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PARACONSISTENT COMPATIBILITY*

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Abstract

In this paper, I present two adaptive logics for paraconsistent compatibility. The consequence relation defined by these logics leads from a (possibly inconsistent) set of premises to all the sentences that are compatible with them. Their proof theory is dynamic, but is proven sound and complete with respect to a static semantics. For the consistent case, both logics lead to exactly the same results as the logics for classical compatibility that were presented in [11].

It is shown that paraconsistent compatibility cannot be defined with respect to a monotonic paraconsistent logic, but only with respect to an inconsistency-adaptive logic. The paper contains modal versions of two well-studied inconsistency-adaptive logics. These modal versions form the basis for the logics for paraconsistent compatibility, but are also interesting with respect to other applications.

1. *Introduction*

Over the last two decades, philosophers of science as well as logicians and computer scientists showed an ever growing interest in the dynamics of reasoning. Obvious examples include the literature on belief change and that on non-monotonic logics. Other examples are related to the fact that, in a multitude of domains, attention shifted from the statical properties of the products of reasoning processes to the dynamical properties of the reasoning processes themselves. In erotetic logic, for instance, the focus is no longer on the abstract relations between questions and answers, but on the way in

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which questions are generated (and suppressed) by sets of declarative sentences and/or other questions. Analogously, in the literature on explanation and on induction, considerable attention is now paid to hypothesis formation and hypothesis withdrawal.

What all these studies have in common is that (explicitly or implicitly) the concept of compatibility plays a key role in it. For instance, the question whether or not A is compatible with one's beliefs makes the difference between belief expansion and belief revision. Also, a minimal requirement for a question to be generated from a set of declarative sentences is that its presuppositions are not incompatible with them, and an important constraint for inductive hypotheses is that they are compatible with the available data as well as with each other.

In [11], two logics are presented, called COM and COM*, that capture the concept of *classical compatibility*. Thus, where CL stands for Classical Logic, both logics lead from a set of sentences Γ to the set of sentences that are CL-compatible with Γ. A sentence A is said to be CL-compatible with a set of premises Γ iff $\Gamma \nvdash_{\mathsf{CL}} \sim A$. What this comes to, semantically, is that A is true in some CL-model of Γ. The only difference between the logics concerns the way in which they handle inconsistent sets of premises. According to COM, everything is compatible with an inconsistent set of premises, according to COM* nothing is compatible with it.

The importance of COM and COM* is that they offer a *proof theory* for compatibility. They thus allow one to *reason* from a set of premises to the sentences that are compatible with them. What is special about the proof theory is that it is *dynamic*. This is related to the fact that compatibility is *non-monotonic* (q is compatible with $\{p\}$, but not with $\{p, \sim q\}$), and that, moreover, at the predicative level, there is no positive test for it. As is shown in [11], the dynamic proof theory warrants that, even for undecidable fragments, one obtains a sensible and rational estimate of which sentences are compatible with the Γ under consideration.

To make matters as transparent as possible, both logics are formulated in a modal way. Where $\Gamma^\Box$ stands for $\{\Box A \mid A \in \Gamma\}$, $\Gamma^\Box \vdash_{\mathsf{COM}^*} \Diamond A$ is taken to express that A is compatible with Γ, and $\Gamma^\Box \vdash_{\mathsf{COM}^*} \sim\Diamond A$ that A is incompatible with Γ. This is motivated by the fact that A is CL-compatible with Γ iff A is true in some CL-model of Γ, and hence, iff A is *possible* in view of Γ. As the members of Γ are true in *all* CL-models of Γ, it is easily observed that A is true in some CL-model of Γ iff $\Diamond A$ is true in some S5-model of $\Gamma^\Box$.

The results presented in [11] are formulated within the adaptive logics programme. The first adaptive logic was designed by Diderik Batens around 1980 (see [1]) and was meant to interpret inconsistent sets of premises 'as consistently as possible'. As we shall see below, one of the main strengths of adaptive logics is that they provide a unified framework for the formal study of consequence relations that are non-monotonic and/or dynamic[1]—see [7] and [8] for recent introductions to the topic.

The plot behind both logics is to assume that any formula A is compatible with Γ *unless* Γ explicitly prevents so—that is, unless A is incompatible with Γ or, what comes to the same, unless $\Gamma^{\Box} \vDash_{\mathsf{S5}} {\sim}\Diamond A$.

Semantically, this is realized by making a *selection* of the S5-models of $\Gamma^{\Box}$. Intuitively, those S5-models of $\Gamma^{\Box}$ are selected that verify a formula of the form ${\sim}\Diamond A$ iff it is 'unavoidable' in view of $\Gamma^{\Box}$ (that is, iff it is true in all S5-models of $\Gamma^{\Box}$). For example, some S5-models of $\{\Box p\}$ verify ${\sim}\Diamond q$ and others verify ${\sim}\Diamond{\sim}q$—this is the reason why neither $\Diamond q$ nor $\Diamond{\sim}q$ is an S5-consequence of $\{\Box p\}$. However, as neither ${\sim}\Diamond q$ nor ${\sim}\Diamond{\sim}q$ are unavoidable in view of $\Gamma^{\Box}$, the $\mathsf{COM}^{(*)}$-models of $\{\Box p\}$ falsify both, and hence, verify $\Diamond q$ as well as $\Diamond{\sim}q$.

Given some set of premises $\Gamma^{\Box}$, COM and COM^* select the same subset of S5-models. The only difference between the two logics concerns the way in which the semantic consequence relation is defined: A is a COM-consequence of $\Gamma^{\Box}$ iff all COM-models of $\Gamma^{\Box}$ verify A and a COM^*-consequence of $\Gamma^{\Box}$ iff $\Gamma^{\Box}$ *has* COM^*-models and all of them verify A.

For consistent sets of premises, both logics lead to adequate results. For instance, it is shown in [11] that the following equivalences hold:

if Γ is consistent, then

$$\Gamma^{\Box} \vdash_{\mathsf{COM}} \Diamond A \text{ iff } \Gamma^{\Box} \vdash_{\mathsf{COM}^*} \Diamond A \text{ iff } \Gamma^{\Box} \nvdash_{\mathsf{S5}} {\sim}\Diamond A \text{ iff } \Gamma \nvdash_{\mathsf{CL}} {\sim}A$$

which is exactly what we expect for classical compatibility. However, neither of them leads to adequate results for the inconsistent case: that everything is compatible with an inconsistent set of premises is just as unsatisfactory as that nothing is compatible with it.

Some readers may not be convinced here. To some, the question which sentences are compatible with an inconsistent set of premises may even seem nonsensical. After all, the natural response to an inconsistent theory seems to be that one looks for a consistent replacement *before* one starts wondering about its possible extensions. The important thing to remember, however, is that, both in the sciences and in everyday life, resolving some inconsistency may be far from evident. And, in lack of a consistent replacement, it is a better strategy to continue working with the inconsistent theory than to

[1] I say that a consequence relation is dynamic if the mere analysis of the premises may lead to the withdrawal of previously derived conclusions. Not all dynamic consequence relations are non-monotonic. In [6] it is shown, for instance, that the pure logic of relevant implication can be characterized by a dynamic proof theory.

prematurely revise it—see [24], [12], [20], [21], [14], and [19] for examples from the history of the sciences that illustrate this.

Both COM and COM* are inadequate to handle cases like this. For instance, even if one's beliefs turn out to be inconsistent in some respects, one will neither completely refrain from changing these beliefs nor adding anything what so ever to them. Analogously, when confronted with an inconsistent theory, one will neither stop asking questions nor start generating arbitrary ones—apart from the fact that an inconsistent theory may still lead to important open problems, asking the right kind of questions may be helpful in devising a consistent alternative for it. The upshot is that, even if one is convinced that (everything else being equal) a consistent theory is preferable to an inconsistent one, it may be important that one is able to distinguish between what is and what is not compatible with an inconsistent set of premises.

The aim of this paper is to generalize the results from [11] to the inconsistent case. The resulting logics will be called COMP^r and COMP^m. The difference between the two systems concerns the way in which they interpret inconsistent sets of premises—see below. I shall present the semantics as well as the (dynamic) proof theory for both logics, and prove soundness and completeness. As many of the properties and proofs are analogous for both systems, I shall use COMP^a as a generic name.

Like for the consistent case, the logics COMP^r and COMP^m will be formulated in modal terms, and $\Gamma^\Box \vdash_{\mathsf{COMP}^a} \Diamond A$ will be used to express that A is compatible with the (possibly inconsistent) set Γ.

An important constraint for the generalization is that a contradiction $A \wedge {\sim}A$ should only be considered as compatible with an inconsistent set of premises if A behaves inconsistently with respect to it. For instance, if $\Gamma = \{p, {\sim}p, q\}$, there is no reason to assume that, in addition to p, also q or, for instance, r behaves inconsistently. Hence, neither $q \wedge {\sim}q$ nor $r \wedge {\sim}r$ should be regarded as compatible with Γ (even if both r and ${\sim}r$ are).

Another important constraint is that, for the consistent case, both logics should lead to the same results as $\mathsf{COM}^{(*)}$. For reasons that will be explained in the next section, meeting this constraint requires that paraconsistent compatibility is defined, not with respect to the original theory, but with respect to an interpretation of it that is 'as consistent as possible'. Technically, this will be realized by making a combination of two adaptive logics: one that interprets possibly inconsistent theories as consistently as possible (that is, an inconsistency-adaptive logic) and one that defines the compatibility relation.

As we shall see below, there are different ways to interpret inconsistent sets of premises 'as consistently as possible' (even with respect to one and the same paraconsistent logic) and hence, different accounts of paraconsistent compatibility. COMP^r and COMP^m are meant to capture the accounts offered by the inconsistency-adaptive logics P^r and P^m.[2]

As COMP^r and COMP^m are defined with respect to sets of premises of the form $\Gamma^{\Box}$, their design requires the formulation of modal versions of P^r and P^m. In view of the similarity with ideas underlying $\mathsf{S5}$, these modal versions will be called $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$.

So, one of the logics incorporated in COMP^r (respectively, COMP^m) is $\mathsf{S5P}^r$ (respectively, $\mathsf{S5P}^m$). The other is a modal adaptive logic that defines the compatibility relation. This logic will be called COMP and is shared by both COMP^r and COMP^m.

I shall proceed as follows. After discussing the reasons why paraconsistent compatibility cannot be defined with respect to a (monotonic) paraconsistent logic, but only with respect to an inconsistency-adaptive logic (Section 2), I shall present a brief introduction to adaptive logics (Section 3). Next, I shall discuss the inconsistency-adaptive logics P^r and P^m (Sections 4 and 5) and elaborate their modal versions (Sections 6 and 7). The logics COMP^r and COMP^m will be presented in Sections 8 and 9. In Section 10, I shall compare the present account of paraconsistent compatibility with some alternatives. I shall end with some conclusions and open problems (Section 11).

2. *The Problem*

In order to generalize the results from [11] to the inconsistent case, we first need a clear idea of what it means that A is compatible with an inconsistent set of premises. In view of the present-day literature on paraconsistent logic, obtaining such an idea seems entirely straightforward.

Consider, for instance, $\Gamma = \{p, \sim p, q\}$. In that case, it seems natural that $\sim q$ is considered as *incompatible* with Γ, but that p is considered as compatible with it (despite the fact that, according to most paraconsistent logics, also $\sim p$ follows from Γ). What this seems to come to is that, on the one hand, the definition of compatibility should refer to a paraconsistent logic rather than to classical logic and, on the other hand, that it should be adequate to handle cases in which both A and $\sim A$ follow from the premises. So, where PL stands for some (standard) paraconsistent logic,[3] it seems that A

[2] The most complete presentation of P^r and P^m can be found in [3]. In many other papers on the subject, the logics are called $\mathsf{ACLuN1}$ and $\mathsf{ACLuN2}$.

[3] I say that a paraconsistent logic is *standard* if it defines a derivability relation that is monotonic, reflexive and transitive.

is (paraconsistently) compatible with a possibly inconsistent set of premises Γ iff A 'behaves inconsistently'[4] with respect to Γ or $\Gamma \nvdash_{\mathsf{PL}} \sim A$.

There are, however, at least three problems with this approach. The first is that, unlike for the consistent case, the definition cannot be matched by an intuitive semantic account. Whatever system one chooses for PL, it will not hold true that A is true in some PL-model of Γ iff A 'behaves inconsistently' with respect to Γ or $\Gamma \nvdash_{\mathsf{PL}} \sim A$. The reason is that, for any Γ that has PL-models, *any* sentence (that is non-contradictory and free of classical negation) is bound to be true in some of them.[5] For many paraconsistent logics, it even holds true that, for an arbitrary Γ, *any* sentence is true in some model of Γ. This is the case, for instance, for the logic P° (the full positive fragment of CL plus $A \vee \sim A$): some P°-models of $\{p\}$ verify both p and $\sim p$, others verify q as well as $\sim q$, and some even verify *all* sentences.

The second problem is that, for consistent sets of premises, the resulting concept of paraconsistent compatibility does not coincide with the concept of classical compatibility—which is the least one expects from a generalization. The reason is that all standard paraconsistent logics necessarily invalidate *Ex Falso Quodlibet* by invalidating some other inference rules of CL (for instance, either Disjunctive Syllogism or Addition).[6] Thus, if $\Gamma = \{p, \sim p \vee \sim q\}$ and PL invalidates Disjunctive Syllogism, $\Gamma \nvdash_{\mathsf{PL}} \sim q$, and hence, q would be PL-compatible (but not CL-compatible) with Γ; if PL invalidates Addition, $\Gamma \nvdash_{\mathsf{PL}} p \vee r$, and hence, $\sim(p \vee r)$ would be PL-compatible with it.

The third problem is that, even for inconsistent sets of premises, the resulting concept of paraconsistent compatibility classifies too many sentences as compatible with them. Also this is related to the fact that standard paraconsistent logics necessarily invalidate some inference rules of CL. Suppose, for instance, that $\Gamma = \{p, \sim p, p \vee \sim q, r \vee \sim s, \sim r\}$ and that PL invalidates Disjunctive Syllogism. In that case, $\Gamma \nvdash_{\mathsf{PL}} \sim q$ and $\Gamma \nvdash_{\mathsf{PL}} \sim s$, and hence,

[4] The precise meaning of this phrase varies from one paraconsistent logic to another. At this point, however, it is sufficient to grasp the intuitive idea. An exact definition of "behaving inconsistently" will be presented below.

[5] The language of many paraconsistent logics includes only a (weak) paraconsistent negation. In some cases, however, the language contains both a paraconsistent negation and the classical negation. This is the case, for instance, for the paraconsistent logic P that forms the basis for P^r and P^m and that is presented in Section 4. I use the name P° to refer to the fragment of P that is free of classical negation.

[6] This does not necessarily hold true for non-standard paraconsistent systems. In [9], it is shown, for instance, that it is possible to design a proof procedure that *only* invalidates *Ex Falso Quodlibet* (while validating all other inference rules of CL, including both Addition and Disjunctive Syllogism). The price to be paid, however, is that the resulting derivability relation is not transitive.

both q and s would be considered as PL-compatible with Γ. This, however, does not seem to be justified. If both p and $\sim p$ are true and "$\vee$" is interpreted classically, then $p \vee \sim q$ does not warrant that $\sim q$ is true, and hence, it seems reasonable that q is considered to be compatible with Γ. This, however, does not hold true for s. As there is no reason to suspect that r behaves inconsistently, there is no reason to suspect that $\sim s$ may be false. Or, put differently, in any interpretation of Γ that is 'as consistent as possible', r will behave consistently and $\sim s$ will be true. Hence, s should not be considered as compatible with Γ.

The observation in the last paragraph provides us at once with an intuitively attractive solution to each of the three problems: paraconsistent compatibility should be defined, not with respect to the original set of premises, but with respect to an interpretation of it that is *as consistent as possible*.

This is the line that will be pursued here: where APL is the inconsistency-adaptive logic based on some standard paraconsistent logic PL, A will be said to be compatible with a possibly inconsistent set of premises Γ iff A behaves inconsistently with respect to Γ or $\Gamma \nvdash_{\mathsf{APL}} \sim A$. As we shall see below, this nicely matches a semantic characterization: A is true in some APL-model of Γ iff A behaves inconsistently with respect to Γ or $\Gamma \nvdash_{\mathsf{APL}} \sim A$. It also warrants that, for the consistent case, the notion of paraconsistent compatibility coincides with that of classical compatibility.

As mentioned in the introduction, I shall define paraconsistent compatibility with respect to the inconsistency-adaptive logics P^r and P^m. These are not the only inconsistency-adaptive logics currently available (alternatives can be found in, for instance, [15] and [22]). They are chosen because they constitute paradigmatic cases, not only to understand the functioning of an inconsistency-adaptive logic, but also with respect to the meta-theory.

For some sets of premises, P^m leads to a richer consequence set than P^r (see below for an explanation). Consequently, the latter sometimes classifies *more* sentences as compatible with a set of premises Γ than the former. For instance, where $\Gamma = \{p, q, \sim p \vee \sim q, \sim p \vee r, \sim q \vee r, \}$, r is a P^m-consequence of Γ, but not a P^r-consequence of it. Hence, $\sim r$ is P^r-compatible with Γ, but not P^m-compatible with it. Which account of compatibility is best suited depends on the application context (see also Section 10).

3. *Some Basics of Adaptive Logics*

What all adaptive logics have in common is that they interpret sets of premises 'as normally as possible'. They differ from each other in the way that this phrase is specified. Inconsistency-adaptive logics, for instance, interpret sets of premises 'as consistently as possible'; ambiguity-adaptive logics interpret them 'as non-ambiguously as possible'.

Note that 'normality' is used here as a technical term. It does not refer to some standard of 'good reasoning' (CL, for instance), but to a set of presuppositions that are, in some application context, considered as desirable but defeasible. When engaging in a conversation, for instance, one of the presuppositions may be that the utterances of one's interlocutor are informative and relevant to the topic of the conversation. This presupposition is constitutive for the context at issue: it helps to define the conditions under which a conversation is considered as rational. It is, however, not an absolute rule of conduct: when proven false in a particular instance (for instance, some of the utterances cannot possibly be interpreted in a way that makes them relevant to the discussion), it will be abandoned.

So, relative to their application context, adaptive logics define some set of defeasible presuppositions and interpret sets of premises *as much as possible* in accordance with these. As an immediate consequence of this, adaptive logics share another important characteristic: the validity of some of their inference rules is *context-dependent*. For instance, the inconsistency-adaptive logics P^r and P^m allow that A is derived from $A \vee B$ and $\sim B$, but *only if* B behaves consistently with respect to the (background) premises. Thus, where $\Gamma = \{p, \sim p, p \vee q, \sim r, r \vee s\}$, both P^r and P^m validate the application of Disjunctive Syllogism to $\sim r$ and $r \vee s$, but invalidate the application of that same rule to $\sim p$ and $p \vee q$. The reason is that, if disjunction is interpreted classically, the justification of Disjunctive Syllogism is dependent on the consistency presupposition (if both B and $\sim B$ are true, then $A \vee B$ is true, even if A is false), and precisely this presupposition is defeasible in an inconsistency-adaptive logic. So, if the justification of an inference rule relies on a defeasible presupposition, it is dependent on the context (that is, the formulas to which one wants to apply the rule) whether its application is validated or not.[7]

Another way to put all this is that, in an adaptive logic, a specified set of formulas is assumed to be *false*, unless and until proven otherwise. Returning to the example from the previous paragraph, P^r and P^m validate the application of Disjunctive Syllogism to $\sim r$ and $r \vee s$ because they assume that $r \wedge \sim r$ is false; they invalidate the application of Disjunctive Syllogism to $\sim p$ and $p \vee q$ because the assumption that $p \wedge \sim p$ is false cannot be maintained in view of Γ.

[7] Because of the context-dependency of some of their inference rules, adaptive logics tend to be non-monotonic. There are, however, some exceptions. One of the examples is the adaptive logic presented in [5] to reconstruct the so-called "weak consequence relation" of Rescher and Manor (see [23]).

Formally, adaptive logics are characterized in terms of four elements: an upper limit logic, a lower limit logic, a set of abnormalities (usually denoted by Ω) and an adaptive strategy.

The upper limit logic and the lower limit logic are monotonic systems, and the former is always an extension of the latter. For instance, all currently available inconsistency-adaptive logics have CL as their upper limit logic and some paraconsistent fragment of CL (possibly extended with classical negation) as their lower limit logic. The upper limit logic thus incorporates not only the presuppositions of the lower limit logic, but also some additional ones. These are the defeasible presuppositions: they define the 'normal' situation, and are only abandoned in 'abnormal contexts'.

The set of abnormalities Ω consists of the formulas that are supposed to be false, unless and until proven otherwise. In all currently available adaptive logics, the abnormalities are delineated by a certain logical form. For instance, the set of abnormalities of an inconsistency-adaptive logic is characterized by the form $\exists(A \wedge {\sim}A)$, in which $\exists A$ abbreviates the existential closure of A.[8] Which members of Ω behave abnormally with respect to some set of premises Γ is determined by the lower limit logic. Thus, the phrase "unless and until proven otherwise" refers to the lower limit logic.

From what is said in the previous paragraphs, it may seem that abnormalities are assumed to be false, unless they are derivable (by the lower limit logic) from the set of premises. Although this holds true for some adaptive logics, the situation is usually a bit more complicated. This is related to the fact that, for most lower limit logics, a set of premises may entail a disjunction of abnormalities, without entailing any of its disjuncts. For instance, according to the paraconsistent logic P, $(p \wedge {\sim}p) \vee (q \wedge {\sim}q)$ is entailed by $\{p \vee q, {\sim}p, {\sim}q\}$, but $p \wedge {\sim}p$ and $q \wedge {\sim}q$ are not.

In line with the conventions from [7], disjunctions of abnormalities will be called *Dab-formulas* and an expression of the form $Dab(\Delta)$ will refer to $\bigvee(\Delta)$, in which Δ is a (finite) subset of the set of abnormalities. The Dab-formulas that are derivable by the lower limit logic from the set of premises Γ are called the Dab-consequences of Γ. $Dab(\Delta)$ is called a *minimal* Dab-consequence of Γ iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is a Dab-consequence of Γ. If $Dab(\Delta)$ is a minimal Dab-consequence of Γ, it can be inferred from Γ that some member of Δ behaves abnormally, but it cannot be inferred which one. Hence, except for the case where Δ is a singleton for every minimal Dab-consequence of Γ, there are different ways to interpret abnormal theories 'as normally as possible'.

[8] For some adaptive logics, the set of abnormalities does not comprise all formulas of the form at issue, but only those that satisfy some restriction. This will not hold true for the adaptive logics discussed in this paper.

It is in view of this fact that an *adaptive strategy* is needed. Intuitively, the adaptive strategy specifies what it means that the presuppositions of the upper limit logic are followed *as much as possible* or, what comes the same, that the members of the set of abnormalities are supposed to be false *unless and until proven otherwise*. The two basic strategies are the *Reliability* strategy and the *Minimal Abnormality* strategy.

According to the Reliability strategy, a formula is considered to behave abnormally with respect to a set of premises Γ iff it is a disjunct of some minimal Dab-consequence of Γ. The Minimal Abnormality strategy leads to a somewhat richer consequence set. This derives from the fact that it considers only the *minimal sets* of abnormally behaving formulas. For instance, where the minimal Dab-consequences of Γ are $Dab\{A, B\}$ and $Dab\{B, C\}$, the Minimal Abnormality strategy only considers situations in which the set of abnormally behaving formulas is either $\{B\}$ or $\{A, C\}$. The Reliability strategy also considers the situation in which the set of abnormally behaving formulas is $\{A, B, C\}$.

Whenever the lower limit logic warrants that, for any set of premises, every minimal Dab-consequence consists of a single abnormality, the Minimal Abnormality strategy and the Reliability strategy coincide. As we shall see below, this holds true for the logic COMP.

An important constraint in the design of an adaptive logic is that extending the lower limit logic with the requirement that the abnormalities are not logically possible should result in the upper limit logic (see [8] for a motivation). In view of this constraint, all currently available adaptive logics can be defined by specifying their lower limit logic, their adaptive strategy and their set of abnormalities.

Semantically, an adaptive logic is obtained by selecting, for each set of premises Γ, a subset of the models of the lower logic. Intuitively, those models are selected that are 'as normally as possible' in view of Γ. So, depending on the strategy, one may obtain a different set of models for Γ. (The Minimal Abnormality strategy selects in general a smaller set of models than the Reliability strategy.)

The proof theory of adaptive logics is dynamic in a strong sense: formulas that, at some stage of a proof, are considered as derived may at a later stage no longer be considered as such. This is related to the fact that it is allowed that inferences are made on the basis of one's best insights in the premises *at a certain stage*. For instance, if at a certain stage in a P^r-proof from Γ, both p and $\sim p \vee q$ are derived in it, it is allowed that q is added to it. If it turns out, however, that p behaves inconsistently with respect to Γ (for instance, because $p \wedge \sim p$ has *explicitly* been derived in the proof), then q will no longer be considered as derived. The precise mechanism by which this is realized will become clear in the subsequent sections.

One of the advantages of the dynamic proof theory is related to the fact that, in general, non-monotonic consequence relations are not only undecidable, but even lack a positive test. Where such absolute criteria are missing, the best one can hope for is a rational estimate of which sentences are derivable and which are not. As is shown in [2], the proof theory of adaptive logics not only provides such an estimate, but moreover warrants that, with each new step in the proof, this estimate becomes *better* (in a clear and measurable sense). The adequacy of the dynamic proof theory may also be seen from the fact that it can be proven sound and complete with respect to a (static) semantics.

4. *The Paraconsistent logic* P

In this section, I briefly discuss the logic P which constitutes the basis of the inconsistency-adaptive systems P^r and P^m.

The language $\mathcal{L}$ of P is as the standard predicative language for CL, except that it contains two symbols for negation: "$\sim$" and "$\neg$". The former stands for a paraconsistent negation, the latter is the standard negation of CL. The language also includes "$\bot$". In most applications that involve P, it is assumed that, in the proofs and the premises, negation is formalized by "$\sim$". In those cases, the main function of "$\neg$" is to simplify the meta-theoretic proofs. Below, we shall see, however, that the classical negation is useful in characterizing the logics COMP^r and COMP^m and that it leads to an elegant definition of paraconsistent compatibility.

The relation between the two negations is straightforward. Where $\mathsf{neg}A$ is used to refer to a formula of the form $\neg A$ or $\sim A$, "$\neg$" is semantically characterized by

(i) $v_M(\mathsf{neg}A) = 1$ if $v_M(A) = 0$ (negation-completeness), *and*
(ii) $v_M(\mathsf{neg}A) = 0$ if $v_M(A) = 1$ (consistency).

The meaning of the paraconsistent "$\sim$" is obtained by dropping (ii). Hence, $\neg A$ entails $\sim A$, but not *vice versa*.

Let $\mathcal{S}, \mathcal{C}, \mathcal{P}^r, \mathcal{F}$ and $\mathcal{W}$ stand for, respectively, the sets of sentential letters, individual constants, predicate letters of rank r, (open and closed) formulas, and wffs (closed formulas) of $\mathcal{L}$.

To simplify the semantic handling of quantifiers, the language $\mathcal{L}$ is extended to the pseudo-language $\mathcal{L}^+$ by introducing a set of pseudo-constants $\mathcal{O}$ that has at least the cardinality of the largest model one wants to consider. Let $\mathcal{W}^+$ denote the set of wffs of $\mathcal{L}^+$ (in which $\mathcal{C} \cup \mathcal{O}$ plays the role played by $\mathcal{C}$ in $\mathcal{L}$) and let ${}^{\sim}\mathcal{W}^+ = \{\sim A \mid \sim A \in \mathcal{W}^+\}$.

A P-model M is a couple $\langle D, v\rangle$, in which D is a set and v an assignment function. Every such model is an interpretation of $\mathcal{W}^+$, and hence of $\mathcal{W}$, which is what we are interested in. The assignment function v is defined by:

C1.1 $v : \mathcal{S} \longrightarrow \{0, 1\}$
C1.2 $v : \mathcal{C} \cup \mathcal{O} \longrightarrow D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C1.3 $v : \mathcal{P}^r \longrightarrow \wp(D^r)$ (the power set of the r-th Cartesian product of D)
C1.4 $v : {}^{\sim}\mathcal{W}^+ \longrightarrow \{0, 1\}$

The valuation function, $v_M : \mathcal{W}^+ \longrightarrow \{0, 1\}$, determined by M is defined by:

C2.1 where $A \in \mathcal{S}$, $v_M(A) = v(A)$
C2.2 $v_M(\bot) = 0$
C2.3 $v_M(\pi^r \alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C2.4 $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C2.5 $v_M({\sim}A) = 1$ iff $v_M(A) = 0$ or $v({\sim}A) = 1$
C2.6 $v_M(\neg A) = 1$ iff $v_M(A) = 0$
C2.7 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C2.8 $v_M((\forall\alpha)A(\alpha)) = 1$ iff $v_M(A(\beta)) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$.

The other logical constants are defined as usual. A is true in a P-model M (M verifies A) iff $v_M(A) = 1$; $\Gamma \vDash_{\mathsf{P}} A$ iff all P-models that verify all members of Γ also verify A; $\vDash_{\mathsf{P}} A$ (A is valid) iff it is verified by all P-models.

An axiomatization for P is obtained by extending the full positive fragment of CL by the following two axiom schemas and definition:

A1 $A \vee {\sim}A$
A2 $\bot \supset A$
D¬ $\neg A =_{df} A \supset \bot$.

I refer to [3] for the Soundness and Completeness proofs:

Theorem 1: $\Gamma \vdash_{\mathsf{P}} A$ *iff* $\Gamma \vDash_{\mathsf{P}} A$.

5. *The Inconsistency-Adaptive Logics* P^r *and* P^m

The lower limit logic of P^r and P^m is P and their upper limit logic is CL. The set of abnormalities Ω is the same for both logics: $\Omega = \{\exists(A \wedge {\sim}A) \mid A \in \mathcal{F}\}$. Given the set Ω, P^r and P^m are obtained from P by, respectively, the Reliability strategy and the Minimal Abnormality strategy.

For properties and definitions that are common to both logics, I shall use P^a as a generic name. Henceforth, $Dab(\Delta)$ will refer to the disjunction $\bigvee(\Delta)$, where $\Delta \subset \Omega$; Dab-formulas and (minimal) Dab-consequences are defined as in Section 3 (with respect to P).

To characterize the semantics of P^r and P^m, I first define the *abnormal part* of a P-model:

Definition 1: *For any* P*-model* M*,* $Ab(M) = \{A \in \Omega \mid v_M(A) = 1\}$.

The set of formulas that are unreliable with respect to Γ is defined by:

Definition 2: $U(\Gamma) = \bigcup\{\Delta \mid Dab(\Delta)$ *is a minimal* Dab*-consequence of* $\Gamma\}$.

For any set of premises Γ, two different kinds of P-models are distinguished:

Definition 3: *A* P*-model* M *of* Γ *is* reliable *iff* $Ab(M) \subseteq U(\Gamma)$.

Definition 4: *A* P*-model* M *of* Γ *is* minimally abnormal *iff there is no* P*-model* M' *of* Γ *such that* $Ab(M') \subset Ab(M)$.

Note that it does not make sense to say that a P-model is reliable (minimally abnormal), but only that it is a reliable (minimally abnormal) model *of* some set of premises Γ. Below, I shall sometimes use the term "P^r-model of Γ" (respectively, "P^m-model of Γ") to refer to a reliable (respectively, minimally abnormal) model of Γ.

The semantic consequence relations are defined with respect to the selected models:

Definition 5: $\Gamma \vDash_{\mathsf{P}^r} A$ *iff* A *is true in all reliable models of* Γ.

Definition 6: $\Gamma \vDash_{\mathsf{P}^m} A$ *iff* A *is true in all minimally abnormal models of* Γ.

As the language of P contains "$\neg$", not all sets of premises have P-models. It has been shown, however, that every set of premises that has P-models, also has P^a-models. It has also been shown that whenever a P-model M is not selected as a P^a-model of Γ, there is some other P-model M' that *is* selected and that is (in the set-theoretical sense) less abnormal.[9] I refer to [4] for the proofs:

Theorem 2: *If* M *is a* P*-model of* Γ *but not a* P^a*-model of* Γ*, then there is a* P^a*-model* M' *of* Γ *such that* $Ab(M') \subset Ab(M)$. (*Strong Reassurance*)

[9] As is shown in [4], adaptive logics that lack this property lead to counterintuitive results.

Corollary 1: *If* Γ *has* P*-models, it also has* P^a*-models.* (*Reassurance*)

In view of the subsequent sections, I also list two properties that relate the different types of adaptive models to one another as well as to the set of unreliable formulas. Their proofs can be found in [8]:

Theorem 3: *For every* $A \in U(\Gamma)$*, there is a minimally abnormal model of* Γ *that verifies* A*.*

Theorem 4: *Every minimally abnormal model of* Γ *is a reliable model of* Γ*.*

Before we turn to the proof theory, it is important to note that P^r and P^m, like all adaptive logics, cannot be characterized by a set of theorems. This is related to the fact that their models cannot be defined independently of a set of premises, and hence, that they do not have valid formulas of their own: $Cn_{\mathsf{P}^a}(\emptyset) = Cn_{\mathsf{P}}(\emptyset)$ and the intersection of $Cn_{\mathsf{P}^a}(\Gamma)$ for all Γ is $Cn_{\mathsf{CL}}(\emptyset)$.

However, as is shown in [8], their proof theory can be characterized by a simple format that is the same for all adaptive logics. The motor for the proof theory is provided by the following theorem—its proof can be found in [3]:

Theorem 5: $B_1, \ldots, B_n \vdash_{\mathsf{CL}} A$ *if and only if there is a finite* $\Delta \subseteq \Omega$ *such that* $B_1, \ldots, B_n \vdash_{\mathsf{P}} A \vee Dab(\Delta)$. (*Derivability Adjustment Theorem*)

Theorem 5 warrants that whenever A is CL-derivable from $B_1, \ldots, B_n$, A is P-derivable from $B_1, \ldots, B_n$ or certain formulas behave abnormally with respect to $B_1, \ldots, B_n$. This naturally suggests that, in the dynamic proofs, we derive A from $B_1, \ldots, B_n$, on the condition that no member of Δ behaves abnormally.

In line with this, the proof theory of both P^r and P^m is characterized by three rules (a premise rule PREM, an unconditional rule RU, and a conditional rule RC) and one marking definition. The rules are the same for both logics, the marking definition (which is determined by the adaptive strategy) is different.

The proofs themselves look like those of any other logic, except that every line has a *condition* attached to it. Thus, lines in a dynamic proof have five elements: (i) a line number, (ii) the formula A that is derived, (iii) the line numbers of the formulas from which A is derived, (iv) the rule by which A is derived, and (v) the condition. (An example of a dynamic proof can be found in Section 9.)

Here are the rules that govern P^a-proofs from Γ:

PREM If $A \in \Gamma$, then one may add to the proof a line consisting of

(i) the appropriate line number,
(ii) A,
(iii) "$-$",
(iv) "PREM", and
(v) $\emptyset$.

RU If $B_1, \ldots, B_n \vdash_{\mathsf{P}} A$ $(n \geq 0)$, and $B_1, \ldots, B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) A,
(iii) the line numbers of the B_i,
(iv) "RU", and
(v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $B_1, \ldots, B_n \vdash_{\mathsf{P}} A \vee Dab(\Delta)$ $(n \geq 0)$, and $B_1, \ldots, B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) A,
(iii) the line number of the B_i,
(iv) "RC", and
(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

Let us now turn to the marking definitions. A Dab-formula $Dab(\Delta)$ will be said to be a *minimal Dab-formula at stage s of a proof* iff, at that stage, $Dab(\Delta)$ occurs in the proof on the empty condition and, for any $\Delta' \subset \Delta$, $Dab(\Delta')$ does not occur in the proof on the empty condition.

The marking for P^r requires that we define, for any stage s of a P^r-proof from Γ, the set $U_s(\Gamma)$ of formulas that are *unreliable at that stage*:

Definition 7: $U_s(\Gamma) = \bigcup\{\Delta \mid Dab(\Delta)$ *is a minimal Dab-formula at stage s of the proof*$\}$.

Definition 8: *Marking for P^r: Line i is marked at stage s of a proof from Γ iff, where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

The marking definition for P^m is a bit more complicated.[10] Given a P^m-proof from Γ, we first define, for each stage s of the proof, the sets $\Phi^\circ_s(\Gamma)$ and $\Phi^\star_s(\Gamma)$:

Definition 9: $\Phi^\circ_s(\Gamma) = \{\phi \subset \Omega \mid \phi$ *contains one disjunct of each minimal* Dab*-formula at stage* s *of the proof*$\}$.

Definition 10: $\Phi^\star_s(\Gamma) = \{Cn_{\mathsf{P}}(\phi) \cap \Omega \mid \phi \in \Phi^\circ_s\}$.

Next, we define the set $\Phi_s(\Gamma)$:

Definition 11: $\Phi_s(\Gamma) = \{\phi \in \Phi^\star_s \mid$ *there is no* $\phi' \in \Phi^\star_s$ *such that* $\phi \supset \phi'\}$.

Finally, marking is defined with respect to the set $\Phi_s(\Gamma)$:

Definition 12: *Marking for* P^m*: Line* i *is marked at stage* s *of a proof from* Γ *iff, where* A *is derived on the condition* Δ *at line* i*, (i) there is no* $\phi \in \Phi_s(\Gamma)$ *such that* $\phi \cap \Delta = \emptyset$*, or (ii) for some* $\phi \in \Phi_s(\Gamma)$*, there is no line at which* A *is derived on a condition* Θ *for which* $\phi \cap \Theta = \emptyset$*.*

A formula is said to be *derived at stage s* in a P^a-proof from Γ iff A is the second element of a line that is not marked in the proof at that stage. In addition to this, a notion of *final derivability* is defined:

Definition 13: A *is finally derived on line* i *of a* P^a*-proof from* Γ *iff (i)* A *is the second element of line* i*, (ii) line* i *is not marked in the proof, and (iii) any extension of the proof in which line* i *is marked may be further extended in such a way that line* i *is unmarked.*

It has been shown that, if A is finally derivable from Γ, then *any* P^a-proof from Γ may be extended in such a way that A is finally derived in it (see [3]). This warrants that the dynamics of the proofs is sensible: 'in the end', different dynamic proofs lead to the same set of finally derived conclusions.

Definition 14: $\Gamma \vdash_{\mathsf{P}^a} A$ $(A$ *is finally* P^a*-derivable from* $\Gamma)$ *iff* A *is finally derived on some line of a* P^a*-proof from* Γ.

I refer to [3] for the Soundness and Completeness proofs:

[10] I refer to [7] for an intuitive account of this definition and for some more explanation. The marking for minimal abnormality will also be illustrated in Section 9.

Theorem 6: $\Gamma \vdash_{\mathsf{P}^a} A$ *iff* $\Gamma \vDash_{\mathsf{P}^a} A$.

6. *The Lower and Upper Limit of* $\mathsf{S5P}^r$ *and* $\mathsf{S5P}^m$

As explained in Section 1, the logics COMP^r and COMP^m are defined in modal terms. This is why we need modal versions of the inconsistency-adaptive logics P^r and P^m. These modal versions are presented in the next section. In this section, I discuss their lower and upper limit logic.

The lower limit logic of both systems is S5P which is a modal version of the paraconsistent logic P. The relation between S5P and P is as that between S5 and CL. The semantics for S5P is obtained in a straightforward way from the one for P.[11]

Let $\mathcal{L}^M$ be the standard modal language (including $\bot$) and let $\mathcal{L}^{M+}$ be obtained from $\mathcal{L}^M$ in the same way as $\mathcal{L}^+$ is obtained from $\mathcal{L}$. Let $\mathcal{W}^{p+}$ be the set of primitive wffs of $\mathcal{L}^+$ (those wffs of $\mathcal{L}^+$ that contain no other logical constants than identity). Unless explicitly stated otherwise, the letter Γ will refer to sets of wffs of the *non-modal* language $\mathcal{L}$.

An S5P-model is a couple $\mathcal{M} = \langle \Sigma, M_0 \rangle$ in which Σ is a *set* of P-models and $M_0 \in \Sigma$. The valuation determined by an S5P-model $\mathcal{M}$ is defined by the following clauses:

MC1 where $A \in \mathcal{W}^{p+}$ or $A \in {}^{\sim}\mathcal{W}^{+}$, $v_{\mathcal{M}}(A, M_i) = v_{M_i}(A)$
MC2 $v_{\mathcal{M}}(\neg A, M_i) = 1$ iff $v_{\mathcal{M}}(A, M_i) = 0$
MC3 $v_{\mathcal{M}}(\sim\Box A, M_i) = 1$ iff $v_{\mathcal{M}}(\neg\Box A, M_i) = 1$
MC4 $v_{\mathcal{M}}(\sim\Diamond A, M_i) = 1$ iff $v_{\mathcal{M}}(\neg\Diamond A, M_i) = 1$
MC5 $v_{\mathcal{M}}(A \wedge B, M_i) = 1$ iff $v_{\mathcal{M}}(A, M_i) = v_{\mathcal{M}}(B, M_i) = 1$
MC6 $v_{\mathcal{M}}((\forall\alpha)A(\alpha), M_i) = 1$ iff $v_{\mathcal{M}}(A(\beta), M_i) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$
MC7 $v_{\mathcal{M}}(\Box A, M_i) = 1$ iff $v_{\mathcal{M}}(A, M_j) = 1$ for all $M_j \in \Sigma$.

The other logical constants are defined as usual. Semantic consequence and validity are as usual (in terms of truth in M_0).

In view of the above semantics, the following theorem is easily proved:

Theorem 7: *Where* $A \in \mathcal{W}$, $\Gamma^{\Box} \vDash_{\mathsf{S5P}} \Box A$ *iff* $\Gamma \vDash_{\mathsf{P}} A$.

Proof. For the left-right direction, suppose that $\Gamma \nvDash_{\mathsf{P}} A$. In that case, there is a P-model M of Γ that falsifies A. Let $\mathcal{M} = \langle \{M\}, M \rangle$. It is easily seen that $\mathcal{M}$ is an S5P-model of $\Gamma^{\Box}$ that falsifies $\Box A$. But then, $\Gamma^{\Box} \nvDash_{\mathsf{S5P}} \Box A$.

[11] The underlying idea is similar to the one used in [11] to design a new semantics for S5—see also below.

For the right-left direction, suppose that $\Gamma^{\Box} \nvDash_{\mathsf{S5P}} \Box A$. In that case, there is an S5P-model $\mathcal{M} = \langle \Sigma, M_0 \rangle$ such that, for some $M \in \Sigma$, M is a P-model of Γ and M falsifies A. Hence, $\Gamma \nvDash_{\mathsf{P}} A$. □

An axiomatization for S5P is obtained by adding the following to P:

MA1 $\Box A \supset A$
MA2 $\Box(A \supset B) \supset (\Box A \supset \Box B)$
MA3 $\Diamond A \supset \Box\Diamond A$
MA4 $\sim\Box A \supset \neg\Box A$
MA5 $\sim\Diamond A \supset \neg\Diamond A$
NEC If $\vdash_{\mathsf{P}} A$ then $\vdash_{\mathsf{S5P}} \Box A$
D◇ $\Diamond A =_{df} \neg\Box\neg A$.

The easy Soundness proof is left to the reader:

Theorem 8: *If* $\Gamma \vdash_{\mathsf{S5P}} A$*, then* $\Gamma \vDash_{\mathsf{S5P}} A$*.*

The Completeness proof proceeds as the one presented in [11] for S5:

Theorem 9: *If* $\Gamma \vDash_{\mathsf{S5P}} A$*, then* $\Gamma \vdash_{\mathsf{S5P}} A$*.*

Proof. Suppose that $\Gamma \nvdash_{\mathsf{S5P}} A$ (where Γ is a set of wffs of $\mathcal{L}^M$). Consider a denumerable $\mathcal{O}' \subset \mathcal{O}$ and let $\mathcal{L}^{M'}$ be obtained from $\mathcal{L}^M$ by extending $\mathcal{C}$ to $\mathcal{C} \cup \mathcal{O}'$. Let $B_1, B_2, \ldots$ be a list of all wffs of $\mathcal{L}^{M'}$ such that, if $B_i = (\exists\alpha)C(\alpha)$ then $B_{i+1} = C(\beta)$ for some $\beta \in \mathcal{O}'$ that does not occur in $B_1, \ldots, B_i$. Define:

$$\begin{aligned} \Delta_0 &= Cn_{\mathsf{S5P}}(\Gamma) \\ \Delta_{i+1} &= Cn_{\mathsf{S5P}}(\Delta_i \cup \{B_{i+1}\}) \text{ if } A \notin Cn_{\mathsf{S5P}}(\Delta_i \cup \{B_{i+1}\}), and \\ \Delta_{i+1} &= \Delta_i \text{ otherwise} \\ \Delta &= \Delta_0 \cup \Delta_1 \cup \ldots \end{aligned}$$

Consider a function $f : \mathcal{C} \cup \mathcal{O} \longrightarrow \mathcal{C} \cup \mathcal{O}'$ such that $f(\alpha) = \alpha$ for all $\alpha \in \mathcal{C} \cup \mathcal{O}'$, and extend it to wffs by defining $f(A)$ as the result of replacing in A any $\alpha \in \mathcal{C} \cup \mathcal{O}$ by $f(\alpha)$. Finally, define $\Delta^\star = \{A \mid f(A) \in \Delta\}$. It is easily seen that $\Delta \subset \Delta^\star$ and that $\Delta^\star$ is closed under S5P-derivability.

Define $\Theta = \{A \mid A$ is a wff of $\mathcal{L}^+; \Box A \in \Delta^\star\}$, and $\Lambda = \{A \mid A$ is a wff of $\mathcal{L}^+; A \in \Delta^\star\}$. It is easily observed that $\Theta \subseteq \Lambda$ (if $\Box C \in \Delta^\star$, then $C \in \Delta^\star$), and that Λ defines a unique P-model (for any wffs A and B of $\mathcal{L}^+$, if $A \notin \Lambda$, then $\sim A \in \Lambda$; $\neg A \in \Lambda$ iff $A \notin \Lambda$; $A \wedge B \in \Lambda$ iff $A, B \in \Lambda$;

...; $(\exists\alpha)A(\alpha) \in \Lambda$ iff $A(\beta) \in \Lambda$ for some $\beta \in \mathcal{C} \cup \mathcal{O}$). Let M_0 be the P-model defined by Λ, Σ the set of *all* P-models of Θ, and $\mathcal{M} = \langle \Sigma, M_0 \rangle$.

Each of the following is easily proved:

(i) $\mathcal{M}$ is an S5P-model (as $\Theta \subseteq \Lambda$, $M_0 \in \Sigma$),
(ii) $\mathcal{M}$ is an S5P-model of $\Delta^\star$ and hence of Δ,
(iii) $\mathcal{M}$ verifies Γ,
(iv) $\mathcal{M}$ falsifies A,

Hence, $\Gamma \nvDash_{\mathsf{S5P}} A$ as desired. □

Corollary 2: $\Gamma \vdash_{\mathsf{S5P}} A$ *iff* $\Gamma \vDash_{\mathsf{S5P}} A$.

Let us now turn to S5 which is the upper limit logic of $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$. The semantics of S5 can be obtained by selecting a subset of the S5P-models:

Definition 15: *An* S5P*-model* $\mathcal{M} = \langle \Sigma, M_0 \rangle$ *is an* S5*-model iff, for every* $M = \langle D, v \rangle \in \Sigma$, $v(\sim A) = 1$ *iff* $v_M(A) = 0$.

It is easily observed that, for every thus defined S5-model, Σ consists of a set of CL-models (the language contains two negations, but Definition 15 warrants that they have the same meaning). Henceforth, "S5-model" will always refer to a model as defined here.

The proof of the following theorem is analogous to that of Theorem 7:

Theorem 10: *Where* $A \in \mathcal{W}$, $\Gamma^{\Box} \vDash_{\mathsf{S5}} \Box A$ *iff* $\Gamma \vDash_{\mathsf{CL}} A$.

The above S5-semantics is somewhat peculiar. Nevertheless, it is shown in [11] that the semantics is sound and complete with respect to the predicative version of S5 with the Barcan formula:

Theorem 11: $\Gamma \vdash_{\mathsf{S5}} A$ *iff* $\Gamma \vDash_{\mathsf{S5}} A$.

7. *The Modal Adaptive Logics* $\mathsf{S5P}^r$ *and* $\mathsf{S5P}^m$

In view of their intended application, $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$ will only be defined for sets of premises of the form $\Gamma^{\Box}$. I shall use $\mathsf{S5P}^a$ as a generic name.

The semantics of both logics is obtained by making a selection of the S5P-models of $\Gamma^{\Box}$ and by defining their semantic consequence relation with respect to the selected models. In view of the format of the S5P-models, the selection criterion can be defined in a very simple and intuitive way:

Definition 16: *An* $\mathsf{S5P}$*-model* $\mathcal{M} = \langle \Sigma, M_0 \rangle$ *is an* $\mathsf{S5P}^a$*-model of* $\Gamma^\Box$ *iff, for every* $M \in \Sigma$, M *is a* P^a*-model of* Γ.

Definition 17: $\Gamma^\Box \vDash_{\mathsf{S5P}^a} A$ *iff* A *is true in all* $\mathsf{S5P}^a$*-models of* $\Gamma^\Box$.

As for P^r and P^m, the dynamic proof theory of $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$ is based on a specific relation between derivability by the upper limit logic $\mathsf{S5}$, and derivability by the lower limit logic $\mathsf{S5P}$. The proof is immediate in view of Theorems 10, 5 and 7:

Theorem 12: $\Gamma^\Box \vdash_{\mathsf{S5}} \Box A$ *iff there is some* $\Delta \subset \Omega$ *such that* $\Gamma^\Box \vdash_{\mathsf{S5P}} \Box(A \vee Dab(\Delta))$.

The proof format is as that for P^a and the inference rules are analogous:

PREM If $A \in \Gamma^\Box$, then one may add to the proof a line consisting of

(i) the appropriate line number,
(ii) A,
(iii) "–",
(iv) "PREM", and
(v) $\emptyset$.

NRU If $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} A$ $(n \geq 0)$, and $\Box B_1, \ldots, \Box B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) A,
(iii) the line numbers of the $\Box B_i$,
(iv) "NRU", and
(v) $\Delta_1 \cup \ldots \cup \Delta_n$.

NRC If $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} \Box(A \vee Dab(\Delta))$ $(n \geq 0)$, and $\Box B_1, \ldots, \Box B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) $\Box A$,
(iii) the line number of the $\Box B_i$,
(iv) "NRC", and
(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

A formula of the form $\Box(Dab(\Delta))$ will be called an $NDab$-formula. An $NDab$-formula $\Box(Dab(\Delta))$ will be said to be a *minimal NDab-formula at stage s of a proof* iff, at that stage, $\Box(Dab(\Delta))$ occurs in the proof on the empty condition and, for any $\Delta' \subset \Delta$, $\Box(Dab(\Delta'))$ does not occur in the proof on the empty condition. Let $U(\Gamma^{\Box})$ be the union of all Δ such that $\Box(Dab(\Delta))$ is a minimal $NDab$-consequence of $\Gamma^{\Box}$.

In view of the marking for $\mathsf{S5P}^r$, I define the set $U_s(\Gamma^{\Box})$:

Definition 18: $U_s(\Gamma^{\Box}) = \bigcup\{\Delta \mid \Box(Dab(\Delta))$ *is a minimal* $NDab$*-formula at stage* s *of the proof*$\}$.

Definition 19: *Marking for* $\mathsf{S5P}^r$*: Line* i *is* r*-marked at stage* s *of a proof from* $\Gamma^{\Box}$ *iff, where* Δ *is its condition,* $\Delta \cap U_s(\Gamma^{\Box}) \neq \emptyset$*.*

The definition of the set $\Phi_s(\Gamma^{\Box})$ is analogous to that of $\Phi_s(\Gamma)$ and proceeds in terms of the $NDab$-consequences and the logic $\mathsf{S5P}$:

Definition 20: $\Phi^{\circ}_s(\Gamma^{\Box}) = \{\phi \subset \Omega \mid \phi$ *contains one disjunct of each minimal* $NDab$*-formula at stage* s *of the proof*$\}$.

Definition 21: $\Phi^{\star}_s(\Gamma^{\Box}) = \{Cn_{\mathsf{S5P}}(\phi) \cap \Omega \mid \phi \in \Phi^{\circ}_s\}$.

Definition 22: $\Phi_s(\Gamma^{\Box}) = \{\phi \in \Phi^{\star}_s \mid$ *there is no* $\phi' \in \Phi^{\star}_s$ *such that* $\phi \supset \phi'\}$.

Definition 23: *Marking for* $\mathsf{S5P}^m$*: Line* i *is* m*-marked at stage* s *of a proof from* $\Gamma^{\Box}$ *iff, where* A *is derived on the condition* Δ *at line* i*, (i) there is no* $\phi \in \Phi_s(\Gamma^{\Box})$ *such that* $\phi \cap \Delta = \emptyset$*, or (ii) for some* $\phi \in \Phi_s(\Gamma^{\Box})$*, there is no line at which* A *is derived on a condition* Θ *for which* $\phi \cap \Theta = \emptyset$*.*

As for P^a, a formula is said to be derived at a stage s of an $\mathsf{S5P}^a$-proof from $\Gamma^{\Box}$ iff A is the second element of a line that is not marked in the proof at that stage. Also the definition of final derivability is analogous:

Definition 24: A *is finally derived on line* i *of an* $\mathsf{S5P}^a$*-proof from* $\Gamma^{\Box}$ *iff (i)* A *is the second element of line* i*, (ii) line* i *is not marked in the proof, and (iii) any extension of the proof in which line* i *is marked may be further extended in such a way that line* i *is unmarked.*

Definition 25: $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} A$ *(*A *is finally* $\mathsf{S5P}^a$*-derivable from* $\Gamma^{\Box}$*) iff* A *is finally derived on some line of an* $\mathsf{S5P}^a$*-proof from* $\Gamma^{\Box}$*.*

The following theorems relate the consequence relation and the derivability relation of $\mathsf{S5P}^a$ to those of P^a. They will be used to prove (in an indirect way) Soundness and Completeness for $\mathsf{S5P}^a$.

Theorem 13: *Where* $A \in \mathcal{W}$, $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \Box A$ *iff* $\Gamma \vDash_{\mathsf{P}^a} A$.

Proof. In view of Definition 16, the proof is completely analogous to that of Theorem 7 (any occurrences of $\mathsf{S5P}$ and P can be replaced by $\mathsf{S5P}^a$ and P^a, respectively). □

Theorem 14: *Where* $A \in \mathcal{W}$, $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} \Box A$ *iff* $\Gamma \vdash_{\mathsf{P}^a} A$.

Proof. For the first direction, suppose that the antecedent holds true. It follows, in view of Definition 25 and the form of the members of $\Gamma^{\Box} \cup \{\Box A\}$, that there is an $\mathsf{S5P}^a$-proof from $\Gamma^{\Box}$ in which (i) $\Box A$ is finally derived on some line i, and (ii) the second element of each line contains a formula of the form $\Box B$ (for $B \in \mathcal{W}$). Transform this proof by deleting every modal operator that occurs in it. In view of Theorem 7 and the Completeness of $\mathsf{S5P}^a$ and P^a, each of the following is easily proved:

(i) every line in the transformed proof is obtained by the application of an inference rule of P^a (by an obvious induction on the length of the proof),
(ii) for any j, Δ is the condition of line j in the original proof iff Δ is the condition of line j in the transformed proof,
(iii) for any s, $B \in U_s(\Gamma^{\Box})$ iff $B \in U_s(\Gamma)$,
(iv) for any s, $B \in \Phi_s(\Gamma^{\Box})$ iff $B \in \Phi_s(\Gamma)$,
(v) the transformed proof is a P^a-proof from Γ,
(vi) A occurs on line i of the transformed proof,
(vii) line i is not marked in the transformed proof,
(viii) any extension of the transformed proof in which line i is marked may be further extended such that i is unmarked.

It follows that there is a P^a-proof from Γ in which A is finally derived, and hence, that $\Gamma \vdash_{\mathsf{P}^a} A$.

The proof for the other direction proceeds by an analogous procedure. □

Given the Soundness and Completeness of P^a (see Theorem 6), the last two theorems immediately entail a restricted form of Soundness and Completeness for $\mathsf{S5P}^a$:

Theorem 15: *Where* $A \in \mathcal{W}$, $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \Box A$ *iff* $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} \Box A$.

In view of the design of COMP^a, I also prove a stronger form of Soundness and Completeness (for conclusions of any logical form). Let $\Gamma^{\star}$ stand for the set $\{\Box A \mid A \in \mathcal{W}; \Gamma^{\Box} \vdash_{\mathsf{S5P}^a} \Box A\}$.

The proof for the first lemma is immediate in view of Theorem 15 and Corollary 2:

Lemma 1: $\Gamma^\star \vDash_{\mathsf{S5P}} A$ *iff* $\Gamma^\star \vdash_{\mathsf{S5P}} A$.

Lemma 2: $\Gamma^\Box \vDash_{\mathsf{S5P}^a} A$ *iff* $\Gamma^\star \vDash_{\mathsf{S5P}} A$.

Proof. For the first direction, suppose that $\Gamma^\Box \vDash_{\mathsf{S5P}^a} A$. It follows that A is verified in all $\mathsf{S5P}^a$-models of $\Gamma^\Box$, and hence, in all P^a-models of Γ. But then, A is verified in all $\mathsf{S5P}$-models that verify $\Box B$ for every $B \in \{A \mid \Gamma \vDash_{\mathsf{P}^a} A\}$ or, what comes to the same (in view of Theorem 13), for every $\Box B \in \Gamma^\star$. Hence, $\Gamma^\star \vDash_{\mathsf{S5P}} A$.

For the other direction, suppose that $\Gamma^\Box \nvDash_{\mathsf{S5P}^a} A$. It follows that there is an $\mathsf{S5P}^a$-model $\mathcal{M}$ of $\Gamma^\Box$ such that $\mathcal{M}$ verifies every member of $\Gamma^\star$ but falsifies A. As all $\mathsf{S5P}^a$-models of $\Gamma^\Box$ are $\mathsf{S5P}$-models, it follows that $\Gamma^\star \nvDash_{\mathsf{S5P}} A$. □

Lemma 3: $\Gamma^\Box \vdash_{\mathsf{S5P}^a} A$ *iff* $\Gamma^\star \vdash_{\mathsf{S5P}} A$.

Proof. If A is of the form $\Box B$, the left-right direction obviously obtains. So, suppose that A is of some other form and that $\Gamma^\Box \vdash_{\mathsf{S5P}^a} A$. It follows that A is finally derived on some line i in an $\mathsf{S5P}^a$-proof from $\Gamma^\Box$. By an inspection of the inference rules, it also follows that i is written down by the application of NRU. But then, there are $\Box B_1, \ldots, \Box B_n$ such that $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} A$, and $\Box B_1, \ldots, \Box B_n$ are finally derived in the proof. Hence, in view of the monotonicity of $\mathsf{S5P}$, $\Gamma^\star \vdash_{\mathsf{S5P}} A$.

For the right-left direction, suppose that $\Gamma^\star \vdash_{\mathsf{S5P}} A$. In view of the Compactness of $\mathsf{S5P}$, it follows that there are $\Box B_1, \ldots, \Box B_n$ such that every $\Box B_i$ is $\mathsf{S5P}^a$-derivable from $\Gamma^\Box$ and $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} A$. Hence, there is an $\mathsf{S5P}^a$-proof from $\Gamma^\Box$ in which $\Box B_1, \ldots, \Box B_n$ are finally derived and in which A is added by the rule NRU. It follows that A is finally derived in the proof, and hence, that $\Gamma^\Box \vdash_{\mathsf{S5P}^a} A$. □

Theorem 16: $\Gamma^\Box \vDash_{\mathsf{S5P}^a} A$ *iff* $\Gamma^\Box \vdash_{\mathsf{S5P}^a} A$.

Proof. Immediate in view of Lemmas 1, 2 and 3. □

8. *The Semantics of* COMP^a

As explained in Section 2, the logics COMP^r and COMP^m are meant to capture the idea that A is compatible with a possibly inconsistent set of premises Γ iff A is true in *some* interpretation of Γ that is *as consistent as possible*. So, given the inconsistency-adaptive logics that were chosen to interpret this phrase, $\Diamond A$ should be a COMP^a-consequence of $\Gamma^\Box$ iff A is true in some P^a-model of Γ. This may be realized by making the following selection:

Definition 26: *An* $\mathsf{S5P}$*-model* $\mathcal{M} = \langle \Sigma, M_0 \rangle$ *is a* COMP^a*-model of* $\Gamma^\Box$ *iff* Σ *is the set of* all P^a*-models of* Γ.

If Γ has several P^a-models, then $\Gamma^\Box$ has several COMP^a-models (they differ from each other in the P^a-model that is chosen as M_0). It is easily observed, however, that the following property obtains for all *fully modal wffs* of $\mathcal{L}^M$ (all wffs of the form $\Box A$ and $\Diamond A$ and all wffs obtained from these by the formation rules of $\mathcal{L}$):

Theorem 17: *If* A *is a fully modal wff of* $\mathcal{L}^M$*, then* A *is verified by* some COMP^a*-model of* $\Gamma^\Box$ *iff* A *is verified by* all COMP^a*-models of* $\Gamma^\Box$.

The semantic consequence relation is defined in the usual way:

Definition 27: $\Gamma^\Box \vDash_{\mathsf{COMP}^a} A$ *iff all* COMP^a*-models of* $\Gamma^\Box$ *verify* A.

Given the semantics of $\mathsf{S5P}$, Definition 26 warrants that $\Diamond A$ is verified by a COMP^a-model of $\Gamma^\Box$ iff A is verified by *some* P^a-model of Γ. Hence, in view of Definition 27, we have:

Theorem 18: *Where* $A \in \mathcal{W}$, $\Gamma^\Box \vDash_{\mathsf{COMP}^a} \Diamond A$ *iff* A *is verified by* some P^a*-model of* Γ.

which is exactly what we want.

The proof of the following theorem proceeds by an obvious inspection of the different semantics:

Theorem 19: $\Gamma^\Box \vDash_{\mathsf{COMP}^a} \Box A_1 \vee \ldots \vee \Box A_n$ *iff* $\Gamma^\Box \vDash_{\mathsf{S5P}^a} \Box A_1 \vee \ldots \vee \Box A_n$ (*for* $n \geq 1$).

The following two theorems lead to definitions of paraconsistent compatibility (in terms of the classical negation and the paraconsistent negation, respectively):

Theorem 20: *For every* Γ *that has* P*-models and every* $A \in \mathcal{W}$, $\Gamma^\Box \vDash_{\mathsf{COMP}^a} \Diamond A$ *iff* $\Gamma^\Box \nvDash_{\mathsf{S5P}^a} \Box\neg A$ *iff* $\Gamma \nvDash_{\mathsf{P}^a} \neg A$.

Proof. The first equivalence is immediate in view of Theorem 19 and the fact that, whenever Γ has P-models, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ iff $\Gamma^{\Box} \nvDash_{\mathsf{COMP}^a} \Box\neg A$. The second follows from Theorem 13. $\Box$

Theorem 21: *For every Γ that has* P*-models and every $A \in \mathcal{W}$, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ iff $[A \wedge {\sim}A \in U(\Gamma^{\Box})$ or $\Gamma^{\Box} \nvDash_{\mathsf{S5P}^a} \Box{\sim}A)]$ iff $[A \wedge {\sim}A \in U(\Gamma)$ or $\Gamma \nvDash_{\mathsf{P}^a} {\sim}A]$.*

Proof. In view of Theorem 13 and the definitions of $U(\Gamma)$ and $U(\Gamma^{\Box})$, it suffices to show that $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ iff $A \wedge {\sim}A \in U(\Gamma)$ or $\Gamma \nvDash_{\mathsf{P}^a} {\sim}A$.

Suppose first that $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ (for $A \in \mathcal{W}$). It follows (by Theorem 18) that A is true in some P^a-model M of Γ. Hence, in view of the P^a-semantics, $Dab(\Delta \cup \{A \wedge {\sim}A\})$ is a minimal Dab-consequence of Γ (for some $\Delta \subset \Omega$) or M falsifies ${\sim}A$. In the former case, $A \wedge {\sim}A \in U(\Gamma)$; in the latter, $\Gamma \nvDash_{\mathsf{P}^a} {\sim}A$.

Suppose next that $A \wedge {\sim}A \in U(\Gamma)$. In view of Theorems 3 and 4, A is true in some P^m-model of Γ, and hence, in some P^r-model of Γ. But then, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$.

Suppose finally that $\Gamma \nvDash_{\mathsf{P}^a} {\sim}A$. It again follows that A is true in some P^a-model of Γ (in view of the fact that every P-model verifies A or ${\sim}A$). $\Box$

In Section 2, I explained that A should be considered as compatible with a possibly inconsistent Γ iff A 'behaves inconsistently' with respect to Γ or $\Gamma \nvdash {\sim}A$. The last theorem leads to an exact definition of this idea: A is compatible with Γ iff $A \wedge {\sim}A \in U(\Gamma)$ or $\Gamma \nvdash_{\mathsf{P}^a} {\sim}A$.

To see that this definition leads to adequate results for both P^r and P^m, consider, for instance, $\Gamma = \{p, q, {\sim}p \vee {\sim}q\}$. In that case, $p \wedge {\sim}p$ as well as $q \wedge {\sim}q$ are members of $U(\Gamma)$, and hence, are classified as compatible with Γ (both according to P^r and P^m). This is as it should be: $p \wedge {\sim}p$ is true in some P^r-model of Γ as well as in some P^m-model of Γ (analogously for $q \wedge {\sim}q$). The difference between the two systems shows in the fact that $\Gamma \nvdash_{\mathsf{P}^r} {\sim}((p \wedge {\sim}p) \wedge (q \wedge {\sim}q))$, but $\Gamma \vdash_{\mathsf{P}^m} {\sim}((p \wedge {\sim}p) \wedge (q \wedge {\sim}q))$. As $(p \wedge {\sim}p) \wedge (q \wedge {\sim}q) \wedge {\sim}((p \wedge {\sim}p) \wedge (q \wedge {\sim}q)) \notin U(\Gamma)$, it follows that $(p \wedge {\sim}p) \wedge (q \wedge {\sim}q)$ is compatible with Γ according to P^r, but *not* according to P^m. However, also this is as it should be: $(p \wedge {\sim}p) \wedge (q \wedge {\sim}q)$ is true in some P^r-models of Γ, but is false in all of its P^m-models.

Where $\Omega^{\Box} = \{\Box A \mid A \in \mathcal{W}\}$, the following theorem relates the consequence relation of COMP^a to that of $\mathsf{S5P}^a$:

Theorem 22: *If Γ has* P*-models, then $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ iff, for some finite $\Delta \subset \Omega^{\Box}$, $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \bigvee(\{\Diamond A\} \cup \Delta)$ and $\Gamma^{\Box} \nvDash_{\mathsf{S5P}^a} \bigvee(\Delta)$.*

Proof. Suppose first that $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$. It follows that $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \Diamond A \vee \Box\neg A$ (in view of the fact that $\Diamond A \vee \Box\neg A$ is valid in S5P) and that $\Gamma^{\Box} \nvDash_{\mathsf{S5P}^a} \Box\neg A$ (by Theorem 20).

Suppose next that, for some finite $\Delta \subset \Omega^{\Box}$, $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \bigvee(\{\Diamond A\} \cup \Delta)$ and $\Gamma^{\Box} \nvDash_{\mathsf{S5P}^a} \bigvee(\Delta)$. As the COMP^a-models of $\Gamma^{\Box}$ are a subset of its $\mathsf{S5P}^a$-models, it follows that $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \bigvee(\{\Diamond A\} \cup \Delta)$. In view of Theorem 19, it also follows that $\Gamma^{\Box} \nvDash_{\mathsf{COMP}^a} \bigvee(\Delta)$. But then, as all COMP^a-models of $\Gamma^{\Box}$ verify $\bigvee(\{\Diamond A\} \cup \Delta)$ and some of them falsify $\bigvee(\Delta)$, some COMP^a-models of $\Gamma^{\Box}$ verify $\Diamond A$. Hence, in view of Theorem 17, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$. □

In view of the Soundness and Completeness of $\mathsf{S5P}^a$, Theorem 22 seems to provide an obvious basis for the proof theory of COMP^a. The (conditional) rule it suggests is this: if $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}^a} \bigvee(\{\Diamond A\} \cup \Delta)$ (for $\Delta \subset \Omega^{\Box}$) and $\Box B_1, \ldots, \Box B_n$ are derived in a proof from $\Gamma^{\Box}$, then $\Diamond A$ may be derived in the proof on the condition that $\bigvee(\Delta)$ is not $\mathsf{S5P}^a$-derivable from $\Gamma^{\Box}$. However, as there is no positive test for $\mathsf{S5P}^a$-derivability, the applicability of this rule would be undecidable. This is why I shall now present an alternative semantics for COMP^a that proceeds entirely in terms of the S5P-models of $\Gamma^{\Box}$.

The alternative semantics will reveal that COMP^r and COMP^m are composed of two adaptive logics. On the one hand, there is $\mathsf{S5P}^r$ (respectively $\mathsf{S5P}^m$). On the other hand, there is the logic COMP which is obtained from S5P in a similar way as COM is obtained from S5. Where $\Gamma^\star$ is defined as before, the relation between the different systems is given by $\Diamond A \in Cn_{\mathsf{COMP}^a}(\Gamma^{\Box})$ iff $\Diamond A \in Cn_{\mathsf{COMP}}(\Gamma^\star)$.

Like $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$, COMP is only defined with respect to sets of premises of the form $\Gamma^{\Box}$ (where all members of Γ are non-modal). The idea behind COMP is to assume that $\Diamond A$ is true, unless $\Gamma^{\Box}$ explicitly prevents so, that is, unless $\Box\neg A$ is S5P-derivable from $\Gamma^{\Box}$. In line with this, the set of abnormalities of COMP consists of all formulas of the form $\Box\neg A$, where A is a member of $\mathcal{W}$.[12] I shall use the term "□-abnormalities" to refer

[12] Note that in view of the semantics of S5P, $\Box\neg A$ is S5P-derivable from $\Gamma^{\Box}$ iff $\sim\Diamond A$ is. Hence, like for COM, the set of abnormalities could be characterized by the form $\sim\Diamond A$ (see [11]). The present format is chosen because it leads to a more transparent approach for the paraconsistent case. Note also that in order to satisfy the usual relations between the lower limit logic, the set of abnormalities and the upper limit logic (see Section 3), valid formulas should be excluded from the set of abnormalities (if not, extending S5P with the requirement that all abnormalities are logically false results in a trivial—that is, Post-inconsistent—system). However, as is explained in [11], the present approach not only leads to a simpler semantics, but also facilitates the design of the proof theory. The upper limit logic of COMP is obtained by extending S5P with the requirement that all logically contingent members of $\Omega^{\Box}$ are logically false.

to formulas of this form and denote their set by $\Omega^{\Box}$.

The lower limit logic of COMP is S5P. The $\Box$-abnormal part of an S5P-model $\mathcal{M}$ is defined by:

Definition 28: *For any* S5P*-model* $\mathcal{M}$, $Ab^{\Box}(\mathcal{M}) = \{A \in \Omega^{\Box} \mid v_{\mathcal{M}}(A) = 1\}$.

and the set of $\Box$-abnormalities that are unavoidable in view of $\Gamma^{\Box}$ by:

Definition 29: $Ab^{\Box}(\Gamma^{\Box}) = \{A \in \Omega^{\Box} \mid \Gamma^{\Box} \vDash_{\mathsf{S5P}} A\}$.

The COMP-models of $\Gamma^{\Box}$ are obtained by selecting its S5P-models that verify $\Box\neg A$ iff $\Box\neg A$ is unavoidable in view of $\Gamma^{\Box}$:

Definition 30: *An* S5P*-model* $\mathcal{M}$ *is a* COMP*-model of* $\Gamma^{\Box}$ *iff* $Ab^{\Box}(\mathcal{M}) = Ab^{\Box}(\Gamma^{\Box})$.

and the semantic consequence relation is defined accordingly:

Definition 31: $\Gamma^{\Box} \vDash_{\mathsf{COMP}} A$ *iff all* COMP*-models of* $\Gamma^{\Box}$ *verify* A.

When applied to sets of premises that are free of classical negation, COMP suffers from all three problems discussed in Section 2. However, when applied to $\Gamma^{\star}$, it leads to adequate results. The reason is that $\Gamma^{\star}$ includes $\Box\neg A$, whenever A is false in all interpretations of Γ that are as consistent as possible. Consider, for instance, $\Gamma^{\Box} = \{\Box p, \Box q, \Box({\sim}p \vee {\sim}q), \Box({\sim}p \vee r), \Box({\sim}q \vee r), \Box s\}$. If $\mathsf{S5P}^r$ is chosen to define $\Gamma^{\star}$, the latter includes $\Box\neg(r \wedge {\sim}r)$ as well as $\Box\neg(s \wedge {\sim}s)$, and hence, $\Diamond(r \wedge {\sim}r)$, $\Diamond(s \wedge {\sim}s)$ and $\Diamond{\sim}s$ are false in *all* S5P-models of $\Gamma^{\star}$. If $\mathsf{S5P}^m$ is chosen, $\Gamma^{\star}$ moreover includes $\Box\neg((p \wedge {\sim}p) \wedge (q \wedge {\sim}q))$. In that case, also $\Diamond{\sim}r$ is false in all S5P-models of $\Gamma^{\star}$. I leave it to the reader to check that $\Diamond(r \wedge {\sim}r)$ as well as $\Diamond(s \wedge {\sim}s)$ are COMP-consequences of $\Gamma^{\Box}$ (but not of $\Gamma^{\star}$).

I now show that the proper combination of $\mathsf{S5P}^a$ and COMP leads to a semantics that, for all conclusions of the form $\Diamond A$, where $A \in \mathcal{W}$, is equivalent to the semantics of COMP^a. Whether the two semantics are in general equivalent is immaterial for the purposes of this paper (we are only interested in the question whether some A is compatible with some Γ, and hence, whether some $\Diamond A$ is a semantic consequence of $\Gamma^{\Box}$).

Lemma 4: $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \Box\neg A$ *iff* $\Gamma^{\star} \vDash_{\mathsf{S5P}} \Box\neg A$.

Proof. The first direction is obvious. For the other direction, suppose that $\Gamma^{\Box} \nvDash_{\mathsf{S5P}^a} \Box\neg A$. It follows that $\Box\neg A$ is false in some $\mathsf{S5P}^a$-model of $\Gamma^{\Box}$, and hence, in some $\mathsf{S5P}$-model of $\Gamma^{\star}$. □

Lemma 5: *If an* $\mathsf{S5P}$*-model* $\mathcal{M}$ *is a* COMP^a*-model of* $\Gamma^{\Box}$*, it is a* COMP*-model of* $\Gamma^{\star}$*.*

Proof. Suppose that $\mathcal{M}$ is a COMP^a-model of $\Gamma^{\Box}$. It follows that, for every $A \in \mathcal{W}$, $\mathcal{M}$ verifies $\Box\neg A$ iff $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \Box\neg A$ (in view of Theorems 19 and 17). Hence, $Ab^{\Box}(\mathcal{M}) = \{\Box\neg A \mid \Box\neg A \in \Gamma^{\star}\} = Ab^{\Box}(\Gamma^{\star})$ (in view of Lemma 4). But then, $\mathcal{M}$ is a COMP-model of $\Gamma^{\star}$. □

Theorem 23: *Where* $A \in \mathcal{W}$, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ *iff* $\Gamma^{\star} \vDash_{\mathsf{COMP}} \Diamond A$*.*

Proof. For the left-right direction, suppose that $\Gamma^{\star} \nvDash_{\mathsf{COMP}} \Diamond A$. It follows that $\Diamond A$ is false, and hence $\Box\neg A$ is true, in some $\mathsf{S5P}$-model $\mathcal{M}$ that verifies $\Box\neg B$ iff $\Gamma^{*} \vDash_{\mathsf{S5P}} \Box\neg B$, and hence, iff $\Gamma^{\Box} \vDash_{\mathsf{S5P}^a} \Box\neg B$ (in view of Lemma 4). But then, $\neg A$ is true in all P-models M that verify $\neg B$ iff $\Gamma \vDash_{\mathsf{P}^a} \neg B$ (in view of Theorem 13). Hence, $\neg A$ is true in all P^a-models of Γ. It follows that $\Diamond A$ is false in all COMP^a-models of $\Gamma^{\Box}$ (by Definition 26).

The right-left direction immediately follows from Lemma 5. □

To end this section, I show that Strong Reassurance holds with respect to $\mathsf{S5P}^a$ and that Reassurance holds with respect to the lower limit logic $\mathsf{S5P}$.

Theorem 24: *If* $\mathcal{M}$ *is an* $\mathsf{S5P}^a$*-model of* $\Gamma^{\Box}$ *but not a* COMP^a*-model of* $\Gamma^{\Box}$*, then there is a* COMP^a*-model* $\mathcal{M}'$ *of* $\Gamma^{\Box}$ *such that* $Ab(\mathcal{M}') \subset Ab(\mathcal{M})$*.*

Proof. Suppose that $\mathcal{M} = \langle \Sigma, M_0 \rangle$ is an $\mathsf{S5P}^a$-model of $\Gamma^{\Box}$, but not a COMP^a-model of $\Gamma^{\Box}$. In view of the semantics of $\mathsf{S5P}^a$, Σ is a set of P^a-models. It follows that there is a $\mathcal{M}' = \langle \Sigma', M_0 \rangle$ such that Σ' is the set of *all* P^a-models, and hence, that there is a COMP^a-model of $\Gamma^{\Box}$. As $\mathcal{M}$ is an $\mathsf{S5P}^a$-model of $\Gamma^{\Box}$, it verifies all members of $\{\Box\neg A \mid \Box\neg A \in \Gamma^{\star}\} = Ab^{\Box}(\Gamma^{\star}) = Ab^{\Box}(\mathcal{M}')$. Hence, $Ab^{\Box}(\mathcal{M}') \subseteq Ab^{\Box}(\mathcal{M})$. As $\mathcal{M}$ is not a COMP^a-model of $\Gamma^{\Box}$, $Ab^{\Box}(\mathcal{M}') \neq Ab^{\Box}(\Gamma^{\star})$. But then, $Ab^{\Box}(\mathcal{M}') \subset Ab^{\Box}(\mathcal{M})$. □

Theorem 25: *If* $\Gamma^{\Box}$ *has* $\mathsf{S5P}$*-models, it also has* COMP^a*-models.*

Proof. If $\Gamma^{\Box}$ has $\mathsf{S5P}$-models, Γ has P^a-models (in view of the semantics of $\mathsf{S5P}$ and Corollary 1). But then, $\Gamma^{\Box}$ has COMP^a-models (in view of the semantics of $\mathsf{S5P}^a$ and Theorem 24). □

9. *The Dynamic Proof Theory of* COMP^a

As both COMP^r and COMP^m consist of two adaptive logics (that have a different set of abnormalities), their proof theory includes two different conditional rules (one related to $\mathsf{S5P}^r$, respectively $\mathsf{S5P}^m$) and one related to COMP. Each of the two logics also has two different marking definitions. However, as $\mathsf{S5P}$ is the lower limit logic of $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$ as well as of COMP, there is only one unconditional rule.

I shall use $Dab(\Delta)$ in the same way as before (to refer to a disjunction of members of Ω), and $Dab^{\Box}(\Delta)$ to refer to a disjunction of members of $\Omega^{\Box}$.

Although there are two different kinds of abnormalities involved, the proof format of COMP^a is exactly the same as that of all other adaptive logics: as the two conditional rules are defined with respect to abnormalities of a different logical form, there is no need to introduce two sets of conditions.

The premise rule is as for $\mathsf{S5P}^a$ and is therefore not repeated here. Also the unconditional rule (NRU*) and the first conditional rule (NRC1) are as for $\mathsf{S5P}^a$, except that some additional restriction is added (in order to simplify the meta-proofs):

NRU* If $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} A$ $(n \geq 0)$, $\Box B_1, \ldots, \Box B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, and $\Delta_1 \cup \ldots \cup \Delta_n \cap \Omega^{\Box} = \emptyset$, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) A,
(iii) the line number of the $\Box B_i$,
(iv) "NRU*", and
(v) $\Delta_1 \cup \ldots \cup \Delta_n$.

NRC1 If $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} \Box(A \vee Dab(\Delta))$ $(n \geq 0)$, $\Box B_1, \ldots, \Box B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, and $\Delta_1 \cup \ldots \cup \Delta_n \cap \Omega^{\Box} = \emptyset$, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) $\Box A$,
(iii) the line number of the $\Box B_i$,
(iv) "NRC1", and

(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

The second conditional rule allows one to introduce formulas of the form $\Diamond A$ to the proof. Note that also this rule refers to $\mathsf{S5P}$:

NRC2 If $\Box B_1, \ldots, \Box B_n \vdash_{\mathsf{S5P}} \Diamond A \vee Dab^{\Box}(\Delta)$ $(n \geq 0)$, and $\Box B_1, \ldots,$ $\Box B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) $\Diamond A$,
(iii) the line number of the $\Box B_i$,
(iv) "NRC2", and
(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

The following rule (which is obviously derivable) leads to proofs that are more interesting from a heuristic point of view:

RD If $B_1, \ldots, B_n \vdash_{\mathsf{S5P}} A$, and $B_1, \ldots, B_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:

(i) the appropriate line number,
(ii) A,
(iii) the line number of the B_i,
(iv) "RD", and
(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

The above rules are the same for COMP^r and COMP^m. The difference between the two systems shows again only in the marking definitions.

The marking definition related to NRC1 (called r-marking and m-marking, respectively) is exactly the same as for $\mathsf{S5P}^a$. Thus, r-marking for COMP^r is given by Definition 19 and m-marking for COMP^m by Definition 23. Also, $U_s(\Gamma^{\Box})$ and $\Phi_s(\Gamma^{\Box})$ are defined in exactly the same way as for $\mathsf{S5P}^a$ (both proceed in terms of the $NDab$-formulas that are derived *on the empty condition*).

The marking definition related to NRC2 (called c-marking) is the same for both COMP^r and COMP^m. What is special about c-marking is that it proceeds in terms of formulas that are (possibly) derived on a *non-empty* condition. This, however, is not surprising. As COMP is defined 'on top of' $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$, a formula that is introduced by means of NRC2 should be withdrawn as soon as it turns out to be incompatible with one of the $\mathsf{S5P}^a$-consequences (or put more accurately, as soon as it turns out to be incompatible with one of the formulas that is, at that stage of the proof, considered as $\mathsf{S5P}^a$-derived). As we shall see below, the interplay between the two different kinds of marking causes the proof theory for paraconsistent compatibility to be much more dynamic than that for classical compatibility.

In view of the definition of c-marking, we first need to define the set $Ab_s(\Gamma^{\Box})$. In the case of COMP^r, respectively COMP^m, $Ab_s(\Gamma^{\Box})$ consists of all members of $\Omega^{\Box}$ that occur, at that stage of the proof, on a line that is not r-marked, respectively m-marked.

Definition 32: *Marking for* COMP*: Line i is c-marked at stage s of a proof from $\Gamma^{\Box}$ iff, where Δ is its condition, $\Delta \cap Ab_s(\Gamma^{\Box}) \neq \emptyset$.*

In the following definitions, I say that a line is *marked* iff it is r-marked, m-marked or c-marked:

Definition 33: *A is derived at stage s in a COMP^a-proof from $\Gamma^{\Box}$ iff A is the second element of a line that is not marked in the proof at that stage.*

Definition 34: *A is finally derived on line i of a COMP^a-proof from $\Gamma^{\Box}$ iff (i) A is the second element of line i, (ii) line i is not marked in the proof, and (iii) any extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 35: $\Gamma^{\Box} \vdash_{\mathsf{COMP}^a} A$ *(A is finally COMP^a-derivable from $\Gamma^{\Box}$) iff A is finally derived on some line of a COMP^a-proof from $\Gamma^{\Box}$.*

As the definition for c-marking refers to lines that are not r/m-marked (consider the definition for $Ab_s(\Gamma)$), the best way to perform the marking is by deleting all marks (whenever a line is added to the proof), and by adding the r/m-marks before the c-marks.

Let us look at a very simple example of a dynamic proof to understand what the proof theory comes to. To save space, I shall illustrate the proof theory of both COMP^r and COMP^m with one and the same proof. Suppose that these are the premises:

1	$\Box(p \wedge q)$	–	PREM	$\emptyset$
2	$\Box({\sim}p \vee {\sim}q)$	–	PREM	$\emptyset$
3	$\Box({\sim}p \vee {\sim}r)$	–	PREM	$\emptyset$
4	$\Box({\sim}q \vee {\sim}r)$	–	PREM	$\emptyset$
5	$\Box s$	–	PREM	$\emptyset$

In view of line 5, the rule NRC2 allows one to add the following line:

6	$\Diamond(r \wedge s)$	5	NRC2	$\{\Box\neg r\}$

This expresses the hypothesis that $r \wedge s$ is compatible with the (non-modal forms of the) premises. However, in view of lines 1 and 3, one may infer $\neg r$, on the condition that both $p \wedge \sim p$ and $r \wedge \sim r$ behave normally:

7	$\Box\neg r$	1, 3	NRC1	$\{p \wedge \sim p,\ r \wedge \sim r\}$

At this stage, the condition of line 6 is no longer fulfilled: $\Box\neg r \in Ab_7(\Gamma)$. So, at stage 7, line 6 is c-marked, and hence, the formula that occurs on it is no longer considered as derived (independent of whether the above proof is a COMP^r-proof or a COMP^m-proof).

Suppose, however, that one then adds the following line:

8	$\Box((p \wedge \sim p) \vee (q \wedge \sim q))$	1, 2	NRU*	$\emptyset$

According to the best insights in the premises at this stage of the proof, both $p \wedge \sim p$ and $q \wedge \sim q$ behave abnormally with respect to the premises. In view of the fact that $U_8(\Gamma) = \{p \wedge \sim p,\ q \wedge \sim q\}$, line 7 is r-marked at this stage. Moreover, as $\Phi_8(\Gamma) = \{\{p \wedge \sim p\}, \{q \wedge \sim q\}\}$, line 7 is also m-marked (in view of (ii) in Definition 23). As a consequence, line 6 is no longer c-marked, and hence, is again considered as derived in the proof.[13]

There is, however, a second way to infer $\neg r$ from the premises:

9	$\Box\neg r$	1, 4	NRC1	$\{q \wedge \sim q,\ r \wedge \sim r\}$

At this stage, the marking for COMP^r and for COMP^m no longer proceeds in the same way. Line 9 is r-marked (in view of line 8), but is not m-marked. Moreover, line 7 is still r-marked, but is no longer m-marked. It follows that, if the above is a COMP^r-proof from $\Gamma^\Box$, line 6 is still not c-marked. However, if it is a COMP^m-proof, line 6 is again c-marked. So, this is how the proof looks like at stage 9:

1	$\Box(p \wedge q)$	–	PREM	$\emptyset$
2	$\Box(\sim p \vee \sim q)$	–	PREM	$\emptyset$
3	$\Box(\sim p \vee \sim r)$	–	PREM	$\emptyset$
4	$\Box(\sim q \vee \sim r)$	–	PREM	$\emptyset$
5	$\Box s$	–	PREM	$\emptyset$
6	$\Diamond(r \wedge s)$	5	NRC2	$\{\Box\neg r\}$ ✓$_c$ (for COMP^m only)
7	$\Box\neg r$	1, 3	NRC1	$\{p \wedge \sim p,\ r \wedge \sim r\}$ ✓$_r$
8	$\Box((p \wedge \sim p) \vee (q \wedge \sim q))$	1, 2	NRU*	$\emptyset$
9	$\Box\neg r$	1, 4	NRC1	$\{q \wedge \sim q,\ r \wedge \sim r\}$ ✓$_r$

I leave it to the reader to check that, for COMP^r, the formula on line 6 is finally derived and that $\Diamond(s \wedge \sim s)$ is not finally derivable in the proof (neither for COMP^r nor for COMP^m), but that $\Diamond(p \wedge \sim p)$ is.

[13] This is an important difference with the adaptive logics for classical compatibility from [11]. The dynamics of their proofs is very limited: once a formula is marked, it remains marked in any extension of the proof.

Let us now turn to the Soundness and Completeness proofs. In view of the intended application of COMP^a, I only prove Soundness and Completeness for conclusions of the form $\Diamond A$.

I begin with two lemmas the proof of which immediately follows by an inspection of the proof theory:

Lemma 6: *A can be finally derived on a condition Δ such that $\Delta \cap \Omega^{\Box} = \emptyset$ iff $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} A$.*

Lemma 7: *Where $\Delta \subset \Omega$, $\Delta' \subset \Omega^{\Box}$, and $\Delta' \neq \emptyset$, $\Diamond A$ can be finally derived on a condition $\Delta \cup \Delta'$ iff $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} \Diamond A \vee Dab^{\Box}(\Delta')$ and $\Gamma^{\Box} \nvdash_{\mathsf{S5P}^a} Dab^{\Box}(\Delta')$.*

In the following proof, I freely rely on the Soundness and Completeness results for $\mathsf{S5P}^a$.

Theorem 26: *For every $A \in \mathcal{W}$, if $\Gamma^{\Box} \vdash_{\mathsf{COMP}^a} \Diamond A$ then, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$.*

Proof. Suppose that $\Gamma^{\Box} \vdash_{\mathsf{COMP}^a} \Diamond A$. It follows that $\Diamond A$ is finally derived on some line i of a COMP^a-proof from $\Gamma^{\Box}$. Where $\Delta \subset \Omega$ and $\Delta' \subset \Omega^{\Box}$, let the condition of line i be $\Delta \cup \Delta'$.

If $\Delta' = \emptyset$, $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} \Diamond A$ (by Lemma 6). Hence, as the COMP^a-models of $\Gamma^{\Box}$ are a subset of its $\mathsf{S5P}^a$-models, $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$.

If $\Delta' \neq \emptyset$, $\Gamma^{\Box} \vdash_{\mathsf{S5P}^a} \Diamond A \vee Dab^{\Box}(\Delta')$ and $\Gamma^{\Box} \nvdash_{\mathsf{S5P}^a} Dab^{\Box}(\Delta')$ (by Lemma 7). Hence, all $\mathsf{S5P}^a$-models of $\Gamma^{\Box}$ verify $\Diamond A \vee Dab^{\Box}(\Delta')$, but some of them falsify $Dab^{\Box}(\Delta')$. It follows that all COMP^a-models of $\Gamma^{\Box}$ falsify $Dab^{\Box}(\Delta')$, and hence, that all of them verify $\Diamond A$. □

Theorem 27: *For every $A \in \mathcal{W}$, if $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$ then, $\Gamma^{\Box} \vdash_{\mathsf{COMP}^a} \Diamond A$.*

Proof. If Γ does not have P-models, the theorem obviously holds (in view of $\{\Box A, \Box\neg A\} \vdash_{\mathsf{S5P}} \bot$).

So, suppose that Γ has P-models and that $\Gamma^{\Box} \vDash_{\mathsf{COMP}^a} \Diamond A$. It follows, by Theorem 20, that $\Gamma^{\Box} \nvdash_{\mathsf{S5P}^a} \Box\neg A$. As $\vdash_{\mathsf{S5P}} \Diamond A \vee \Box\neg A$, RC2 allows one to start a proof from $\Gamma^{\Box}$ by writing down a line (with line number 1) that has $\Diamond A$ as its second element and $\{\Box\neg A\}$ as its fifth. It is easily seen that line 1 is neither r-marked nor m-marked in any extension of the proof and that any extension of the proof in which line 1 is c-marked may be further extended such that line 1 is not c-marked. This proves that $\Diamond A$ is finally derived in some COMP^a-proof from $\Gamma^{\Box}$, and hence, that $\Gamma^{\Box} \vdash_{\mathsf{COMP}^a} \Diamond A$. □

10. *Some Alternatives*

The inconsistency-adaptive logics P^r and P^m are based on a very weak paraconsistent logic. For instance, the *De Morgan properties* are not valid in P, and *Replacement of Identicals* is only valid outside the scope of a negation. P also does not spread inconsistencies. For example, some P-models verify both $p \wedge q$ and $\sim(p \wedge q)$, but falsify any other inconsistency.

This has specific consequences for the accounts of paraconsistent compatibility that are based on P^r and P^m. Consider, for instance, $\Gamma = \{p, \sim p\}$. As $\sim\sim p \wedge \sim\sim\sim p \notin U(\Gamma)$ and $\Gamma \vdash_{\mathsf{P}^a} \sim\sim\sim p$, $\sim\sim p$ is *not* P^a-compatible with Γ. More generally, any formula obtained by prefixing an *even* number of "$\sim$" to p is incompatible with Γ. However, any formula obtained by prefixing an *odd* number of "$\sim$" to p is compatible with it. The reason is that for any such A (p preceded by an odd number of paraconsistent negations), $\Gamma \nvdash_{\mathsf{P}^a} \sim A$.

As is shown in [10], the logic P may be enriched in several ways. Evidently, every enrichment will have effects on the corresponding notion of paraconsistent compatibility. For instance, if one chooses an inconsistency-adaptive logic based on an enrichment of P that validates $A \; / \sim\sim A$, any formula obtained by prefixing a series of "$\sim$" to p will be compatible with $\{p, \sim p\}$. In general, enriching P will have the effect that *more* sentences are compatible with a given set of premises. The reason is that enriching P results in a poorer inconsistency-adaptive logic. Consider, for instance, $\Gamma = \{\sim(p \wedge q), p \wedge q, \sim p \vee r\}$. All P^a-models of Γ falsify $\sim p$ and verify r (as P does not spread inconsistencies). Hence, r is a P^a-consequence of Γ and $\sim r$ is not P^a-compatible with it. However, where P^+ stands for an enrichment of P that validates all *De Morgan* properties, some P^+-models verify $\sim p$ as well as $\sim r$, and hence, p, $\sim p$, r and $\sim r$ are all $\mathbf{P}^{+a}$-compatible with Γ.[14]

The situation is somewhat different if one chooses a paraconsistent system that validates Disjunctive Syllogism as well as all other 'analysing' rules—an example is the logic AN from [15]. In that case, one obtains an inconsistency-adaptive logic that is much richer than P^r and P^m. For instance, where $\Gamma = \{\sim(p \wedge q), p \wedge q, \sim p \vee r\}$, ANA (the adaptive logic based on AN) not only entails p and $\sim p$, but also r and $\sim\sim r$. Hence, p and

[14] Readers that want to experiment with different enrichments of P and the corresponding inconsistency-adaptive logics are referred to the computer program LaC (designed by Alex Klijn and free downloadable from `http://logica.UGent.be/centrum/writings/programs.php`). The program not only allows one to check whether, according to P, P^r or P^m, some sentence is a semantic consequence of some set of premises, but also to 'compose' alternative systems and to check semantic consequence for these.

$\sim p$ are ANA-compatible with Γ (because they behave inconsistently with respect to it), but $\sim r$ is not.

Still a different situation obtains when one chooses an inconsistency-adaptive logic that is based on a non-adjunctive paraconsistent system. An example is the logic $\mathbf{DL}^r$ (see [18]). This logic, which is an adaptive version of Jaśkowski's system $\mathsf{D_2}$, invalidates Adjunction for all sentences that behave inconsistently with respect to the premises (thus preventing the derivation of contradictions), but validates it for all others. Thus, where $\Gamma = \{p \wedge q, \sim p, r\}$, $p \wedge \sim p$ is not $\mathbf{DL}^r$-derivable from Γ, but $q \wedge r$ is.

In [16], $\mathbf{DL}^r$ is combined with COM to reconstruct the notion of *pragmatic truth* as developed in the partial structures approach of Newton da Costa and associates (see, for instance, [13]). The resulting logic (called APT) forms an alternative for the logic of paraconsistent compatibility presented in this paper. An important difference, however, is that APT does not allow to distinguish between inconsistent information and incomplete information. For instance, where $\Gamma = \{p, \sim p\}$, there is no way to decide from the APT-consequence set that Γ is inconsistent with respect to p, but incomplete with respect to q: p, $\sim p$, q and $\sim q$ are all APT-consequences of Γ, and any complex formula derivable for p is also derivable for q. In the case of COMP^a, however, $p \wedge \sim p$ is derivable from Γ, but $q \wedge \sim q$ is not.

11. *In Conclusion*

The logics of paraconsistent compatibility presented in this paper have several attractive properties. To begin with, they offer a characterization of paraconsistent compatibility that is as intuitive as that for classical compatibility, both from a definitorial and a semantic point of view. Moreover, for the consistent case, they lead to exactly the same results as the logics for classical compatibility that were presented in [11]. Finally, for the inconsistent case, they only classify a sentence as compatible with a set of premises Γ if it is true in some interpretation of Γ that is as consistent as possible.

Another important characteristic of the results presented in this paper is that they are highly generic. The procedure by which the logics COMP^r and COMP^m are obtained from the inconsistency-adaptive logics P^r and P^m is easily adjustable for other inconsistency-adaptive logics. This is especially important in view of the fact that, for the inconsistent case, different contexts ask for different logics of compatibility. For instance, in some contexts, there may be reasons to assume that the number of inconsistencies is minimal. In other contexts, there may be reasons to be much more cautious. Still in other contexts, there may be reasons to interpret the information in the richest way possible (despite the inconsistencies)—see [17] for examples. Which sentences are considered as compatible with the available information and which not will be different in each of these contexts.

The paper also raises some open problems. One is to formulate criteria to decide which logic of paraconsistent compatibility is suited for which application context. Another series of problems is related to the logics $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$. Here, they are used as a basis for the logics COMP^r and COMP^m. However, as is discussed in, for instance, [7], many non-monotonic consequence relations are most easily captured by a modal adaptive logic. Because of this, it is worthwhile to further elaborate the logics $\mathsf{S5P}^r$ and $\mathsf{S5P}^m$. What seems most urgent in this respect is to generalize the two logics (for arbitrary sets of premises) and to reformulate them in the usual format of adaptive logics. Once such a formulation is available, the results from [8] will allow to prove, in a completely standard way, a whole series of properties.[15]

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REFERENCES

[1] Diderik Batens. Dynamic dialectical logics. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.
[2] Diderik Batens. Blocks. The clue to dynamic aspects of logic. *Logique et Analyse*, 150–152:285–328, 1995. Appeared 1997.
[3] Diderik Batens. Inconsistency-adaptive logics. In Ewa Orłowska, editor, *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*, pages 445–472. Physica Verlag (Springer), Heidelberg, New York, 1999.
[4] Diderik Batens. Minimally abnormal models in some adaptive logics. *Synthese*, 125:5–18, 2000.
[5] Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000 (appeared in 2002).
[6] Diderik Batens. A dynamic characterization of the pure logic of relevant implication. *Journal of Philosophical Logic*, 30:267–280, 2001.
[7] Diderik Batens. A general characterization of adaptive logics. *Logique et Analyse*, 173-174-175:45–68, 2001. Appeared 2003.

[15] Unpublished papers in the reference section are available at the internet address http://logica.UGent.be/centrum/writings.

[8] Diderik Batens. The need for adaptive logics in epistemology. To appear.
[9] Diderik Batens. A paraconsistent proof procedure based on classical logic. To appear.
[10] Diderik Batens and Kristof De Clercq. A rich paraconsistent extension of full positive logic. To appear.
[11] Diderik Batens and Joke Meheus. The adaptive logic of compatibility. *Studia Logica*, 66:327–348, 2000.
[12] Bryson Brown. How to be realistic about inconsistency in science. *Studies in History and Philosophy of Science*, 21:281–294, 1990.
[13] Newton C. A. da Costa, Otávio Bueno, and Steven French. The logic of pragmatic truth. *Journal of Philosophical Logic*, 27:603–620, 1998.
[14] Joke Meheus. Adaptive logic in scientific discovery: the case of Clausius. *Logique et Analyse*, 143–144:359–389, 1993 (appeared in 1996).
[15] Joke Meheus. An extremely rich paraconsistent logic and the adaptive logic based on it. In Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, editors, *Frontiers of Paraconsistent Logic*, pages 189–201. Research Studies Press, Baldock, UK, 2000.
[16] Joke Meheus. An adaptive logic for pragmatic truth. In Walter A. Carnielli, Marcello E. Coniglio, and Itala M. Loffredo D'Ottaviano, editors, *Paraconsistency. The Logical Way to the Inconsistent*, pages 167–185. Marcel Dekker, New York, 2002.
[17] Joke Meheus, editor. *Inconsistency in Science*. Kluwer, Dordrecht, 2002.
[18] Joke Meheus. An adaptive logic based on Jaśkowski's D2. *Journal of Philosophical Logic*, to appear.
[19] Nancy Nersessian. Inconsistency, generic modeling, and conceptual change in science. In Meheus [17], pages 197–211.
[20] John Norton. The logical inconsistency of the old quantum theory of black body radiation. *Philosophy of Science*, 54:327–350, 1987.
[21] John Norton. A paradox in Newtonian gravitation theory II. In Meheus [17], pages 185–195.
[22] Graham Priest. Minimally inconsistent LP. *Studia Logica*, 50:321–331, 1991.
[23] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.
[24] Joel Smith. Inconsistency and scientific reasoning. *Studies in History and Philosophy of Science*, 19:429–445, 1988.