^{\mathrm{P}}b$, and X is P-limited$_a$ iff, for every $b \in X$, b is either P-maximal$_a$ in X or there is a $c \in X$ such that c is P-maximal$_a$ in X and $c\mathrm{S}^{\mathrm{P}}b$. A frame F is P-limited$_a$ iff every $X \subseteq W$ is P-limited$_a$; F is limited$_a$ iff it is P-limited$_a$ for every $\mathrm{P} \in \mathcal{P}_a$; and F is limited iff F is limited$_a$ for every $a \in W$.

Corollary 11: P *is characterized by the class of limited (perhaps reflexive, transitive, standard) multiple preference frames.*

This follows from Theorem 7 as Corollary 5 followed from Theorem 1 since the canonical frame is limited. (For any $a \in W$ and any $\mathrm{P} \in \mathcal{P}_a$, $\mathrm{P} = \mathrm{P}_a^A$ for some formula A. Take any $X \subseteq W$ and consider $b \in X$. If $OA \notin a$, then b is P-maximal$_a$ in X. If $OA \in a$ and $X \cap [A] = \emptyset$, then again b is P-maximal$_a$ in X. If $OA \in a$ and $X \cap [A] \neq \emptyset$, then if $b \in [A]$ then too b is P-maximal$_a$ in X, but if $b \notin [A]$ then for any $c \in X \cap [A]$, c is P-maximal$_a$ in X and $c\mathrm{S}^{\mathrm{P}}b$.)[12]

The multiple preference models that yield Theorem 7 are interesting in their own right, as we will see in Part II of this work [4]. Here, however, they serve primarily as a key step toward the principle result of this section, namely, that the original simple preference semantics over the general class of frames is adequate for the logic P. This follows because, for any multiple preference model, there is an equivalent simple preference model. To prove this, however, is a bit complicated. The complication is required to collapse the multiple preference relations of the first model into a single preference relation for the second. This cannot be simply the union of the multiple relations, for that might create connections amongst points where there should be none. For example, one relation $\mathrm{P} \in \mathcal{P}_a$ might rank b strictly higher than c, while a different relation $\mathrm{Q} \in \mathcal{P}_a$ ranks c strictly higher than b. In their union, however, b and c would stand equally. This could change the truth value of an O formula from the original multiple preference model to the simple model that is supposed to be derived from it. To prevent that, we will

[12] In a similar vein, corresponding to Corollary 6, P is characterized by the class of multiple preference frames that are two-tiered with respect to each $\mathrm{P} \in \mathcal{P}_a$, for every $a \in W$. This follows from the correspondence between the multiple preference semantics described here and the multiple relational semantics presented in [3] and also given by Schotch and Jennings [10], just as the simple standard preference semantics corresponds to the simple Kripke semantics for SDL.

take the points of the simplified model to be pairs consisting of the worlds of the original model together with an ordering relation from that model, or an index indicating it, which will allow for the separation of multiple counterparts of b and c. Given such constructed points, we can define multiple preference rankings that are entirely disjoint from each other. Those can then be collapsed without loss into a single preference relation to yield the result we want. Thus, in the new model there will be distinct points b' and b'' corresponding to b (exactly the same formulas will hold at them as at b under the original model) and similarly c' and c'' corresponding to c, such that under the relation P' corresponding to P, $b'\mathrm{P}'c'$, while under the relation Q'' corresponding to Q, $c''\mathrm{Q}''b''$ but neither $b'\mathrm{P}'c''$ nor $c''\mathrm{Q}''b'$, so that under the union of the relations, b' rates higher than c' and c'' rates higher than b'' but b' is not equal to c'' nor b'' to c'. Instead they are incommensurable.

Theorem 12: P *is sound and complete with respect to the class of all simple preference frames.*

Proof: Soundness as usual. For completeness, let $M = \langle F, v\rangle$ be a model on a multiple preference frame $F = \langle W, \mathcal{P}\rangle$ in which every $\mathrm{P} \in \mathcal{P}_a$ is reflexive on its field (which, in light of Lemma 8, could be stipulated to be the whole of W). We now derive an equivalent simple preference model $M^* = \langle F^*, v^*\rangle$. First, let each relation $\mathrm{P} \subseteq W \times W$ bear a distinct index i and let I be the set of these indexes. For notation, P^j will be the relation that bears the index $j \in I$. Then, define a set of points $W^* = \{\langle a, i\rangle : a \in W \text{ and } i \in I\}$. Next, for each relation $\mathrm{P}^j \subseteq W \times W$, define a corresponding relation on W^*, thus:

$$\mathrm{P}^{*j} = \{\langle b^*, c^*\rangle : \text{there are } b, c \in W \text{ such that } b^* = \langle b, j\rangle \;\&\; c^* = \langle c, j\rangle \;\&\; b\mathrm{P}^j c\}$$

We note that, for any $j \in I$, if $b^* \in \mathcal{F}\mathrm{P}^{*j}$, then $b^* = \langle b, j\rangle$, for some $b \in W$, and hence, when $j \neq k$, P^{*j} and P^{*k} are disjoint in the sense that their fields do not overlap.

Observation 1: *For all $j, k \in I$, if $j \neq k$ then $\mathcal{F}\mathrm{P}^{*j} \cap \mathcal{F}\mathrm{P}^{*k} = \emptyset$.*

This is obvious from the definition of the relations. It will play an important role below.

For each $a \in W$ and $i \in I$, let

$$\mathcal{P}^*_{\langle a,i\rangle} = \{\mathrm{P}^* : \text{there is a } j \in I \text{ such that } \mathrm{P}^* = \mathrm{P}^{*j} \;\&\; \mathrm{P}^j \in \mathcal{P}_a\}.$$

Then let

$$\mathrm{P}^*_{\langle a,i\rangle} = \bigcup \mathcal{P}^*_{\langle a,i\rangle}$$

We now define $F^* = \langle W^*, \mathrm{P}^* \rangle$ in which P^* assigns each $\langle a, i \rangle \in W^*$ the relation $\mathrm{P}^*_{\langle a,i \rangle}$. Let $M^* = \langle F^*, v^* \rangle$ where

$$v^*(p) = \{\langle a, i \rangle : a \in v(p) \;\&\; i \in I\}$$

Lemma 13: *M^* is a model on a simple preference frame.*

Proof: This is virtually trivial, since no conditions are placed on the relations $\mathrm{P}^*_{\langle a,i \rangle}$ except that their fields not be empty, which follows immediately from the non-emptiness of the fields of the relations in $\mathcal{P}_a$. It is worth noting for later, however, that

Observation 2: *For all $\langle a, i \rangle \in W^*$, (a) $\mathrm{P}^*_{\langle a,i \rangle}$ is reflexive on its field, and (b) if every relation $\mathrm{P}^j \in \mathcal{P}_a$ is transitive, then $\mathrm{P}^*_{\langle a,i \rangle}$ is transitive as well.*

Reflexivity is immediate from reflexivity for the relations of $\mathcal{P}_a$ in F. Transitivity follows from the transitivity of the relations $\mathrm{P}^j \in \mathcal{P}_a$ and the disjointness of the corresponding relations P^{*j}. Thus, if $b^*\mathrm{P}^*_{\langle a,i \rangle}c^*$ and $c^*\mathrm{P}^*_{\langle a,i \rangle}d^*$, then there is a $\mathrm{P}^{*j} \in \mathcal{P}^*_{\langle a,i \rangle}$ such that $b^*\mathrm{P}^{*j}c^*$ and a $\mathrm{P}^{*k} \in \mathcal{P}^*_{\langle a,i \rangle}$ such that $c^*\mathrm{P}^{*k}d^*$. But then $b^* = \langle b, j \rangle$ and $c^* = \langle c, j \rangle$ and $b\mathrm{P}^j c$, and also $c^* = \langle c, k \rangle$ and $d^* = \langle d, k \rangle$ and $c\mathrm{P}^k d$. Thus, $k = j$, and so $d^* = \langle d, j \rangle$ and $c\mathrm{P}^j d$, whence $b\mathrm{P}^j d$ by transitivity of P^j. Therefore, $\langle b, j \rangle \mathrm{P}^{*j} \langle d, j \rangle$, and then $b^*\mathrm{P}^*_{\langle a,i \rangle}d^*$, as required. Note too that $\mathrm{P}^*_{\langle a,i \rangle}$ is not connected, since, for any $b, c \in W$, if $j \neq k$ neither $\langle b, j \rangle \mathrm{P}^*_{\langle a,i \rangle} \langle c, k \rangle$ nor $\langle c, k \rangle \mathrm{P}^*_{\langle a,i \rangle} \langle b, j \rangle$, although both $\langle b, j \rangle$ and $\langle c, k \rangle$ are in the field of $\mathrm{P}^*_{\langle a,i \rangle}$.

Lemma 14: *For every formula $A \in L_o$, every $a \in W$, and every $i \in I$, $M, a \models_{MP} A$ iff $M^*, \langle a, i \rangle \models_{P} A$.*

Proof: By induction on A. Obvious when $A = p$. We assume the lemma for all formulas up to $A = OB$ and demonstrate that case. (a) Suppose $M, a \models_{MP} OB$, i.e., there is a $\mathrm{P} \in \mathcal{P}_a$ such that $M, \mathrm{P} \models_{MP} B$. $\mathrm{P} = \mathrm{P}^j$ for some $j \in I$. Thus there is a $b \in \mathcal{F}\mathrm{P}^j$ and $M, b \models_{MP} B$ and for all c such that $c\mathrm{P}^j b$, $M, c \models_{MP} B$. By reflexivity, $b\mathrm{P}^j b$, so $\langle b, j \rangle \mathrm{P}^{*j} \langle b, j \rangle$ and $\langle b, j \rangle \mathrm{P}^*_{\langle a,i \rangle} \langle b, j \rangle$, and hence $\langle b, j \rangle \in \mathcal{F}\mathrm{P}^*_{\langle a,i \rangle}$. Moreover, $M^*, \langle b, j \rangle \models_{P} B$, by the inductive hypothesis. Suppose then some $c^* \in W^*$ such that $c^*\mathrm{P}^*_{\langle a,i \rangle} \langle b, j \rangle$. There is then a relation $\mathrm{P}^{*k} \in \mathcal{P}^*_{\langle a,i \rangle}$ such that $c^*\mathrm{P}^{*k} \langle b, j \rangle$. Since $\langle b, j \rangle \in \mathcal{F}\mathrm{P}^{*k}$, $k = j$, and so $c^*\mathrm{P}^{*j} \langle b, j \rangle$, and then $c^* = \langle c, j \rangle$. Given $\langle c, j \rangle \mathrm{P}^{*j} \langle b, j \rangle$, $c\mathrm{P}^j b$. (It was to enable this conclusion that disjoint relations

P^* were formed out of the relations $\mathrm{P} \in \mathcal{P}_a$.) From this, $M, c \models_{MP} B$, and so $M^*, \langle c, j\rangle \models_P B$, i.e., $M^*, c^* \models_P B$, by the inductive hypothesis, which suffices for $M^*, \langle a, i\rangle \models_P OB$. (b) Suppose $M^*, \langle a, i\rangle \models_P OB$, so that there is a $b^* \in \mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$ such that $M^*, b^* \models_P B$ and for all $c^* \in W^*$, if $c^*\mathrm{P}^*_{\langle a,i\rangle}b^*$ then $M^*, c^* \models_P B$. With $b^* \in \mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$, there is a $j \in I$ such that $\mathrm{P}^{*j} \in \mathcal{P}^*_{\langle a,i\rangle}$ and $b^* \in \mathcal{F}\mathrm{P}^{*j}$, which implies that $b^* = \langle b, j\rangle$ for some $b \in W$, and that $\langle b, j\rangle \mathrm{P}^{*j} \langle b, j\rangle$ by Observation 2 above, and so $b\mathrm{P}^j b$ and $b \in \mathcal{F}\mathrm{P}^j$. Further, $M, b \models_{MP} B$, by the inductive hypothesis. Suppose then a c such that $c\mathrm{P}^j b$. By definition, $\langle c, j\rangle \mathrm{P}^{*j} \langle b, j\rangle$, whence $\langle c, j\rangle \mathrm{P}^*_{\langle a,i\rangle} \langle b, j\rangle$, and so $M^*, \langle c, j\rangle \models_P B$, which gives $M, c \models_{MP} B$, by the inductive hypothesis. This suffices for $M, a \models_{MP} OB$, to complete the lemma.

Theorem 12, the principle result of this section, now follows from Theorem 7.b and Lemma 14. Suppose A is not provable in P. Then there is a reflexive multiple preference model that falsifies A, by Theorem 7.b. By Lemma 14 the derived simple preference model also falsifies A, hence A is not valid in all simple preference models.

Corollary 15: P *is sound and complete with respect to the class of all simple preference frames that are reflexive, or transitive, or both.*

From Observation 2, if all the relations of $\mathcal{P}_a$ are reflexive or transitive then $\mathrm{P}^*_{\langle a,i\rangle}$ will be reflexive or transitive as well.[13]

Just as SDL is characterized by the class of limited standard preference frames, Corollary 5, and P is characterized by the class of limited multiple preference frames, Corollary 11, so too

[13] Reflexivity here is reflexivity on the field of each relation P_a, or, in the proof, $\mathrm{P}^*_{\langle a,i\rangle}$. To demonstrate completeness with respect to all frames that are not only reflexive on the fields of their relations P_a but with respect to all of W, i.e., for the field of each such relation to be W, modify the proof of Theorem 12 so that, instead of the relations $\mathrm{P}^*_{\langle a,i\rangle}$ defined there, it uses relations $\mathrm{P}^{\#}_{\langle a,i\rangle}$, where

$$\langle b, j\rangle \mathrm{P}^{\#}_{\langle a,i\rangle} \langle c, k\rangle \text{ iff } \langle b, j\rangle \mathrm{P}^*_{\langle a,i\rangle} \langle c, k\rangle \text{ or } \mathrm{P}^k \notin \mathcal{P}_a$$

Then the proof of the theorem will go through in much the same way, and if all relations in $\mathcal{P}_a$ are reflexive on W (cf. Lemma 8), then $\mathrm{P}^{\#}_{\langle a,i\rangle}$ is reflexive on W^*, and if all relations in $\mathcal{P}_a$ are transitive then so is $\mathrm{P}^{\#}_{\langle a,i\rangle}$. Thus, at present, it does not matter whether the field of a point's preference relation be the whole of W. For the logics of Sections 3 and 4 below this does make a difference, however. There we do not want to require such a global condition; hence we have not imposed it here.

Corollary 16: P *is characterized by the class of limited (perhaps reflexive, transitive) simple preference frames.*

Consider a limited reflexive (and perhaps transitive) multiple preference frame $F = \langle W, \mathcal{P}\rangle$, and let $F^* = \langle W^*, \mathrm{P}^*\rangle$ be the simple preference frame derived from F as for Theorem 12. Then F^* is limited. For consider, for any $a^* \in W^*$, any $X \subseteq W^*$, and suppose $b^* \in X$. $b^* = \langle b, j\rangle$ for some $b \in W$ and $j \in I$. Let $Y = \{c : \langle c, j\rangle \in X\}$. $b \in Y$. Since F is limited$_a$, Y is P^j-limited$_a$, and so either b is P^j-maximal$_a$ in Y or else there is a $c \in Y$ such that c is P^j-maximal$_a$ in Y and $c\mathrm{S}^{\mathrm{P}^j}b$. In the first case, $b^* = \langle b, j\rangle$ will be P^{*j}-maximal$_{\langle a,i\rangle}$ in X, and so maximal$_{\langle a,i\rangle}$ in X with respect to $\mathrm{P}^*_{\langle a,i\rangle}$. For consider any $d^* = \langle d, k\rangle \in X$, and suppose for *reductio* that $\langle d, k\rangle\mathrm{S}^*_{\langle a,i\rangle}\langle b, j\rangle$. If $j = k$, then $d \in Y$. Also $\langle d, j\rangle\mathrm{P}^*_{\langle a,i\rangle}\langle b, j\rangle$ and not-($\langle b, j\rangle\mathrm{P}^*_{\langle a,i\rangle}\langle d, j\rangle$). From the first, $\langle d, j\rangle\mathrm{P}^{*j}\langle b, j\rangle$ and so $d\mathrm{P}^j b$. From the latter, not-($\langle b, j\rangle\mathrm{P}^{*j}\langle d, j\rangle$), and so not-($b\mathrm{P}^j d$). Thus $d\mathrm{S}^{\mathrm{P}^j}b$, contrary to b's being P^j-maximal$_a$ in Y. If, on the other hand $j \neq k$, then it could not be that $\langle d, k\rangle\mathrm{P}^*_{\langle a,i\rangle}\langle b, j\rangle$, and so not $\langle d, k\rangle\mathrm{S}^*_{\langle a,i\rangle}\langle b, j\rangle$, as supposed. Similarly, in the second case, $\langle c, j\rangle$ will likewise be P^{*j}-maximal$_{\langle a,i\rangle}$ in X and so maximal$_{\langle a,i\rangle}$ with respect to $\mathrm{P}^*_{\langle a,i\rangle}$, and since $c\mathrm{S}^{\mathrm{P}^j}b$, $\langle c, j\rangle\mathrm{S}^{\mathrm{P}^{*j}}\langle b, j\rangle$, and so $\langle c, j\rangle\mathrm{S}^*_{\langle a,i\rangle}\langle b, j\rangle$, as required for X to be limited$_{\langle a,i\rangle}$. Since F^*, derived from a limited F, is limited, Corollary 11 implies the present corollary. Any non-theorem of P can be falsified in a model on a limited reflexive (and perhaps transitive) multiple preference frame F, and so falsified on the equivalent limited (reflexive, transitive) simple preference frame F^* derived from F as demonstrated above.

3. *Standard Dyadic Deontic Logic*

Historically, the pattern of the preference semantics for deontic logic was introduced to give an account of the concept of conditional obligation, statements of the sort 'given that A, it ought to be that B'. (Cf. the references in footnote 1.) In this section we demonstrate the adequacy of this framework to two forms of dyadic deontic logic. On the one hand, we have a logic of conditional obligation that corresponds to SDL for the monadic obligation operator. On the other hand, we have a logic of preference itself, that represents in the object language the relation that makes the models work. The two dyadic logics are closely related. Both are determined by the class of standard preference frames, and their fundamental concepts are interdefinable. In the next section we will look at weaker versions of these systems that are determined by the class of all reflexive and transitive, but not necessarily connected, frames. These stand to the monadic deontic logic P as the standard dyadic logics of this section stand to SDL.

Let us begin with the logic of conditional obligation. Instead of the monadic operator O of Sections 1 and 2, let the language, L_{co}, contain a binary operator $O(-/-)$ such that $O(B/A)$ is well-formed whenever A and B are. As indicated above, '$O(B/A)$' is read 'given that A, it ought to be that B'. Such formulas are interpreted in the present preference semantics according to the rule

(P-CO) $M, a \models_{P} O(B/A)$ iff there is a $b \in \mathcal{F}\mathrm{P}_a$ such that $M, b \models_{P} A \wedge B$ and for any c such that $c\mathrm{P}_a b$ and $M, c \models_{P} A$, $M, c \models_{P} B$

which corresponds to the original monadic rule (P-O) except for restricting the range of worlds to be considered for preference comparisons to those where the antecedent A obtains. That is the import of *conditional* obligation.

One might also be interested in a sense of conditional obligation in which 'Ought(B/A)' could be true even when A is impossible or contradictory, which is excluded by (P-CO). This would allow, for example, Ought($\perp/\perp$), and Ought(A/A) in general, to be valid. For this, one could introduce an operator $O'(-/-)$ such that

(P-CO') $M, a \models_{P} O'(B/A)$ iff either there is no $b \in \mathcal{F}\mathrm{P}_a$ such that $M, b \models_{P} A$ or there is a $b \in \mathcal{F}\mathrm{P}_a$ such that $M, b \models_{P} A \wedge B$ and for any c such that $c\mathrm{P}_a b$ and $M, c \models_{P} A$, $M, c \models_{P} B$

$O'(B/A)$ could, however, be defined in terms of $O(B/A)$ thus: $O'(B/A) =_{df} O(\top/A) \rightarrow O(B/A)$. Alternatively, given $O'(B/A)$ as primitive, $O(B/A)$ could be defined by $O(B/A) =_{df} \neg O'(\perp/A) \wedge O'(B/A)$. (Cf. [8] p. 5.) Because of these equivalences, it is immaterial which notion one adopts. All the results below for logics of $O(B/A)$ apply *mutatis mutandis* to corresponding logics of $O'(B/A)$. The same is true in the next section.[14]

[14] The pattern of (P-CO), or (P-CO'), contrasts with another rule of evaluation that is widely used for conditional obligation, namely that $O(B/A)$ is true iff there are worlds at which A is true and all the best (maximal) A-worlds are worlds where B is true, or that $O'(B/A)$ is true iff all the best (maximal) A-worlds are B-worlds. The latter is the form of rule proposed by Hansson [5] p. 144. (Thus his conditional obligation corresponds to $O'(B/A)$.) This other kind of rule, however, requires the Limit Assumption (LA), described below, to be satisfied by all models if it is to produce desired results. Thus, without LA, the principle of cautious monotony, $(O(B/A) \wedge O(C/A)) \rightarrow O(C/A \wedge B)$, is not valid under the Hansson sort of rule, whereas it is valid for all standard models that do satisfy this condition. By contrast, the present rules (P-CO) and (P-CO') do not require LA, although, as will be shown, imposing such a condition would not change the set of valid formulas, unlike the case with the Hansson forms. For models satisfying LA, the two kinds of evaluation rules are equivalent; they verify exactly the same formulas. Without the assumption, they can diverge. The rule for the monadic operator that corresponds to this other pattern of

Let SDDL (standard dyadic deontic logic)[15] be the class of formulas containing PC and closed under *modus ponens* with the additional axioms and rules:

(RCE) If $\vdash A \leftrightarrow A'$ then $\vdash O(B/A) \leftrightarrow O(B/A')$
(RCM) If $\vdash B \rightarrow C$ then $\vdash O(B/A) \rightarrow O(C/A)$
(CK) $O(B \rightarrow C/A) \rightarrow (O(B/A) \rightarrow O(C/A))$
(CD) $O(B/A) \rightarrow \neg O(\neg B/A)$
(CN) $O(\top/\top)$
(C$O\wedge$) $O(B/A) \rightarrow O(A \wedge B/A)$
(trans) $((A \geq B) \wedge (B \geq C)) \rightarrow (A \geq C)$

where in L_{co}

$$A \geq B =_{df} \neg O(\neg A/A \vee B)$$

$A \geq B$ provides a way of representing the weak preference ordering in terms of conditional obligation; such formulas should be read 'A is at least as good (desirable, valuable, etc.) as B. Given their definition, these formulas are evaluated according to the derived rule

(P-$\geq$) $M, a \models_P A \geq B$ iff for every c such that $c \in \mathcal{F}\mathrm{P}_a$ and $M, c \models_P B$, there is a b such that $b\mathrm{P}_a c$ and $M, b \models_P A$

Soon we will explore taking '$\geq$' as primitive and defining $O(-/-)$ in terms of it. First, however, let us list some principles that are derivable from SDDL; most of these are useful when establishing the results below.

(D.1) If $\vdash A$ then $\vdash O(A/A)$
(D.2) $\vdash O(\top/A) \leftrightarrow O(A/A)$
(D.2a) $\vdash O(B/A) \rightarrow O(A/A)$
(D.3) $\vdash (O(A/C) \wedge O(B/C)) \rightarrow O(A \wedge B/C)$
(D.4) $\vdash \neg O(\neg A/A)$
(D.5) $\vdash O(A/B \vee C) \rightarrow (O(A/B) \vee O(A/C))$
(D.6) $\vdash O(B/A) \leftrightarrow \neg((A \wedge \neg B) \geq (A \wedge B))$
(D.7) $\vdash A \geq A$

interpretation amounts to the rule of evaluation in the Kripke-style relational semantics; OA would be true just in case all the best $\top$-worlds, i.e., all the best worlds in W, are A-worlds, where LA implies that there are such worlds.

[15] It might be a misnomer to call this 'standard' dyadic deontic logic since there is far less standardization in this area than in monadic deontic logic, and more room for variation. I call it standard because it corresponds so directly to SDL as manifest through its models, models on 'standard' frames. SDDL is the logic CD of van Fraassen's [12], where it receives a similar axiomatization, but a somewhat different semantics. It is also discussed by Lewis, with a rather different axiomatization, in [7] Ch. 6 (under the name VN) and in [8], where it is given various alternative — though in a sense equivalent — semantics, including one that is virtually the same as the present account.

(D.8) $\vdash (A \geq B) \vee (B \geq A)$
(D.9) $\vdash \neg(\bot \geq \top)$
(D.10) $\vdash A \geq \bot$
(D.11) $\vdash ((A \geq B) \wedge (A \geq C)) \rightarrow (A \geq (B \vee C))$
(DR.1) If $\vdash A \rightarrow B$ then $\vdash B \geq A$
(DR.2) If $\vdash A \rightarrow (B \vee C)$ then $\vdash (B \geq A) \vee (C \geq A)$
(DR.2gen) If $\vdash A \rightarrow (B_1 \vee \cdots \vee B_n)$ then $\vdash (B_1 \geq A) \vee \cdots \vee (B_n \geq A)$

Proofs of these are easy enough that they can be left to the reader. Some, but not all, require (CK) and (CD).

The principles of SDDL look like conditionalized versions of the principles of SDL. The connection between the two is made even closer by defining formulas OA of SDL as $O(A/\top)$. Then it is apparent that the original evaluation rule (P-O) is derivable from (P-CO) via the definition, and all the theorems of SDL are derivable in SDDL. Like SDL, SDDL is characterized by the class of all standard preference frames.

Before demonstrating that, however, it will be useful to bring forward the logic of preference itself. For this, we take the language $L_{\geq}$ to contain, along with the vocabulary of classical logic, the binary connective '$\geq$' as primitive, with $A \geq B$ well-formed when A and B are. Such formulas are evaluated according to the rule (P-$\geq$) above, but this is now stipulated and not derived.

Just as '$\geq$' was defined in terms of '$O(-/-)$ before, so now we can define '$O(-/-)$' in $L_{\geq}$ in terms of '$\geq$', thus:

$$O(B/A) =_{df} \neg((A \wedge \neg B) \geq (A \wedge B))$$

and then the evaluation rule (P-CO) will be derivable from (P-$\geq$). Likewise, the monadic operator O is definable so that OA is equivalent to $\neg(\neg A \geq A)$.

The standard logic of preference, SPref, that corresponds to SDDL is given by these axioms and rules, in addition to PC and closure under *modus ponens*:[16]

(R.1) If $\vdash A \rightarrow B$ then $\vdash B \geq A$
(trans) $((A \geq B) \wedge (B \geq C)) \rightarrow (A \geq C)$
(connex) $(A \geq B) \vee (B \geq A)$
($\geq \vee$) $((A \geq B) \wedge (A \geq C)) \rightarrow (A \geq (B \vee C))$
(poss) $\neg(\bot \geq \top)$

[16] This is equivalent to the axiomatization Lewis gives for VN in terms of comparative possibility, here preferability, in [7], p. 123. Lewis proved completeness for this system in terms of his systems of spheres, from which completeness in terms of preference rankings can be derived. Here we prove the result in terms of preference directly.

Some derivable theorems are

(D.1) $\vdash A \geq A$
(D.2) $\vdash (A \geq B) \leftrightarrow \neg O(\neg A/A \vee B)$
(D.3) $\vdash (A \geq (A \vee B)) \vee (B \geq (A \vee B))$
(D.4) $\vdash A \geq \bot$
(DR.1) If $\vdash A \rightarrow B$ then $\vdash (A \geq C) \rightarrow (B \geq C)$
(DR.2) If $\vdash B \rightarrow C$ then $\vdash (A \geq C) \rightarrow (A \geq B)$
(DR.3gen) If $\vdash A \rightarrow (B_1 \vee \cdots \vee B_n)$ then $\vdash (B_1 \geq A) \vee \cdots \vee (B_n \geq A)$

(Proofs are left to the reader.) (D.2) here is like (D.6) of the preceding list; they correspond to the equivalences between preference formulas and conditional obligation formulas described semantically. That is, (D.2) here reflects the definition of $A \geq B$ in L_{co}, while (D.6) earlier reflects the definition of $O(B/A)$ in $L_{\geq}$. Thus one can move freely between the two regardless of which language one is in.

It might be useful to note that the connective $\geq$ allows for the expression of alethic modalities, or very close counterparts thereto, within the deontic language. Thus, a formula $\bot \geq A$ will hold at a point, a, just in case A holds at no points in the field of the relation P_a, and so represents a sort of impossibility operator. Likewise, necessity, truth in all points in the field of the relation, is represented by $\bot \geq \neg A$ and possibility, truth in some points, by $\neg(\bot \geq A)$. (These are easily verified.)

Theorem 17: SPref *is sound and complete with respect to the class of all standard preference frames.*

Proof: Soundness is routine and can be left to the reader. It is noteworthy, though, that only the axiom (connex) requires that P_a be connected; this is relevant to the next section. For completeness, construct a canonical model as follows: Let $F = \langle W, \mathrm{P}\rangle$ with W the set of all maximal consistent extensions of SPref, as usual. For P first define for every formula $A \in L_{\geq}$ and $a \in W$,

$$\Pi_a A = \{b : \forall B \in b, B \geq A \in a\}$$

Also, for some useful notation, given what was noted above about the notion of necessity, let us define

$$Nec_a = \{\neg A : \bot \geq A \in a\}$$

A world, b, might be considered a possible alternative to a given world, a, if it contains nothing that a considers impossible, or, to say the same thing, if it contains the negation of everything impossible, which is to say, it contains all of Nec_a. Hence, let us define,

$$\Diamond_a b \text{ iff } Nec_a \subseteq b$$

Preference rankings of canonical worlds will range over possible alternatives only. Let

$$\mathrm{P}_a = \{\langle b, c\rangle : \Diamond_a b \;\&\; \Diamond_a c \;\&\; \forall C(c \in \Pi_a C \Rightarrow \exists B(b \in \Pi_a B \;\&\; B \geq C \in a))\}$$

Let P assign P_a to a. Let $M = \langle F, v\rangle$ with

$$v(p) = \{a \in W : p \in a\}$$

as usual.

Before establishing that M is indeed a standard preference model and canonical, it will be useful to have this lemma.

Lemma 18: *For all formulas A, if* $\perp \geq A \notin a$ *and* $X = \{\neg B : B \geq A \notin a\}$ *then* $X \cup Nec_a \cup \{A\}$ *is consistent.*

Proof: Suppose $\perp \geq A \notin a$, and also that $X \cup Nec_a \cup \{A\}$ is not consistent. $X \cup Nec_a$ is not empty, for otherwise $\{A\}$ would be inconsistent and $\vdash A \to \perp$, in which case, by (R.1), $\vdash \perp \geq A$ and $\perp \geq A \in a$, contrary to the initial supposition. Since $X \cup Nec_a$ is not empty, there are $C_1, \ldots, C_n \in X \cup Nec_a$ such that $\vdash (C_1 \wedge \ldots \wedge C_n) \to \neg A$, and so $\vdash A \to (\neg C_1 \vee \cdots \vee \neg C_n)$, and thus $\vdash (\neg C_1 \geq A) \vee \cdots \vee (\neg C_n \geq A)$ by (DR.3gen). So, $(\neg C_1 \geq A) \vee \cdots \vee (\neg C_n \geq A) \in a$. Each C_i is in X or in Nec_a. Suppose that $\neg C_i \geq A \in a$, and suppose $C_i \in X$. Then $C_i = \neg D_i$ where $D_i \geq A \notin a$. So $\neg\neg D_i \geq A \in a$, hence $D_i \geq A \in a$, and also $D_i \geq A \notin a$, a contradiction. Suppose, however, $\neg C_i \geq A \in a$, and $C_i \in Nec_a$. Then $C_i = \neg D_i$ where $\perp \geq D_i \in a$. So $\neg\neg D_i \geq A \in a$, and then $D_i \geq A \in a$. But then $\perp \geq A \in a$, by (trans), contrary to the initial assumption. Hence, in either case, there is a contradiction. Thus, if $\perp \geq A \notin a$, $X \cup Nec_a \cup \{A\}$ must be consistent.

From this completeness will follow quickly.

Lemma 19: *M is a model on a standard preference frame.*

Proof: (i) P_a is not empty: Since $\vdash \neg(\perp \geq \top)$, $\perp \geq \top \notin a$. Hence, by Lemma 18 $\{\neg B : B \geq \top \notin a\} \cup Nec_a \cup \{\top\}$ is consistent, and so $Nec_a \cup \{\top\}$ is consistent, and thus has a maximal consistent extension, b. Trivially, $\Diamond_a b$. Since $C \geq C \in a$ for every C, it is obvious that if, for some C, $b \in \Pi_a C$, there is some B, namely C, such that $b \in \Pi_a B$ and $B \geq C \in a$. Hence $b\mathrm{P}_a b$, and $b \in \mathcal{F}\mathrm{P}_a$. So P_a is not empty. (ii) that P_a is standard, i.e., reflexive, transitive, and connected on its field, is also easily shown, and so left to the reader. (We note, though, that P_a is not necessarily reflexive over the whole of W since there could be points $c \in W$ such that not-$\Diamond_a c$.)

Lemma 20: *For all formulas $A \in L_{\geq}$ and all $a \in W$, $A \in a$ iff $M, a \models_P A$.*

Proof: As usual, by induction on A; we consider only the case where $A = B \geq C$ and suppose the lemma to hold for B and C.

(i) Suppose $B \geq C \in a$ and let c be an arbitrary point in $\mathcal{F}\mathrm{P}_a$ such that $M, c \models_P C$. By the inductive hypothesis, $C \in c$. Since $c \in \mathcal{F}\mathrm{P}_a$, $\Diamond_a c$ and so $\bot \geq C \notin a$. Hence $\bot \geq B \notin a$ by (trans). Let $X = \{\neg D : D \geq B \notin a\}$. By Lemma 18, $X \cup Nec_a \cup \{B\}$ is consistent, and thus has a maximal consistent extension, b. Since $B \in b$, $M, b \models_P B$, by the inductive hypothesis. To show that $b\mathrm{P}_a c$, we have first that $\Diamond_a b$ since $Nec_a \subseteq b$. Second, suppose some E such that $c \in \Pi_a E$. Then $C \geq E \in a$ and so $B \geq E \in a$, by (trans). $b \in \Pi_a B$ since for any $F \in b$, $F \geq B \in a$, otherwise, if $F \geq B \notin a$, then $\neg F \in X$ and $\neg F \in b$, contrary to its consistency. Thus, for any E such that $c \in \Pi_a E$, there is a B such that $b \in \Pi_a B$ and $B \geq E \in a$, which suffices for $M, a \models_P B \geq C$, as required.

(ii) Suppose that $M, a \models_P B \geq C$, so that, for any $c \in \mathcal{F}\mathrm{P}_a$, if $M, c \models_P C$ then there is a b such that $b\mathrm{P}_a c$ and $M, b \models_P B$. Suppose then that $B \geq C \notin a$. $\bot \geq C \notin a$, else $B \geq C \in a$ by (trans) since $\vdash B \geq \bot$ and so $B \geq \bot \in a$. Let $X = \{\neg D : D \geq C \notin a\}$. By Lemma 18, $X \cup Nec_a \cup \{C\}$ is consistent, and so has a maximal consistent extension, c. $\Diamond_a c$ since $Nec_a \subseteq c$, and thus $c\mathrm{P}_a c$, and $c \in \mathcal{F}\mathrm{P}_a$. Since $C \in c$, $M, c \models_P C$ by the inductive hypothesis, and so there is a b such that $b\mathrm{P}_a c$ and $M, b \models_P B$. $c \in \Pi_a C$ since for any $F \in c$, $F \geq C \in a$, since otherwise $\neg F \in X$ and $\neg F \in c$, contrary to its consistency. Therefore, since $b\mathrm{P}_a c$, there is an E such that $b \in \Pi_a E$ and $E \geq C \in a$. Since $B \in b$, by the definition of $\Pi_a E$, $B \geq E \in a$, whence $B \geq C \in a$, contrary to the supposition above. Hence, if $M, a \models_P B \geq C$, $B \geq C \in a$, as required.

These two lemmas suffice for the completeness of SPref, Theorem 17, in the usual way, as for, e.g., Theorem 1.

The preceding also lays the groundwork for the characterization theorem for SDDL itself.

Theorem 21: *SDDL is sound and complete with respect to the class of all standard preference frames.*

Proof: As before, soundness is routine. For completeness, take the same model M as for Theorem 17, with the understanding that $\geq$ is now defined. It will suffice for present purposes to adapt Lemma 20 to the language of SDDL, to show the particular case, under the inductive hypothesis:

Lemma 22: $O(B/C) \in a$ *iff* $M, a \models_P O(B/C)$

Proof: (i) Suppose $O(B/C) \in a$. Then $\neg((C \wedge \neg B) \geq (C \wedge B)) \in a$, by (D. 6) for SDDL, so $(C \wedge \neg B) \geq (C \wedge B) \notin a$. It follows that $\bot \geq (C \wedge B) \notin a$, since $\vdash (C \wedge \neg B) \geq \bot$, so that $(C \wedge \neg B) \geq \bot \in a$; hence if $\bot \geq (C \wedge B) \in a$, $(C \wedge \neg B) \geq (C \wedge B) \in a$, by (trans), contrary to the opening supposition. Let $X = \{\neg D : D \geq (C \wedge B) \notin a\}$. By Lemma 18, $X \cup Nec_a \cup \{C \wedge B\}$ is consistent, and so has a maximal consistent extension, b. Since $Nec_a \subseteq b$, $\Diamond_a b$. Hence $b\mathrm{P}_a b$ and $b \in \mathcal{F}\mathrm{P}_a$. Also, since $C \in b$ and $B \in b$, $M, b \models_P C$ and $M, b \models_P B$, by the inductive hypothesis, so $M, b \models_P C \wedge B$. Now consider any c such that $c\mathrm{P}_a b$ and $M, c \models_P C$. By the inductive hypothesis, $C \in c$. Further, $b \in \Pi_a(C \wedge B)$, for consider any $D \in b$ and suppose that $D \geq (C \wedge B) \notin a$. Then $\neg D \in X$, and so $\neg D \in b$, contrary to its consistency. Hence, for all $D \in b$, $D \geq (C \wedge B) \in a$, which is to say $b \in \Pi_a(C \wedge B)$. Given that, since $c\mathrm{P}_a b$, there is an E such that $c \in \Pi_a E$ and $E \geq (C \wedge B) \in a$. $C \in c$; now suppose $B \notin c$, so that $\neg B \in c$ and thus $C \wedge \neg B \in c$. Since $c \in \Pi_a E$, $(C \wedge \neg B) \geq E \in a$ and so $(C \wedge \neg B) \geq (C \wedge B) \in a$, by (trans). But that contradicts the original assumption. Hence $B \in c$, and so $M, c \models_P B$ by the inductive hypothesis. That suffices for $M, a \models_P O(B/C)$, as required.

(ii) For the converse, suppose that $M, a \models_P O(B/C)$, so that there is a $b \in \mathcal{F}\mathrm{P}_a$ and $M, b \models_P C \wedge B$ and, for every c, if $c\mathrm{P}_a b$ and $M, c \models_P C$, then $M, c \models_P B$. Suppose also, for *reductio*, that $O(B/C) \notin a$. Then, by (D.6), $(C \wedge \neg B) \geq (C \wedge B) \in a$. Since $M, b \models_P C$ and $M, b \models_P B$, $C \in b$ and $B \in b$, by the inductive hypothesis, and so $C \wedge B \in b$. Since $b \in \mathcal{F}\mathrm{P}_a$, $\Diamond_a b$, and so $\bot \geq (C \wedge B) \notin a$. Hence $\bot \geq (C \wedge \neg B) \notin a$ by (trans) as above. Let $X = \{\neg D : D \geq (C \wedge \neg B) \notin a\}$. By Lemma 18, $X \cup Nec_a \cup \{C \wedge \neg B\}$ is consistent, and thus has a maximal consistent extension, c. Since $C \in c$, $M, c \models_P C$ by the inductive hypothesis. And since $Nec_a \subseteq c$, $\Diamond_a c$. Consider any E such that $b \in \Pi_a E$. Since $C \wedge B \in b$, $(C \wedge B) \geq E \in a$. $c \in \Pi_a(C \wedge \neg B)$, since otherwise there would be an $F \in c$ such that $F \geq (C \wedge \neg B) \notin a$, in which case $\neg F \in X$ and $\neg F \in c$, contrary to its consistency. Since $(C \wedge \neg B) \geq (C \wedge B) \in a$, $(C \wedge \neg B) \geq E \in a$, by (trans). That suffices for $c\mathrm{P}_a b$. Hence $M, c \models_P B$, and so $B \in c$, by the inductive hypothesis. But $\neg B \in c$ too, contrary to its consistency. Hence, if $M, a \models_P O(B/C)$, then $O(B/C) \in a$, as required.

This completes the lemma, and so the theorem in the usual way.

In the previous sections we referred to frames that have the property of being limited. This notion came to the fore in David Lewis's discussions of the Limit Assumption as applied to logics of conditionals, including conditional obligation, such as SDDL. (Cf. [7] §1.4, and elsewhere.) In the present framework, this is the assumption that, for any model M, for every formula, A, the set of worlds at which A holds on M — call that $|A|_M$ — is limited$_a$ (for any $a \in W$) in the sense defined at the end of Section 1. Lewis, [7] p. 129, demonstrated that SDDL, among other conditional logics, is characterized by the class of appropriate models that satisfy this assumption, that imposing it adds no new theorems to the logic. He liked to demonstrate this in terms of the structures he called systems of spheres, from which the same result can be derived for models on standard preference frames. Let us now show this directly, working from the canonical model of Theorem 21.

Corollary 23: *SDDL and* SPref *are sound and complete with respect to the class of models on standard preference frames that meet the Limit Assumption.*

Proof: All that is required is to show that the canonical model of Theorems 17 and 21 satisfies the assumption. Consider any $a \in W$ and any formula A, and show that for any b such that $b \in |A|_M$, b is maximal$_a$ in $|A|_M$ or there is a c such that c is maximal$_a$ in $|A|_M$ and $c\mathrm{S}_a b$. Suppose a $b \in |A|_M$ but that b is not maximal$_a$ in $|A|_M$. By Lemmas 20 and 22, $|A|_M = [A]$, so we can move back and forth between these descriptions. Thus, $b \in [A]$, and $A \in b$. If b is not maximal$_a$ in $|A|_M$, there is a c such that $c \in |A|_M$ and $c\mathrm{S}_a b$, i.e., $c\mathrm{P}_a b$ and not-($b\mathrm{P}_a c$). Since $c\mathrm{P}_a b$, both $\Diamond_a b$ and $\Diamond_a c$. It follows that $\bot \geq A \notin a$, since otherwise $\neg A \in Nec_a$ and $\neg A \in b$, contrary to its consistency. Let $X = \{\neg B : B \geq A \notin a\}$. By Lemma 18, $X \cup Nec_a \cup \{A\}$ is consistent, and so has a maximal consistent extension, d. Obviously, $\Diamond_a d$. Also, $d \in \Pi_a A$, for consider any $D \in d$; $D \geq A \in a$ since if $D \geq A \notin a$, $\neg D \in X$ and then $\neg D \in d$, contrary to its consistency. We now show that for all e such that $e \in [A]$ and $\Diamond_a e$, $d\mathrm{P}_a e$. Consider such an e, and suppose $e \in \Pi_a E$ for any E. Since $A \in e$, $A \geq E \in a$. Since $d \in \Pi_a A$, there is thus a D, namely A, such that $d \in \Pi_a D$ and $D \geq E \in a$, which suffices for $d\mathrm{P}_a e$, given both $\Diamond_a d$ and $\Diamond_a e$. It follows that d is maximal$_a$ in $|A|_M$. For, since $A \in d$, $d \in [A]$ and so $d \in |A|_M$, and there can't be any e such that $e \in |A|_M$ and $e\mathrm{S}_a d$, for if there were, then since $e\mathrm{P}_a d$, $\Diamond_a e$ and also not-($d\mathrm{P}_a e$), contrary to what was just established. Hence, d is maximal$_a$ in $|A|_M$. Furthermore, $d\mathrm{S}_a b$, since the preceding establishes that $d\mathrm{P}_a b$; it also establishes that $d\mathrm{P}_a c$. Hence, it cannot be that $b\mathrm{P}_a d$, for if it were, then $b\mathrm{P}_a c$, by transitivity, contrary to the assumption that b is not maximal$_a$ in $|A|_M$ and that c is a world that is strictly better than it. Thus, for any $b \in |A|_M$ either b is itself maximal$_a$ in $|A|_M$ or there is a d that is and $d\mathrm{S}_a b$, as required for $|A|_M$ to be limited$_a$.[17]

It should be apparent from the preceding discussion that SDDL and SPref are equivalent in quite a strong sense. Let us make this precise. Map the formulas of L_{co}, the language of SDDL, into the formulas of $L_{\geq}$, the language of SPref, by the translation function t: When $*$ is any binary truth-functional connective,

$$t(p) = p$$
$$t(\neg A) = \neg t(A)$$
$$t(A * B) = t(A) * t(B)$$
$$t(O(B/A)) = \neg((\neg t(B) \wedge t(A)) \geq (t(B) \wedge t(A)))$$

Then it is easy to establish this

Lemma 24: *For any model M on a standard preference frame and any $a \in W$, for every formula $A \in \mathrm{L}_{co}$, $M, a \models_P A$ iff $M, a \models_P t(A)$,*

which can be easily proved by induction on A. From this, and the preceding results of Theorems 17 and 21, it follows immediately that

Theorem 25: *For every formula $A \in L_{co}$, A is provable in SDDL iff $t(A)$ is provable in* SPref.

The formulas of $L_{\geq}$ can be similarly translated into formulas of L_{co}, thus:

$$s(p) = p$$
$$s(\neg A) = \neg s(A)$$
$$s(A * B) = s(A) * s(B)$$
$$s(A \geq B) = \neg O(\neg s(A)/s(A) \vee s(B))$$

And then

Lemma 26: *For any model M on a standard preference frame and any $a \in W$, for every formula $A \in \mathrm{L}_{\geq}$, $M, a \models_P A$ iff $M, a \models_P s(A)$,*

[17] This result differs from Corollaries 5, 11, and 16 in that those referred to frames that were limited, while this refers to limited models. The Limit Assumption is a semantical condition concerned with the interplay of the modelling relation, $\models$, and the preference relations, P_a, rather than a structural condition on the relations P_a themselves. We can, however, also say that SDDL is characterized by the class of limited standard frames given that Lewis, [7] §6.2, also demonstrated that it is characterized by the class of finite standard frames, and any finite frame is automatically limited.

whence,

Theorem 27: *For every formula $A \in L_{\geq}$, A is provable in* SPref *iff $s(A)$ is provable in SDDL.*

This section has demonstrated that the preference semantics is adequate for both of the standard dyadic deontic logics, SDDL for conditional obligation and SPref for the logic of preference, that may be considered counterparts of the monadic SDL. In the next section the same sort of results are established for dyadic counterparts to the monadic P.

4. *Dyadic P*

The logic P was introduced in Section 2 as a monadic deontic logic that allows for conflicts of obligation. It differs from SDL in containing neither the aggregation principle (C) (or the distribution principle (K)) nor the consistency postulate (D). In the semantics this is achieved by allowing the preference relation assigned to points $a \in W$ not to be connected. Just as there could be conflicts of absolute obligations, so too there could be conflicts of conditional obligations, situations in which it ought to be that B, given that A, and also it ought to be that not-B given the same condition, A. To accommodate such conflicts, the dyadic logic should reject the principle of conditional aggregation (D.3) and the postulates (CK) and (CD) of SDDL.

Let the logic DP be the class of formulas in the language L_{co} that extends PC and is closed under *modus ponens*, and these axioms and rules:

(RCE) If $\vdash A \leftrightarrow A'$ then $\vdash O(B/A) \leftrightarrow O(B/A')$
(RCM) If $\vdash B \rightarrow C$ then $\vdash O(B/A) \rightarrow O(C/A)$
(CN) $O(\top/\top)$
(CP) $\neg O(\bot/A)$
(C$O\wedge$) $O(B/A) \rightarrow O(A \wedge B/A)$
(trans) $((A \geq B) \wedge (B \geq C)) \rightarrow (A \geq C)$
(C$O\vee$) $O(A/B \vee C) \rightarrow (O(A/B) \vee O(A/C))$

These are all contained in SDDL, though (CP) and (C$O\vee$) were there derivable from SDDL's stronger postulates. As before, $A \geq B$ in L_{co} is defined as $\neg O(\neg A/A \vee B)$. Of other principles that were listed as derivable in SDDL, the following are derivable in DP; several are used to establish results below:

(D.1) If $\vdash A$ then $\vdash O(A/A)$
(D.2) $\vdash O(\top/A) \leftrightarrow O(A/A)$
(D.2a) $\vdash O(B/A) \rightarrow O(A/A)$

(D.4)	$\vdash \neg O(\neg A/A)$
(D.6)	$\vdash O(B/A) \leftrightarrow \neg((A \wedge \neg B) \geq (A \wedge B))$
(D.7)	$\vdash A \geq A$
(D.9)	$\vdash \neg(\bot \geq \top)$
(D.10)	$\vdash A \geq \bot$
(D.11)	$\vdash ((A \geq B) \wedge (A \geq C)) \rightarrow (A \geq (B \vee C))$
(DR.1)	If $\vdash A \rightarrow B$ then $\vdash B \geq A$

Missing from the earlier list are the conditional aggregation principle (D.3), the principle of connectedness for $\geq$ (D.8), and importantly the derived rules (DR.2) and its generalization (DR.2gen) for conditional obligation.

DP is characterized by the class of all reflexive, transitive frames. Before demonstrating that, however, it is useful to bring in the logic of preference that corresponds to DP as SPref corresponds to SDDL. Call this system PPref. With formulas from the language $L_{\geq}$ as before, PPref is axiomatized simply by deleting the axiom (connex) from SPref. That is, its postulates are, in addition to PC and *modus ponens*:

(R.1)	If $\vdash A \rightarrow B$ then $\vdash B \geq A$
(trans)	$((A \geq B) \wedge (B \geq C)) \rightarrow (A \geq C)$
($\geq \vee$)	$((A \geq B) \wedge (A \geq C)) \rightarrow (A \geq (B \vee C))$
(poss)	$\neg(\bot \geq \top)$

Of the derivable principles listed for SPref, these

(D.1)	$\vdash A \geq A$
(D.2)	$\vdash (A \geq B) \leftrightarrow \neg O(\neg A/A \vee B)$
(D.4)	$\vdash A \geq \bot$
(DR.1)	If $\vdash A \rightarrow B$ then $\vdash (A \geq C) \rightarrow (B \geq C)$
(DR.2)	If $\vdash B \rightarrow C$ then $\vdash (A \geq C) \rightarrow (A \geq B)$

are derivable in PPref, but (D.3) and (DR.3gen) for preference are not since they require the axiom (connex).

Like DP, PPref is characterized by the class of all reflexive, transitive preference frames. To prove this, we make the same sort of detour through multiple preference models that was applied for the proof of Theorem 12 in Section 2.

Multiple preference frames $F = \langle W, \mathcal{P} \rangle$ are defined exactly as in Section 2. We assume all relations in $\mathcal{P}_a$ are non-empty, and reflexive and transitive on their fields. They may be connected too, but that will not affect the results below. Given a model $M = \langle F, v \rangle$ on such a frame, formulas $O(B/A)$ in L_{co} are interpreted according to the rule:

(MP-CO) $M, a \models_{MP} O(B/A)$ iff there is a $\mathrm{P} \in \mathcal{P}_a$ such that, for some $b \in \mathcal{F}\mathrm{P}$, $M, b \models_{MP} A \wedge B$ and, for any c, if $c\mathrm{P}b$ and $M, c \models_{MP} A$, then $M, c \models_{MP} B$

and formulas $A \geq B$ in $L_{\geq}$ are interpreted according to

(MP-$\geq$) $M, a \models_{MP} A \geq B$ iff, for every $\mathrm{P} \in \mathcal{P}_a$, for all $c \in \mathcal{F}\mathrm{P}$ such that $M, c \models_{MP} B$, there is a b such that $b\mathrm{P}c$ and $M, b \models_{MP} A$

The first of these rules is the conditional counterpart of the rule (MP-O) and asks that the basic understanding of $O(B/A)$ obtain with respect to some relation in the set $\mathcal{P}_a$ assigned to a. With this rule and the definition of $\geq$ in terms of $O(-/-)$, the other rule (MP-$\geq$) for formulas $A \geq B$ is derivable. By the same token, if $\geq$ is taken as primitive and formulas $O(B/A)$ defined as $\neg((A \wedge \neg B) \geq (A \wedge B))$, then the rule (MP-C$O$) is derivable. Hence we should be comfortable moving back and forth between the two dyadic connectives. In what follows we will treat the two together, letting either one be primitive as appropriate.

Theorem 28: *(a)* PPref *is sound and complete with respect to all reflexive, transitive multiple preference frames. (b)* DP *is sound and complete with respect to all reflexive, transitive multiple preference frames. (c)* PPref *and* DP *are also both sound and complete with respect to all standard multiple preference frames.*

Proof: Soundness for both is, as usual, routine and left to the reader. For completeness we define a canonical model in a way that is somewhat analogous to, but not exactly the same as, that for Theorem 7. Let $F = \langle W, \mathcal{P} \rangle$ where W is the set of all maximal consistent extensions of PPref or DP as appropriate, and $\mathcal{P}$ is defined thus: For each formula A in $L_{\geq}$ or L_{co}, as appropriate, define

$$\Theta_a A = \{b : \forall B(\text{if } B \in b \text{ then } \neg A \geq B \notin a)\}$$

Then for each $a \in W$ and each formula A define a binary relation P_a^A such that

$$\mathrm{P}_a^A = \{\langle b, c \rangle : \Diamond_a b \text{ and } \Diamond_a c, \text{ and if } c \in \Theta_a A \text{ then } b \in \Theta_a A\}$$

where $\Diamond_a b$, etc. is defined as for Theorem 17. Let

$$\mathcal{P}_a = \{\mathrm{P} : \mathrm{P} \neq \emptyset \text{ and } \exists A(\mathrm{P} = \mathrm{P}_a^A)\}$$

$\mathcal{P}$ assigns $\mathcal{P}_a$ to a. Let $M = \langle F, v \rangle$ where, as usual,

$$v(p) = \{a : a \in W \text{ and } p \in a\}$$

Lemma 18 that was used to establish that the canonical model for SDDL for Theorem 17 was indeed a model, and canonical, does not hold for PPref or DP. Instead we have,

Lemma 29: *For any formulas A and B, if $\bot \geq B \notin a$ and $A \geq B \notin a$ and $X = \{\neg C : A \geq C \in a\}$ then $X \cup \{B\}$ is consistent.*

Proof: Assume $\bot \geq B \notin a$ and $A \geq B \notin a$, and suppose $X \cup \{B\}$ is not consistent. If X were empty then $\{B\}$ would be inconsistent, and then $\vdash B \to \bot$ and $\vdash \bot \geq B$, by (R.1), so that $\bot \geq B \in a$, contrary to the initial assumption. So suppose X is not empty. Then there are $D_1, \ldots, D_n \in X$ ($n \geq 1$) such that $\vdash (D_1 \wedge \ldots \wedge D_n) \to \neg B$ and thus $\vdash B \to (\neg D_1 \vee \cdots \vee \neg D_n)$, and $\vdash (\neg D_1 \vee \cdots \vee \neg D_n) \geq B$, by (R.1). So $(\neg D_1 \vee \cdots \vee \neg D_n) \geq B \in a$. Each $D_i = \neg C_i$ when $A \geq C_i \in a$, so $C_1 \vee \cdots \vee C_n \geq B \in a$. Also, $(A \geq C_1) \wedge \ldots \wedge (A \geq C_n) \in a$, so $A \geq (C_1 \vee \cdots \vee C_n) \in a$, by ($\geq \vee$) generalized. But then $A \geq B \in a$ by (trans), contrary to the initial assumption. Hence, $X \cup \{B\}$ must be consistent.

Lemma 30: *M is a model on a standard multiple preference frame.*

Proof: (i) There is a non-empty relation in $\mathcal{P}_a$. Since $\vdash \neg(\bot \geq \top)$, and thus $\bot \geq \top \notin a$, by Lemma 29, $\{\neg D : \bot \geq D \in a\} \cup \{\top\}$ is consistent, and so has a maximal consistent extension, $b \in W$. $\Diamond_a b$, and $b\mathrm{P}_a^\top b$, both of which are easily shown. Hence $\mathrm{P}_a^\top$ is not empty and $\mathrm{P}_a^\top \in \mathcal{P}_a$. (ii) That every $\mathrm{P} \in \mathcal{P}_a$ is non-empty is part of the definition of $\mathcal{P}_a$; that they are all reflexive, transitive, and connected on their fields, hence standard, is easily shown, and so left to the reader.

Lemma 31: *For every formula A and every $a \in W$, $A \in a$ iff $M, a \models_{MP} A$.*

Proof: By induction on A. We consider the two cases: (a) $A = B \geq C$ and (b) $A = O(B/C)$, treating each one as primitive in its context, with the rules of derivation as given for the system for that connective. Assume the lemma holds for B and C.

Case (a.i). Suppose $B \geq C \in a$, and let P be any relation in $\mathcal{P}_a$. $\mathrm{P} = \mathrm{P}_a^D$ for some D. Let c be any member of $\mathcal{F}\mathrm{P}_a^D$ such that $M, c \models_{MP} C$. $C \in c$, by the inductive hypothesis. Since $c \in \mathcal{F}\mathrm{P}_a^D$, $\Diamond_a c$; hence, $\bot \geq C \notin a$ and so $\bot \geq B \notin a$ by (trans). Either (1) $c \in \Theta_a D$ or (2) $c \notin \Theta_a D$. If

(1), then, since $C \in c$, $\neg D \geq C \notin a$, and so $\neg D \geq B \notin a$, by (trans). Let $X = \{\neg E : \neg D \geq E \in a\}$. By Lemma 29, $X \cup \{B\}$ is consistent, and so has a maximal consistent extension, b. $b \in \Theta_a D$, for otherwise there would be an $E \in b$ such that $\neg D \geq E \in a$ and then $\neg E \in X$ and $\neg E \in b$, contrary to its consistency. Moreover, $\Diamond_a b$, for if there were an $E \in b$ such that $\bot \geq E \in a$, then, since $\vdash \neg D \geq \bot$ and so $\neg D \geq \bot \in a$, $\neg D \geq E \in a$, and we have just seen that that is not the case. Since $b \in \Theta_a D$, it is trivial that if $c \in \Theta_a D$ then $b \in \Theta_a D$, and so $b\mathrm{P}_a^D c$. Since $B \in b$, $M, b \models_{MP} B$ by the inductive hypothesis. This suffices for $M, a \models_{MP} B \geq C$. In case (2), where $c \notin \Theta_a D$, let $X = Nec_a$. Again by Lemma 29, $X \cup \{B\}$ is consistent. Let b be a maximal consistent extension of $X \cup \{B\}$. $\Diamond_a b$ by definition. It is again trivial that if $c \in \Theta_a D$ then $b \in \Theta_a D$. Hence $b\mathrm{P}_a^D c$. Since $B \in b$, $M, b \models_{MP} B$, by the inductive hypothesis, which suffices again for $M, a \models_{MP} B \geq C$, as required.

Case (a.ii). Suppose $M, a \models_{MP} B \geq C$, but also suppose for *reductio* that $B \geq C \notin a$. It follows that $\bot \geq C \notin a$. For suppose otherwise, then since $\vdash B \geq \bot$, and so $B \geq \bot \in a$, it would follow that $B \geq C \in a$ by (trans), contrary to the supposition above. Since $M, a \models_{MP} B \geq C$, for all $\mathrm{P} \in \mathcal{P}_a$, for every $c \in \mathcal{F}\mathrm{P}$ if $M, c \models_{MP} C$ then there is a b such that $b\mathrm{P}c$ and $M, b \models_{MP} B$. Consider then $\mathrm{P}_a^{\neg B}$. Let $X = \{\neg D : B \geq D \in a\}$. By Lemma 29, $X \cup \{C\}$ is consistent, and thus has a maximal consistent extension, c. $c \in \Theta_a \neg B$; for consider any $E \in c$ and suppose, for *reductio*, $\neg\neg B \geq E \in a$. Then, obviously $B \geq E \in a$, and so $\neg E \in X$. But then $\neg E \in c$, contrary to its consistency. Hence, for any E, if $E \in c$ then $\neg\neg B \geq E \notin a$. Further, $\Diamond_a c$. For suppose otherwise; then there would be an $E \in c$ such that $\bot \geq E \in a$ and then $B \geq E \in a$, since $B \geq \bot \in a$, contrary to what was just said. Since $\Diamond_a c$ and $c \in \Theta_a \neg B$, $c\mathrm{P}_a^{\neg B} c$, so $c \in \mathcal{F}\mathrm{P}_a^{\neg B}$. Hence $\mathrm{P}_a^{\neg B} \neq \emptyset$ and $\mathrm{P}_a^{\neg B} \in \mathcal{P}_a$. $C \in c$, so $M, c \models_{MP} C$, by the inductive hypothesis. Therefore, there is a b such that $b\mathrm{P}_a^{\neg B} c$ and $M, b \models_{MP} B$. $B \in b$ by the inductive hypothesis. By definition of $\mathrm{P}_a^{\neg B}$, since $c \in \Theta_a \neg B$, $b \in \Theta_a \neg B$. Hence, by definition, since $B \in b$, $\neg\neg B \geq B \notin a$. But, of course, $\vdash B \rightarrow \neg\neg B$, so that $\vdash \neg\neg B \geq B$, by (R.1) of **PPref**, and so $\neg\neg B \geq B \in a$, a contradiction. Therefore, if $M, a \models_{MP} B \geq C$, $B \geq C \in a$, as required.

Case (b.i). Suppose $O(B/C) \in a$. Then $\neg((C \wedge \neg B) \geq (C \wedge B)) \in a$, by (D.6) for **DP**, so $(C \wedge \neg B) \geq (C \wedge B) \notin a$. This entails $\bot \geq (C \wedge B) \notin a$, since if $\bot \geq (C \wedge B) \in a$ then since $\vdash (C \wedge \neg B) \geq \bot$, $(C \wedge \neg B) \geq \bot \in a$, and then $(C \wedge \neg B) \geq (C \wedge B) \in a$ by (trans), contrary to what is given. Consider the relation $\mathrm{P}_a^{\neg(C \wedge \neg B)} \in \mathcal{P}_a$. Let $X = \{\neg D : C \wedge \neg B \geq D \in a\}$. By Lemma 29, $X \cup \{C \wedge B\}$ is consistent, and thus has a maximal consistent extension, b. By the arguments of (a.ii) regarding c, $b \in \Theta_a \neg(C \wedge \neg B)$, and

$\diamond_a b$. Hence $b\mathrm{P}_a^{\neg(C\wedge\neg B)}b$ and $b \in \mathcal{F}\mathrm{P}_a^{\neg(C\wedge\neg B)}$. Thus $\mathrm{P}_a^{\neg(C\wedge\neg B)} \neq \emptyset$ and $\mathrm{P}_a^{\neg(C\wedge\neg B)} \in \mathcal{P}_a$. $C \in b$ and $B \in b$, so $M, b \models_{MP} C$ and $M, b \models_{MP} B$, by the inductive hypothesis, and thus $M, b \models_{MP} C \wedge B$. Consider any c such that $c\mathrm{P}_a^{\neg(C\wedge\neg B)}b$ and $M, c \models_{MP} C$. Since $b \in \Theta_a\neg(C\wedge\neg B)$, $c \in \Theta_a\neg(C\wedge\neg B)$. $C \in c$, by the inductive hypothesis. Suppose $B \notin c$; then $\neg B \in c$ and so $C \wedge \neg B \in c$. With $c \in \Theta_a\neg(C \wedge \neg B)$, it follows that $\neg\neg(C \wedge \neg B) \geq (C\wedge\neg B) \notin a$, and hence $(C\wedge\neg B) \geq (C\wedge\neg B) \notin a$, contrary to reflexivity, (D.7) of DP. Hence, $B \in c$, and so $M, c \models_{MP} B$ by the inductive hypothesis. Therefore, there is a $\mathrm{P} \in \mathcal{P}_a$ such that there is a $b \in \mathcal{F}\mathrm{P}$ and $M, b \models_{MP} C \wedge B$ and for all c such that $c\mathrm{P}b$ and $M, c \models_{MP} C$, $M, c \models_{MP} B$, which is to say, $M, a \models_{MP} O(B/C)$, as required.

Case (b.ii). Suppose $M, a \models_{MP} O(B/C)$, but that $O(B/C) \notin a$. Then, by (D.6) for DP, $(C\wedge\neg B) \geq (C\wedge B) \in a$. Also, there is a $\mathrm{P} \in \mathcal{P}_a$ such that for some $b \in \mathcal{F}\mathrm{P}$, $M, b \models_{MP} C \wedge B$ and for all c such that $c\mathrm{P}b$ and $M, c \models_{MP} C$, $M, c \models_{MP} B$. $\mathrm{P} = \mathrm{P}_a^D$ for some D. $M, b \models_{MP} C$ and $M, b \models_{MP} B$, so $C \in b$ and $B \in b$, by the inductive hypothesis. Hence $C \wedge B \in b$. Since $b \in \mathcal{F}\mathrm{P}_a^D$, $\diamond_a b$, and so $\bot \geq (C \wedge B) \notin a$, whence $\bot \geq (C \wedge \neg B) \notin a$, by (trans). Either (1) $b \in \Theta_a D$ or (2) $b \notin \Theta_a D$. In case (1), $\neg D \geq (C\wedge\neg B) \notin a$, for if otherwise, then, by (trans), $\neg D \geq (C\wedge B) \in a$, but, since $C\wedge B \in b$ and $b \in \Theta_a D$, $\neg D \geq (C \wedge B) \notin a$, a contradiction. Let $X = \{\neg E : \neg D \geq E \in a\}$. By Lemma 29, $X \cup \{C \wedge \neg B\}$ is consistent. Let c be a maximal consistent extension of that. $c \in \Theta_a D$ and $\diamond_a c$, as with the argument regarding b under (a.i) above. Hence, $c\mathrm{P}_a^D b$. $C \in c$, so $M, c \models_{MP} C$, by the inductive hypothesis. Then, $M, c \models_{MP} B$, and $B \in c$ by the inductive hypothesis again. But $\neg B \in c$, contrary to its consistency. Hence, $O(B/C) \in a$. In case (2), $X = Nec_a$; $X \cup \{C \wedge \neg B\}$ is consistent, by Lemma 29, and so has a maximal consistent extension, c. $\diamond_a c$ by definition. Since $b \notin \Theta_a D$, $c\mathrm{P}_a^D b$, trivially. As with case (1), $C \in c$, so $M, c \models_{MP} C$ and $M, c \models_{MP} B$, so that $B \in c$. But $\neg B \in c$, contrary to its consistency. Therefore, $O(B/C) \in a$. Hence, in either case, $O(B/C) \in a$, as required.

This completes the Lemma, and so the Theorem for both PPref and DP in the usual way. Following the theme of the previous theorems, we can also say,

Corollary 32: PPref *and* DP *are both sound and complete with respect to the class of models on reflexive, transitive (or standard) multiple preference frames that satisfy the Limit Assumption,*

where here the Limit Assumption is generalized to apply to all relations in $\mathcal{P}_a$ for every $a \in W$. That is, it states that for all A, and for all $\mathrm{P} \in \mathcal{P}_a$, every $b \in |A|_M$ is such that either b is P-maximal$_a$ in $|A|_M$ or there is a c such that c is P-maximal$_a$ in $|A|_M$ and $c\mathrm{S}^{\mathrm{P}}b$. This is established through an argument very similar to that for Corollary 23, but now, supposing that some $b \in |A|_M$ is not P-maximal$_a$, when $\mathrm{P} = \mathrm{P}^B_a$ for some formula B, one takes $X = \{\neg D : \neg B \geq D \in a\}$. Since there is a $c \in |A|_M$ such that $c\mathrm{P}^B_a b$ and not-($b\mathrm{P}^B_a c$), $\Diamond_a b$, whence $\perp \geq A \notin a$, and $c \in \Theta_a B$, whence $\neg B \geq A \notin a$. Therefore, Lemma 29 assures the consistency of $X \cup \{A\}$. Take d to be a maximal consistent extension of that. It fulfills the role of d in the argument for Corollary 23. (It is not hard to show that $\Diamond_a d$ and that $d \in \Theta_a B$.)

The multiple preference framework introduced for Theorem 28 is only a stepping stone on the way to establishing completeness in the framework of simple preference frames. To complete that result, we apply the method that was used in Section 2 to establish completeness for monadic P, reducing multiple preference models to equivalent simple preference models.

Theorem 33: *(a)* PPref *is sound and complete with respect to the class of reflexive, transitive simple preference frames; (b)* DP *is sound and complete with respect to the class of reflexive, transitive simple preference frames.*

Proof: We treat the two parts together. Soundness is simple, and left to the reader. Let $M = \langle F, v\rangle$ be a model on a multiple preference frame $F = \langle W, \mathcal{P}\rangle$ in which every $\mathrm{P} \in \mathcal{P}_a$ is reflexive and transitive on its field. Take M^* to be the model $\langle F^*, v^*\rangle$ derived from M as defined for Theorem 12.

Lemma 34: *M^* is a model on a reflexive, transitive simple preference frame.*

This is merely Lemma 13 of Section 2, enriched by Observation 2 there. We likewise extend Lemma 14 to apply to the grammar of the dyadic systems. We assume this to be true for B and C and show:

Lemma 35: *For every $a \in W$, and every $i \in I$, (a) $M, a \models_{MP} B \geq C$ iff $M^*, \langle a, i\rangle \models_{P} B \geq C$. (b) $M, a \models_{MP} O(B/C)$ iff $M^*, \langle a, i\rangle \models_{P} O(B/C)$.*

Proof: The arguments for (a) and (b) are similar.

(a.i) Suppose that $M, a \models_{MP} B \geq C$, and thus that for every $\mathrm{P} \in \mathcal{P}_a$, for all $c \in \mathcal{F}\mathrm{P}$ if $M, c \models_{MP} C$ then there is a b such that $b\mathrm{P}c$ and $M, b \models_{MP} B$. To show that $M^*, \langle a, i\rangle \models_{P} B \geq C$, let c^* be any member of $\mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$ such that $M^*, c^* \models_{P} C$. Given reflexivity for $\mathrm{P}^*_{\langle a,i\rangle}$, $c^*\mathrm{P}^*_{\langle a,i\rangle}c^*$ and so there is a $\mathrm{P}^{*j} \in \mathcal{P}^*_{\langle a,i\rangle}$ such that $c^*\mathrm{P}^{*j}c^*$, and thus $c^* = \langle c, j\rangle$ for some $c \in W$ and $c\mathrm{P}^j c$, so that $c \in \mathcal{F}\mathrm{P}^j$. Also, $M, c \models_{MP} C$, by the inductive hypothesis. Hence, there is a b such that $b\mathrm{P}^j c$ and $M, b \models_{MP} B$. Since $b\mathrm{P}^j c$, $\langle b, j\rangle \mathrm{P}^{*j} \langle c, j\rangle$ and so $\langle b, j\rangle \mathrm{P}^*_{\langle a,i\rangle} \langle c, j\rangle$. By the inductive hypothesis, $M^*, \langle b, j\rangle \models_P B$. This suffices for $M^*, \langle a, i\rangle \models_P B \geq C$.

(a.ii) Suppose that $M^*, \langle a, i\rangle \models_P B \geq C$, so that for every $c^* \in \mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$ if $M^*, c^* \models_P C$, then there is a b^* such that $b^*\mathrm{P}^*_{\langle a,i\rangle} c^*$ and $M^*, b^* \models_P B$. To show that $M, a \models_{MP} B \geq C$, let P be any relation in $\mathcal{P}_a$. $\mathrm{P} = \mathrm{P}^j$ for some $j \in I$. Let c be any member of $\mathcal{F}\mathrm{P}^j$ such that $M, c \models_{MP} C$. By reflexivity, $c\mathrm{P}^j c$ and so $\langle c, j\rangle \mathrm{P}^{*j} \langle c, j\rangle$ and thus $\langle c, j\rangle \mathrm{P}^*_{\langle a,i\rangle} \langle c, j\rangle$ and $\langle c, j\rangle \in \mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$. Also, $M^*, \langle c, j\rangle \models_P C$, by the inductive hypothesis. Hence there is a b^* such that $b^*\mathrm{P}^*_{\langle a,i\rangle} \langle c, j\rangle$ and $M^*, b^* \models_P B$. Hence, $b^*\mathrm{P}^{*k} \langle c, j\rangle$ for some $k \in I$. But then, since $\langle c, j\rangle \in \mathcal{F}\mathrm{P}^{*k}$, $k = j$ and so $b^*\mathrm{P}^{*j} \langle c, j\rangle$. In that case too, $b^* = \langle b, j\rangle$, and then $\langle b, j\rangle \mathrm{P}^{*j} \langle c, j\rangle$, which implies $b\mathrm{P}^j c$. Moreover, $M, b \models_{MP} B$, by the inductive hypothesis. That suffices for $M, a \models_{MP} B \geq C$.

(b.i) Suppose that $M, a \models_{MP} O(B/C)$, and thus that there is a $\mathrm{P} \in \mathcal{P}_a$ and a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} C \wedge B$ and, for every c, if $c\mathrm{P}b$ and $M, c \models_{MP} C$ then $M, c \models_{MP} B$. For such a P, $\mathrm{P} = \mathrm{P}^j$ for some $j \in I$. With reflexivity, $b\mathrm{P}^j b$ and so $\langle b, j\rangle \mathrm{P}^{*j} \langle b, j\rangle$; hence, $\langle b, j\rangle \mathrm{P}^*_{\langle a,i\rangle} \langle b, j\rangle$ and $\langle b, j\rangle \in \mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$. $M, b \models_{MP} C$ and $M, b \models_{MP} B$, so $M^*, \langle b, j\rangle \models_P C$ and $M^*, \langle b, j\rangle \models_P B$, by the inductive hypothesis, so that $M^*, \langle b, j\rangle \models_P C \wedge B$. Consider any c^* such that $c^*\mathrm{P}^*_{\langle a,i\rangle} \langle b, j\rangle$ and $M^*, c^* \models_P C$. $c^*\mathrm{P}^{*k} \langle b, j\rangle$ for some $k \in I$, but then $k = j$ as above. Hence, $c^*\mathrm{P}^{*j} \langle b, j\rangle$, and then $c^* = \langle c, j\rangle$ and $\langle c, j\rangle \mathrm{P}^{*j} \langle b, j\rangle$, which implies $c\mathrm{P}^j b$. Furthermore, $M, c \models_{MP} C$, by the inductive hypothesis. So $M, c \models_{MP} B$and $M^*, \langle c, j\rangle \models_P B$, by the inductive hypothesis again, so $M^*, c^* \models_{MP} B$, which suffices for $M^*, \langle a, i\rangle \models_P O(B/C)$.

(b.ii) Suppose $M^*, \langle a, i\rangle \models_P O(B/C)$, so that there is a $b^* \in \mathcal{F}\mathrm{P}^*_{\langle a,i\rangle}$ such that $M^*, b^* \models_P C \wedge B$ and for every c^* if $c^*\mathrm{P}^*_{\langle a,i\rangle} b^*$ and $M^*, c^* \models_P C$ then $M^*, c^* \models_P B$. By reflexivity $b^*\mathrm{P}^*_{\langle a,i\rangle} b^*$, so there is a $\mathrm{P}^{*j} \in \mathcal{P}^*_{\langle a,i\rangle}$ and $b^*\mathrm{P}^{*j} b^*$, in which case $b^* = \langle b, j\rangle$, for some $b \in W$, and $b\mathrm{P}^j b$. Thus $b \in \mathcal{F}\mathrm{P}^j$. Also $\mathrm{P}^j \in \mathcal{P}_a$. $M^*, \langle b, j\rangle \models_P C$ and $M^*, \langle b, j\rangle \models_P B$, so $M, b \models_{MP} C$ and $M, b \models_{MP} B$, by the inductive hypothesis, so that $M, b \models_{MP} C \wedge B$. Now consider any c such that $c\mathrm{P}^j b$ and $M, c \models_{MP} C$. Since $c\mathrm{P}^j b$, $\langle c, j\rangle \mathrm{P}^{*j} \langle b, j\rangle$, so $\langle c, j\rangle \mathrm{P}^*_{\langle a,i\rangle} \langle b, j\rangle$. Also, $M^*, \langle c, j\rangle \models_P C$, by the inductive hypothesis, and thus $M^*, \langle c, j\rangle \models_P B$. So $M, c \models_{MP} B$ by the inductive hypothesis again, which suffices for $M, a \models_{MP} O(B/C)$ to complete the lemma.

The theorem now follows from these lemmas just as Theorem 12 followed from Lemmas 13 and 14 in Section 2. If A is not provable in PPref or DP, then by Theorem 28 there is a reflexive, transitive multiple preference model M that falsifies A at a point $a \in W$, and so by the present lemmas there is a reflexive, transitive simple preference model, M^* derived from M that falsifies A at $\langle a, i\rangle$ for any $i \in I$, and so A cannot be valid in all such simple preference frames. Furthermore, just as Corollary 16 followed from Theorem 12, so too

Corollary 36: PPref *and* DP *are sound and complete with respect to the class of models on reflexive, transitive simple preference frames that meet the Limit Assumption.*

This follows from Corollary 32 by an argument very like the one that led to Corollary 16 from Corollary 11.

From the two parts (a) and (b) of Theorem 33, it follows easily that DP and PPref are equivalent in the same sense as SDDL and SPref. That is, with the same translation functions t and s used for Theorems 25 and 27,

Theorem 37: *(a) For all formulas A of L_{co}, A is provable in* DP *iff $t(A)$ is provable in* PPref. *(b) For all formulas A of $L_{\geq}$, A is provable in* PPref *iff $s(A)$ is provable in* DP.

This is proved just as were Theorems 25 and 27.

5. *Conclusion*

This completes the results for this Part. We have shown that the basic preference semantics for deontic logic introduced in Section 1 is adequate for monadic standard deontic logic, SDL, when it is assumed that each world's preference relation is reflexive, transitive and connected on its field. To drop the requirement of connectedness leads to the characterization of the weaker monadic deontic logic P, whose virtue is that it allows for conflicts of obligation. Similar results obtain for dyadic deontic logics. The class of standard preference frames characterizes the standard logic of conditional obligation SDDL and its counterpart logic of preferability SPref, while the class of frames that do not require that the preference relation be connected characterizes dyadic logics that correspond to P, namely DP for conditional obligation and PPref for preferability. These too have the virtue of accommodating conflicts of (conditional) obligation. The proofs of completeness for P and its dyadic counterparts introduced a variant on the original simple preference semantics, namely, the evaluation of deontic formulas in models on what I have called multiple preference frames. Although such frames were introduced here for technical purposes, they also have interest in their own right, for they allow for further distinctions to be drawn amongst deontic modalities. That is the subject of the sequel, Part II of this work [4].[18]

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