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BF, CBF AND LEWIS SEMANTICS

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1. *Introduction*

The Barcan Formula (BF) $\forall x\Box A \rightarrow \Box\forall xA$ and its Converse (CBF) $\Box\forall xA \rightarrow \forall x\Box A$ have been a central topic of discussion since the very beginning of quantified modal logic, which goes back to Ruth Barcan Marcus's paper of 1946, [1]. The problem we shall address in this paper is that of selecting the 'right' semantics so as to fully understand their meanings. BF and CBF are often considered as dual principles: BF corresponds in Kripke semantics, $\mathcal{K}$-semantics, to the condition that inner domains never increase, CBF to the condition that inner domains never decrease. We shall show that this duality is not intrinsic to the meaning of CBF and BF but rather depends on general features of Kripke semantics. A major step forward to the clarification of the meaning of BF was achieved by counterpart semantics, $\mathcal{C}$-semantics. Counterpart semantics was introduced as early as 1988 in [7] in the context of the semantics of relational universes. It has many advantages including the fact that it provides the conceptual tools for a foundational study of quantified modal logics. A detailed description of it together with several completeness theorems can be found in [2]. As we shall see, in counterpart semantics the meaning of BF is well captured, whereas the meaning of CBF still remains opaque. We will introduce a generalization of counterpart semantics that we call *Lewis semantics*[1], $\mathcal{L}$-semantics, to address this problem. We will limit ourselves to modal languages without individual constants or the identity relation because, notwithstanding their central role in counterpart semantics, these are not relevant to the present discussion. In Lewis semantics, as distinct from counterpart semantics (though exactly as in Kripke semantics[2]), any world w is endowed with an *inner* domain D_w and an *outer* domain U_w, $D_w \subseteq U_w$, where D_w represents the set of 'existing' individuals at w, whereas U_w is a subset of the pool of entities that either existed or will exist or will remain forever fictional entities. Other

[1] From David Lewis.

[2] See[3] .

features of Lewis semantics are analogous to those of counterpart semantics, and, in particular, individuals are worldbound, so, in general, no individual exists in different worlds. Individuals have *similia* in different worlds and the task of retracing these is left to the counterpart relation $\mathfrak{C}$. An individual of a world w can have none, one, many *similia* or *counterparts* in any accessible world. This notion of counterpart is crucial to determining the value of modalized formulas: *John has the property of being necessarily A* if all of his counterparts in all accessible worlds have the property A. In more detail, given a formula $\Box A(x_1, \ldots, x_n)$ with *exactly* the free variables $x_1, \ldots x_n$, an n-tuple of individuals satisfies the property $\Box A$ iff all the n-tuples of their respective counterparts satisfy A.

$$\langle a_1, \ldots, a_n \rangle \models_w \Box A(x_1, \ldots, x_n)$$

iff for every v such that wRv and for every counterpart $a_1^*, \ldots, a_n^*$ of $a_1, \ldots, a_n$ in U_v, $\langle a_1^*, \ldots, a_n^* \rangle \models_v A(x_1, \ldots, x_n)$.

Analogously,

$$\langle a_1, \ldots, a_n \rangle \models_w \Diamond A(x_1, \ldots, x_n)$$

iff there is a v such that wRv and there are counterparts $a_1^*, \ldots, a_n^*$ of $a_1, \ldots, a_n$ in U_v, such that $\langle a_1^*, \ldots, a_n^* \rangle \models_v A(x_1, \ldots, x_n)$.

2. *Lewis semantics*

A *Lewis-frame*, *$\mathcal{L}$-frame*, is a quintuple $\mathcal{F} = \langle W, R, D, U, \mathfrak{C} \rangle$, where $W \neq \emptyset$, $R \subseteq W^2$, D and U are functions such that D_w is a set for every $w \in W$, U_w is a set for every $w \in W$ and $D_w \subseteq U_w$. $\mathfrak{C}$ is the counterpart relation: $\mathfrak{C} = \biguplus_{w,v \in W} \{\mathfrak{C}_{\langle w,v \rangle} : wRv\}$, where for any $w, v \in W$, $\mathfrak{C}_{\langle w,v \rangle} \subseteq (D_w \times D_v)$.[3]
A *Lewis-model* $\mathcal{M}$ based on an $\mathcal{L}\text{-}frame$ $\mathcal{F}$ is given by an interpretation function I such that: for any predicate symbol P^n, $I_w(P^n) \subseteq (U_w)^n$.
D_w is the *inner* domain, the domain of variation of the quantifiers whereas U_w is the *outer* domain, the domain of interpretation of the predicate symbols, of the individual constants (if any) and the domain of variation of the variables.

Here are the four classes of frames we will refer to in the sequel.

[3] We take the disjoint union because the same individual may happen to belong to two different domains U_v and U_z and the one in U_v may be a counterpart of some a, whereas the one in U_z is not.

Lewis frames	Counterpart frames
$D_w \subseteq U_w$ no proviso on $\mathfrak{C}$	$D_w = U_w$ no proviso on $\mathfrak{C}$
Kripke frames	**Tarski-Kripke frames**
$D_w \subseteq U_w \neq \emptyset$ wRv implies $U_w \subseteq U_v$ $\mathfrak{C}$ is a totally defined function typically $\mathfrak{C}$ is the subset relation	$D_w = U_w \neq \emptyset$ wRv implies $U_w \subseteq U_v$ $\mathfrak{C}$ is a totally defined function typically $\mathfrak{C}$ is the subset relation

A difficulty with Lewis semantics, as well as with counterpart semantics, is that a language appropriate to talk about $\mathcal{L}$-frames and $\mathcal{L}$-models has to be a language with types. Let us see why. In the truth clause given above, $x_1, \ldots, x_n$ are *exactly*[4] the variables occurring free in A, but $x_1, \ldots, x_n$ can be "too few" or "too many" when we consider subformulas of $A(x_1, \ldots, x_n)$, here is an example: $\exists x_3 Q(x_1, x_2, x_3) \wedge P(x_1)$ contains two free variables, $Q(x_1, x_2, x_3)$ three and $P(x_1)$ just one free variable. Types are needed to solve the problem. Terms have types and consequently formulas have types. From a semantical point of view a type tells us the length of the n-tuple of elements of the domain with respect to which it makes sense either to evaluate a term or to establish if a formula is satisfied or not. Let $x_1, x_2, x_3, \ldots$ be all the variables of the language. In order to see in a simple way how and why types are associated to terms, let us interpret the variables as projection functions. Let π^n_i, $n \geq i$, be the projection function such that $\pi^n_i(a_1, \ldots, a_n) = a_i$. Quite naturally the formulas $P(\pi^2_1(x, y))$ and $P(\pi^1_1(x))$ are synonymous, but the first contains two free variables, whereas the second contains just one free variable. So $P(\pi^2_1(x, y))$ is satisfied or not satisfied by pairs of individuals, whereas $P(\pi^1_1(x))$ by single individuals. Now, according to the given truth clause of modalized formulas, it is not the case that

$$\langle a_1, a_2 \rangle \models_w \Box P(\pi^2_1(x, y)) \qquad \text{iff} \qquad \langle a_1 \rangle \models_w \Box P(\pi^1_1(x))$$

because the worlds where there are counterparts of both a_1 and a_2 are, in general, fewer than the worlds where there are counterparts of a_1.[5]

[4] As we will see in a moment, this proviso cannot be weakened.

[5] This explains also why infinitary assignments will not do.

So we have to distinguish between $\pi_1^2(x, y)$ and $\pi_1^1(x)$. Terms with types do the job. For each variable x_i,

$$x_i^n,$$

$n \geq i$, is a term of type n. Intuitively speaking, x_i^n is a term "containing" the free variables $x_1, \ldots, x_n$.
For any n, and m-tuple of variables of type n, $x_1^n, \ldots, x_m^n$,

$$\langle x_1^n, \ldots, x_m^n \rangle$$

is a *complex term* of type $n \to m$ or 'from type n to type m'. In the following we will use the simpler notation $\langle n : x_1, ..., x_m \rangle$ and we will call such complex terms *projections*. For every n, the empty list of variables of type n, $\langle n : \rangle$, is a projection of type $n \to 0$. Now the notion of well formed formula.

1. If P^n is an n-ary predicate symbol then P^n is a *pure atomic formula* of type n,
2. If A is a wff of type n and $\langle m : x_1, ..., x_n \rangle$ is a projection of type $m \to n$, then $\langle m : x_1, ..., x_n \rangle A$ is a wff of type m,
3. If A and B are wffs of type n, then $\neg A, \quad \Box A, \quad A \vee B$ are wffs of type n.
4. If A is a wff of type $n + 1$, then $\exists x_{n+1} A$ is a wff of type n,

Pure atomic formulas are written also as $P^n(n : x_1, ..., x_n)$. Given a pure atomic formula P^n and a complex term $\langle m : x_{i_1}, ..., x_{i_n} \rangle$ of type $m \to n$, then $\langle m : x_{i_1}, ..., x_{i_n} \rangle P^n$ is an *atomic formula* of type m and, as usual, can be written as $P^n(m : x_{i_1}, ..., x_{i_n})$. $Q^2(3 : x_1, x_3)$ and $Q^2(5 : x_1, x_3)$ are different, for they have different types, whereas $Q^2(2 : x_1, x_3)$ is not well formed because the type is less than the maximum index of the free variables. $\langle m : x_{i_1}, ..., x_{i_n} \rangle A$ is a *substituted formula*. It is expedient to take the operation of substitution as a primitive logical operation. Given a formula A containing the variables $x_1, \ldots, x_n$, substitution applies to all the variables $x_1, \ldots, x_n$ although vacuously to some (none or all) of them (i.e. x_i will be substituted for x_i itself), hence n-tuples of terms (of the same type) have to be considered. Moreover if each of the terms $t_1, \ldots, t_n$ is of type m, the resulting formula will contain the free variables $x_1, \ldots, x_m$ and so it will be of type m.
Quantifying reduces the type by one, so from $Q^2(3 : x_1, x_3)$ we get $\forall x_3 Q^2 (3 : x_1, x_3)$ of type 2, from $Q^2(3 : x_1, x_2)$ we get $\forall x_3 Q^2(3 : x_1, x_2)$ of type 2 (vacuous quantification). $\forall x_1 Q^2(2 : x_1, x_2)$ is not well formed. No collision between free and bound variables can occur: all bound variables have indices greater than the indices of the free variables.

Interpretation of terms and satisfaction in $\mathcal{L}$-models
Projections of type $n \to m$ are interpreted with respect to n-tuples of elements of the domain:

$$\langle a_1, ..., a_n\rangle[n : x_{i_1}, ..., x_{i_m}] = \langle a_{i_1}, ..., a_{i_m}\rangle.$$

Let $\mathcal{M} = \langle W, R, D, U, \mathfrak{C}, I\rangle$ be an $\mathcal{L}$-model. For any $w \in W$, n-tuple $\langle a_1, ..., a_n\rangle$ of elements of U_w and formula A of type n, we define when $\langle a_1, ..., a_n\rangle$ *satisfies* A *at* w *in* $\mathcal{M}$, $\langle a_1, ..., a_n\rangle \models_w A$. By induction on A.

$\langle a_1, ..., a_n\rangle \models_w P^n$	iff	$\langle a_1, ..., a_n\rangle \in I_w(P^n)$
$\langle a_1, ..., a_n\rangle \models_w \langle n : x_{i_1}, ..., x_{i_k}\rangle B$	iff	$\langle a_1, ..., a_n\rangle[n : x_{i_1}, ..., x_{i_k}] \models_w B$
$\langle a_1, ..., a_n\rangle \models_w \neg C$	iff	$\langle a_1, ..., a_n\rangle \not\models_w C$
$\langle a_1, ..., a_n\rangle \models_w C \vee D$	iff	$\langle a_1, ..., a_n\rangle \models_w C$ or $\langle a_1, ..., a_n\rangle \models_w D$
$\langle a_1, ..., a_n\rangle \models_w \exists x_{n+1} C$	iff	for some $b \in D_w$, $\langle a_1, ..., a_n, b\rangle \models_w C$
$\langle a_1, ..., a_n\rangle \models_w \Box C$	iff	for all v such that wRv and for all $a^*_1, \ldots, a^*_n$ in U_v such that $a_i \mathfrak{C} a^*_i$, $1 \leq i \leq n$, $\langle a^*_1, ..., a^*_n\rangle \models_v C$.

A formula A of type n is *true at* w *in* $\mathcal{M}$, $\mathcal{M} \models^n_w A$, iff for any n-tuple $a_1, \ldots, a_n$ of elements of U_w, $\langle a_1, ..., a_n\rangle \models_w A$. A is *valid on* $\mathcal{M}$, $\mathcal{M} \models^n A$, iff $\mathcal{M} \models^n_w A$ for all $w \in W$. A is *valid on a* $\mathcal{L}$*-frame* $\mathcal{F}$, $\mathcal{F} \models^n A$, iff $\mathcal{M} \models^n A$ for every model $\mathcal{M}$ based on $\mathcal{F}$.

Note that, in general, $\langle n : x_{i_1}, \ldots, x_{i_k}\rangle \Box A$ and $\Box\langle n : x_{i_1}, \ldots, x_{i_k}\rangle A$ have different meanings. Take the formula $\langle 1 : x_1, x_1\rangle \Box P(x_1, x_2)$. Then

$\langle a\rangle \models_w \langle 1 : x_1, x_1\rangle \Box P(x_1, x_2)$	iff	$\langle a, a\rangle \models_w \Box P(x_1, x_2)$,
	iff	for all v such that wRv, $\langle a^*, a^{**}\rangle \models_v P(x_1, x_2)$, where a^* and a^{**} are counterparts of a in v.
$\langle a\rangle \models_w \Box\langle 1 : x_1, x_1\rangle \Box P(x_1, x_2)$	iff	for all v such that wRv, $\langle a^*\rangle \models_v \langle 1 : x_1, x_1\rangle P(x_1, x_2)$, where a^* is a counterpart of a in v,
	iff	$\langle a^*, a^*\rangle \models_v P(x_1, x_2)$.

$\langle 1 : x_1, x_1\rangle \Box P(x_1, x_2)$ is a *de re* modality, whereas $\Box\langle 1 : x_1, x_1\rangle P(x_1, x_2)$ (that is just $\Box P(x_1, x_1)$) is a *de dicto* modality.

Here is a list of well known modal formulas

GF $\quad \exists x_{n+1} \Box A \to \Box\exists x_{n+1} A$

D $\quad \Box\langle n+1 : x_1, \ldots, x_n\rangle A \to \langle n+1 : x_1, \ldots, x_n\rangle \Box A$

BF $\quad \forall x_{n+1} \Box A \to \Box\forall x_{n+1} A$

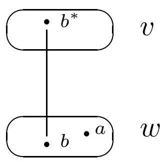
$M \quad \Box\langle n+1 : x_1, x_1, x_2, \ldots, x_{n+2}\rangle A$
$\qquad\qquad \rightarrow \langle n+1 : x_1, x_1, x_2, \ldots, x_{n+2}\rangle \Box A$

$CBF \quad \Box\forall x_{n+1} A \rightarrow \forall x_{n+1} \Box A$

Models and countermodels

We present with a few simple pictures some countermodels to the formulas listed above based on $\mathcal{L}$-frames. The pictures with the comments alongside are self-explanatory. In the pictures the counterpart relation is symmetric.

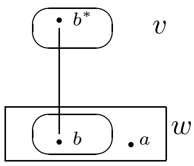
Model I $\quad \mathcal{F} \not\models GF$



$b^* \notin \hat{P}$

$\langle a\rangle \models_w \Box P(x_1), \quad \not\models_w \exists x_1 \Box\, P(x_1) \rightarrow \Box\exists x_1 P(x_1)$

Model II $\quad \mathcal{F} \not\models D$

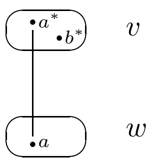


$b^* \notin \hat{P}$

$\langle b, a\rangle \models_w \Box\langle 2 : x_1\rangle P(x_1) \quad \langle b, a\rangle \not\models_w \langle 2 : x_1\rangle \Box P(x_1)$

$\not\models_w \Box\langle 2 : x_1\rangle P(x_1) \rightarrow \langle 2 : x_1\rangle \Box P(x_1)$

Model III $\quad \mathcal{F} \not\models BF$

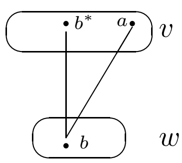


$b^* \notin \hat{P}, \ a^* \in \hat{P}$

$\models_w \forall x_1 \Box P(x_1) \qquad \not\models_v \forall x_1 P(x_1)$

$\not\models_w \forall x_1 \Box P(x_1) \rightarrow \Box\forall x_1 P(x_1)$

Model IV $\quad \mathcal{F} \not\models M$

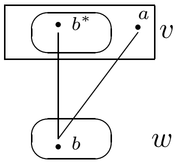


$\langle b^*, b^*\rangle \in \hat{R}, \quad \langle a, a\rangle \in \hat{R}, \quad \langle b^*, a\rangle \notin \hat{R}$

$\langle b\rangle \models_w \Box\langle 1 : x_1, x_1\rangle R(x_1, x_2) \quad \langle b, b\rangle \not\models_w \Box R(x_1, x_2)$

$\langle b\rangle \not\models_w \Box\langle 1 : x_1, x_1\rangle R(x_1, x_2)$
$\qquad\qquad \rightarrow \langle 1 : x_1, x_1\rangle \Box R(x_1, x_2)$

Model V $\quad \mathcal{F} \not\models CBF$



$b^* \in \hat{P} \quad a \notin \hat{P}$

$\langle\rangle \models_v \forall x_1 P(x_1) \qquad \langle b\rangle \not\models_w \Box P(x_1)$

$\langle\rangle \not\models_w \Box\forall x_1 P(x_1) \rightarrow \forall x_1 \Box P(x_1)$

Properties of $\mathfrak{C}$ corresponding to modal formulas[6]

$\mathfrak{C}$ is *totally defined*	iff	if wRv and $a \in D_w$ then there is a $b \in D_v$ such that $a\mathfrak{C}b$	iff	$\mathcal{F} \models GF$
$\mathfrak{C}$ is *u-totally defined*	iff	if wRv and $a \in U_w$ then there is a $b \in U_v$ such that $a\mathfrak{C}b$	iff	$\mathcal{F} \models D$
$\mathfrak{C}$ is *surjective*	iff	if wRv and $b \in D_v$ then there is an $a \in D_w$ such that $a\mathfrak{C}b$	iff	$\mathcal{F} \models BF$
$\mathfrak{C}$ is a *part. function*	iff	if $wRv, a \in U_w, b, c \in U_v$, $a\mathfrak{C}b$ and $a\mathfrak{C}c$ then $b = c$	iff	$\mathcal{F} \models M$
$\mathfrak{C}$ is *preservative*	iff	if wRv, $a \in D_w$, $b \in U_v$ and $a\mathfrak{C}b$ then $b \in D_v$	iff	$\mathcal{F} \models CBF$

Let us prove the 'if' arrows; the 'only if' arrows are trivial. Consider a modal language with a unary predicate letter P and let an $\mathcal{L}$-frame $\mathcal{F}$ be given.

Assume that GF is valid on $\mathcal{F}$. Take an $\mathcal{L}$-model $\mathcal{M}$ based on $\mathcal{F}$, let $w \in W$, $a \in D_w$ and define $I_v(P) = \{b \in D_v : a\mathfrak{C}b\}$, for all v such that wRv. It obtains that $\langle\rangle \models_w \exists x_1 \Box P$. (If $I_v(P) = \emptyset$, then trivially $\langle a\rangle \models_w \Box P(x_1)$.) By hypothesis GF is $\mathcal{L}$-valid, whence $\langle\rangle \models_w \Box\exists x_1 P$. It follows that $\langle\rangle \models_v \exists x_1 P$ for all v such that wRv, whence for some $b \in D_v$, $\langle b\rangle \models_v P$, consequently $I_v(P) \neq \emptyset$, so $\{b \in D_v : a\mathfrak{C}b\} \neq \emptyset$, whence $\mathfrak{C}$ is totally defined.

[6] See [7].

Assume that D is valid on $\mathcal{F}$. Take an $\mathcal{L}$-model $\mathcal{M}$ based on $\mathcal{F}$, let $w \in W$ and $a \in U_w$. For any v, wRv, define $I_v(P) = D_v$ if there is an $a^* \in U_v$ such that $a\mathfrak{C}a^*$; $I_v(P) = \emptyset$, otherwise. Then $\langle a \rangle \models_w \Box\langle 1 :\rangle\forall x_1 P(x_1)$. Therefore via D, $\langle a \rangle \models_w \langle 1 :\rangle\Box\forall x_1 P(x_1)$, so $\langle\,\rangle \models_w \Box\forall x_1 P(x_1)$, hence for any v, wRv, $\langle\,\rangle \models_v \forall x_1 P(x_1)$ and so for any v, wRv, $I_v(P) = D_v$. Consequently, by definition of $I_v(P)$, there is an $a^* \in U_v$ such that $a\mathfrak{C}a^*$, so $\mathfrak{C}$ is u-totally defined.

Assume that BF is valid on $\mathcal{F}$. Take an $\mathcal{L}$-model $\mathcal{M}$ based on $\mathcal{F}$, a world $w \in W$ and for all $v \in W$ such that wRv, define $I_v(P) = \{b \in D_v : a\mathfrak{C}b$ for some $a \in D_w\}$. Then $\langle\rangle \models_w \forall x_1\Box P(x_1)$. Therefore, via BF, $\langle\rangle \models_w \Box\forall x_1 P(x_1)$. Whence for all $v.wRv$. and for all $b \in D_v$, $\langle b \rangle \models_v P(x_1)$. So for all $b \in D_v$ there is an $a \in D_w$ such that $a\mathfrak{C}b$. Consequently $\mathfrak{C}$ is surjective.

Assume that M is valid on $\mathcal{F}$. Let $w \in W$. Take an $\mathcal{L}$-model $\mathcal{M}$ based on $\mathcal{F}$, where $I_v(P^2) = \{\langle a, a \rangle : a \in D_v\}$, for all v such that wRv. It obtains that $\langle a \rangle \models_w \Box\langle 1 : x_1, x_1\rangle P^2(x_1, x_2)$. Whence by M it follows that $\langle a \rangle \models_w \langle 1 : x_1, x_1\rangle\Box P^2(x_1, x_2)$. Therefore $\langle a, a \rangle \models_w \Box P^2(x_1, x_2)$. So for all v, wRv, all a^*, $a^\circ \in D_v$ such that $a\mathfrak{C}a^*$ and $a\mathfrak{C}a^\circ$, $\langle a^*, a^\circ \rangle \models_w P^2(x_1, x_2)$. Therefore $a^* = a^\circ$ by definition of $I_v(P^2)$. Consequently $\{a^* \in U_v : a\mathfrak{C}a^*\}$ either is equal to the empty set or is equal to $\{a^*\}$, so $\mathfrak{C}$ is a partial function.

Assume that CBF is valid on $\mathcal{F}$. Take an $\mathcal{L}$-model $\mathcal{M}$ based on $\mathcal{F}$, let $w \in W$, $a \in D_w$ and define $I_v(P) = D_v$, for all v such that wRv. It obtains that $\langle\rangle \models_w \Box\forall x_1 P(x_1)$. Whence by CBF, $\langle\rangle \models_w \forall x_1\Box P(x_1)$. It follows in particular that, $\langle a \rangle \models_w \Box P(x_1)$; consequently either a has no counterparts in U_v or for all $a^* \in U_v$ such that $a\mathfrak{C}a^*$, $\langle a^* \rangle \models_v P(x_1)$, hence $a^* \in D_v$. Consequently $\mathfrak{C}$ is preservative.

In counterpart semantics validity of BF corresponds to the condition that $\mathfrak{C}$ is surjective, and this condition, as we have seen above, remains the same in Lewis semantics. The situation is different for CBF, in counterpart semantics no condition parallels the property of being 'preservative', CBF seems to be uncontroversial and unassuming. In $\mathcal{C}$-semantics CBF corresponds to the principle that R.Stalnaker calls $QCBF$. "But $QCBF$, a qualified version of the converse of the Barcan formula does seem to be validated without any assumptions about the relationships between the domains of the different possible worlds:

($QCBF$) $\Box\forall\hat{x}\phi \to \forall\hat{x}\Box(Ex \to \phi)$

where E is the predicate of existence (defined as $\exists\hat{y}(x = y)$)). Whatever the relations between the domains, surely if in w it is necessary that everything must satisfy ϕ, then anything that exists in w must satisfy ϕ in every accessible possible world *in which that individual exists*.", see [11], p.18. We can rephrase this quotation by saying ' ... surely if in w it is necessary that everything must satisfy ϕ, then every counterpart of anything that exists in w must satisfy ϕ in every accessible possible world *in which that counterpart exists*.' But this is exactly the meaning of $\Box\forall x\phi \rightarrow \forall x\Box\phi$ in counterpart semantics, consequently CBF is synonymous with $QCBF$.

Let $Q.K$ and $\tau Q.K$ be the formal systems which axiomatize the set of formulas valid on all Tarski-Kripke frames and the set of formulas valid on all counterpart frames, respectively.[7] CBF is a theorem of $Q.K$ and here is its well known proof:

$\forall xA \rightarrow A$	Universal Instantiation
$\Box\forall xA \rightarrow \Box A$	Nec. + Distribution
$\Box\forall xA \rightarrow \forall x\Box A$	$\forall$-Introduction

In $\tau Q.K$ the proof of CBF is one line longer:

$\langle n+1 : x_1, \ldots, x_n\rangle\forall xA \rightarrow A$	Universal Instantiation
$\Box\langle n+1 : x_1, \ldots, x_n\rangle\forall xA \rightarrow \Box A$	Nec. + Distribution
$\langle n+1 : x_1, \ldots, x_n\rangle\Box\forall xA \rightarrow \Box A$	by CD
$\Box\forall xA \rightarrow \forall x\Box A$	$\forall$-Introduction

CD $\quad \langle n+1 : x_1, \ldots, x_n\rangle\Box A \rightarrow \Box\langle n+1 : x_1, \ldots, x_n\rangle A$
(Converse of D)

From a semantical point of view CD says that the worlds where there are counterparts of a given $n+1$-tuple of individuals $a_1, \ldots a_n, a_{n+1}$ are a subset of the worlds where there are counterparts of the n-tuple $a_1, \ldots a_n$. What is noticeable is that CD and CBF are mutually derivable in the presence of the universal instantiation.

Let $\vec{x}$ be $x_1, ..., x_n$ and A be of type $n+1$.

CD implies CBF

$\langle n+1 : \vec{x}\rangle\forall x_{n+1}A \rightarrow A$	Universal Instantiation
$\Box\langle n+1 : \vec{x}\rangle\forall x_{n+1}A \rightarrow \Box A$	Nec.+Distribution
$\langle n+1 : \vec{x}\rangle\Box\forall x_{n+1}A \rightarrow \Box A$	CD
$\Box\forall x_{n+1}A \rightarrow \forall x_{n+1}\Box A$	$\forall$-Introduction

[7] See [2] and [3]. $\tau Q.K$ is called $Q.K^t_*$ in [2].

Now, let $\vec{x}$ be $x_1, ..., x_n$ and A be of type n.

CBF implies CD

$\langle n+1 : \vec{x}\rangle A \rightarrow \langle n+1 : \vec{x}\rangle A$	Taut.
$A \rightarrow \forall x_{n+1}\langle n+1 : \vec{x}\rangle A$	$\forall$-Introduction
$\Box A \rightarrow \Box\forall x_{n+1}\langle n+1 : \vec{x}\rangle A$	Nec.+Distribution
$\Box A \rightarrow \forall x_{n+1}\Box\langle n+1 : \vec{x}\rangle A$	by CBF
$\langle n+1 : \vec{x}\rangle\Box A \rightarrow \langle n+1 : \vec{x}\rangle\forall x_{n+1}$ $\Box\langle n+1 : \vec{x}\rangle A$	Substitut. for Variables
$\langle n+1 : \vec{x}\rangle\Box A \rightarrow \Box\langle n+1 : \vec{x}\rangle A$	by Universal Instantiation

CD is a theorem of $\tau Q.K$ since it is an instance of axiom

$S^{\Box}$ $\qquad \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\Box A \rightarrow \Box\langle m : x_{i_1}, \ldots, x_{i_n}\rangle A,$

so obviously CBF is provable in $\tau Q.K$.

Let us go back to Lewis semantics. Consider the $\mathcal{L}$-frame of model I. The fact that $a \in D_w$ has no counterparts in v does not prevent the validity of CBF, therefore the equation between validity of CBF and 'increasing domains' seems not to hold. The increasing domains condition of $\mathcal{K}$-semantics seems rather to parallel in $\mathcal{L}$-semantics the property of $\mathfrak{C}$ of being totally defined. It is because in Kripke semantics it is assumed from the very beginning that the same individual exists in every related world, i.e. the counterpart relation is the identity relation and it is totally defined, that CBF is intertwined with the increasing domains condition.

3. *The asymmetry between BF and CBF*

The asymmetry between BF and CBF is better seen in the presence of axiom B : $A \rightarrow \Box\Diamond A$.

The system $Q^{\circ}.B$.

The language of $Q^{\circ}.B$ is a standard first-order modal language L and $Q^{\circ}.B$ is given by adding axiom B : $A \rightarrow \Box\Diamond A$ to the system $Q^{\circ}.K$ which is characterized by the class of all Kripke frames. Here are its axioms and rules:

truth-functional tautologies	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
$\forall x_j(\forall x_i A(x_i) \rightarrow A(x_j/x_i))$	$\forall x_i(A \rightarrow B) \rightarrow (\forall x_i A \rightarrow \forall x_i B)$
$\forall x_j\forall x_i A \leftrightarrow \forall x_i\forall x_j A$	$A \rightarrow \forall x_i A$, x_i not free in A

Inference rules : Modus Ponens, Necessitation, Universal Generalization (from A infer $\forall x_i A$).

The system $Q^{\circ}.B$ is valid with respect to the class of symmetric $\mathcal{K}$-frames, but it is unknown if it is also complete with respect to that class. By adding CBF to $Q^{\circ}.B$ we get a system which contains BF among its theorems and which is complete with respect to the class of symmetric $\mathcal{K}$-frames with constant inner domains and constant outer domains.[8]

In the presence of symmetry either validity of the condition that "domains never increase" or validity of the condition that "domains never decrease" leads to the constant domains condition, and so one would expect that adding either CBF or BF would lead to the same system. Instead, by adding BF to $Q^{\circ}.B$ we get a $\mathcal{K}$-incomplete system: CBF is valid on all $\mathcal{K}$-frames for $Q^{\circ}.B + BF$ and still it is not a theorem of $Q^{\circ}.B + BF$.[9]

In order to see this we show how to transform any wff A of L into a wff $\tau_n A$, for some n, of the typed language L^{τ}, such that $Q^{\circ}.B + BF \vdash A$ only if $\tau_n A$ is valid on all symmetric Lewis frames where the counterpart relation is surjective and u-totally defined. Now, take the following instance of CBF: $\Box\forall x_1 P(x_1) \rightarrow \forall x_1 \Box P(x_1)$. As we shall see $\tau_1[\Box\forall x_1 P(x_1) \rightarrow \forall x_1 \Box P(x_1)]$ is $\Box\forall x_2 \langle 2 : x_2\rangle P(x_1) \rightarrow \forall x_2 \langle 2 : x_2\rangle \Box P(x_1)$ and we can easily check that this formula fails on Model V of section 2, in which R is symmetric and $\mathfrak{C}$ is surjective and u-totally defined. Let us check.
$\langle b\rangle \models_w \Box\forall x_2\langle 2 : x_2\rangle P(x_1)$ since $\langle b^*\rangle \models_v \forall x_2\langle 2 : x_2\rangle P(x_1)$ and $\langle a\rangle \models_v \forall x_2\langle 2 : x_2\rangle P(x_1)$, and this is so because $\langle b^*, b^*\rangle \models_v \langle 2 : x_2\rangle P(x_1)$ and $\langle a, b^*\rangle \models_v \langle 2 : x_2\rangle P(x_1)$, in fact $\langle b^*\rangle \models_v P(x_1)$.

But $\langle b\rangle \not\models_w \forall x_2\langle 2 : x_2\rangle \Box P(x_1)$ because $\langle b, b\rangle \not\models_w \langle 2 : x_2\rangle \Box P(x_1)$, because $\langle b\rangle \not\models_w \Box P(x_1)$, since $\langle a\rangle \not\models_v P(x_1)$.
Therefore CBF is not a theorem of $Q^{\circ}.B + BF$.

The rest of the paper is devoted to proving the $\mathcal{K}$-incompleteness of $Q^{\circ}.B + BF$.

4. *$Q^{\circ}.B + BF$ is $\mathcal{K}$-incomplete*

We start by listing some formulas of L^{τ} which are valid on all $\mathcal{L}$-frames.

For any projections $\langle m : x_{i_1}, ..., x_{i_k}\rangle$, $\langle k : x_{j_1}, ..., x_{j_n}\rangle$, and $\langle n : x_{h_1}, ..., x_{h_s}\rangle$ and formulas A, B of type n, C, D of type $n + 1$ and E of type $s + 1$:

[8] See [3]. Here is a proof of BF: $\forall x[\forall x\Box A(x) \rightarrow \Box A(x)]$, $\Box\forall x[\forall x\Box A(x) \rightarrow \Box A(x)]$, $\forall x\Box[\forall x\Box A(x) \rightarrow \Box A(x)]$ by CBF, $\forall x[\Diamond\forall x\Box A(x) \rightarrow \Diamond\Box A(x)]$, $\forall x[\Diamond\forall x\Box A(x) \rightarrow A(x)]$ via B, $\forall x\Diamond\forall x\Box A(x) \rightarrow \forall x A(x)$, $\Diamond\forall x\Box A(x) \rightarrow \forall x A(x)$, $\Box\Diamond\forall x\Box A(x) \rightarrow \Box\forall x A(x)$, $\forall x\Box A(x) \rightarrow \Box\forall x A(x)$, via B.

[9] This result was announced in [3], where Lewis frames are called counterpart Kripke frames.

S^i $\langle n : x_1, ..., x_n\rangle A \leftrightarrow A$
S^S $\langle m : x_{i_1}, ..., x_{i_k}\rangle(\langle k : x_{j_1}, ..., x_{j_n}\rangle A)$
$\leftrightarrow (\langle m : x_{j_1}, ..., x_{j_k}\rangle \circ \langle k : x_{i_1}, ..., x_{i_n}\rangle)A$[10]
$S^\neg$ $\langle k : x_{i_1}, ..., x_{i_n}\rangle(\neg A) \leftrightarrow \neg\langle k : x_{i_1}, ..., x_{i_n}\rangle A$
$S^\vee$ $\langle k : x_{i_1}, ..., x_{i_n}\rangle(A\vee B) \leftrightarrow \langle k : x_{i_1}, ..., x_{i_n}\rangle A\vee\langle k : x_{i_1}, ..., x_{i_n}\rangle B$
$S^\exists$ $\langle k : x_{i_1}, ..., x_{i_n}\rangle(\exists x_{n+1}C) \leftrightarrow \exists x_{k+1}\langle k+1 : x_{i_1}, ..., x_{i_n}, x_{k+1}\rangle C$
$S^\Box$ $\langle k : x_{i_1}, ..., x_{i_n}\rangle\Box B \to \Box\langle k : x_{i_1}, ..., x_{i_n}\rangle B$
τUI° $\forall x_{n+1}(\forall x_{n+2}\langle n+2 : x_{h_1}, ..., x_{h_s}, x_{n+2}\rangle E$
$\to \langle n+1 : x_{h_1}, ..., x_{h_s}, x_{n+1}\rangle E)$
$\forall$-D $\forall x_{n+1}(C \to D) \to (\forall x_{n+1}C \to \forall x_{n+1}D)$
VQ $A \to \forall x_{n+1}\langle n+1 : x_1, \dots, x_n\rangle A$ *Vacuous Quantification*

The following rules preserve $\mathcal{L}$-validity:
Modus ponens, (MP): from A and $A \to B$ infer B,
Necessitation, (N): from A infer $\Box A$.
Generalization, (G): from $\langle n+1 : x_1, \dots, x_{n+1}\rangle C$ infer $\forall x_{n+1}C$,
Substitution for variables, (SV): from A of type n, infer $\langle k : x_{i_1}, ..., x_{i_n}\rangle A$.

A projection $\langle m : x_{i_1}, \dots, x_{i_n}\rangle$ is said to be a *selection* if $\{x_{i_1}, \dots, x_{i_n}\} \subseteq \{x_1, \dots, x_m\}$ and if $j \neq k$ then $x_{i_j} \neq x_{i_k}$. A selection never contains the same variable twice.

A projection $\langle m : x_{i_1}, \dots, x_{i_n}\rangle$ is said to be a *permutation* if it is a selection and $\{x_{i_1}, \dots, x_{i_n}\} = \{x_1, \dots, x_m\}$.

From $S^\Box$ and S^i we get the $\mathcal{L}$-validity of

$$S^p \qquad \langle m : x_{i_1}, \dots, x_{i_m}\rangle\Box A \leftrightarrow \Box\langle m : x_{i_1}, \dots, x_{i_m}\rangle A$$

where $\langle m : x_{i_1}, \dots, x_{i_m}\rangle$ is a permutation.[11]

Let us consider the following generalizations of D and CD, respectively,

$$D^* \qquad \Box\langle m : x_{i_1}, \dots, x_{i_n}\rangle A \to \langle m : x_{i_1}, \dots, x_{i_n}\rangle\Box A$$

[10] Let the complex term $\langle m : x_1, ..., x_n\rangle$ of type $m \to n$ be given. The operation of composition with terms of type n is defined so: if x^n_j is a term of type n, then

$$\langle m : x_{i_1}, ..., x_{i_n}\rangle \circ x^n_j = x^m_{i_j},$$

For any pair of complex terms $\langle m : x_{i_1}, ..., x_{i_n}\rangle$ of type $m \to n$ and $\langle n : x_{j_1}, ..., x_{j_k}\rangle$ of type $n \to k$,
$\langle m : x_{i_1}, ..., x_{i_n}\rangle \circ \langle n : x_{j_1}, ..., x_{j_k}\rangle = \langle m : \langle m : x_{i_1}, ..., x_{i_n}\rangle \circ x_{j_1}, \dots, \langle m : x_{i_1}, ..., x_{i_n}\rangle \circ x_{j_k}\rangle$.

[11] If π is a permutation from m to m there is a permutation π^* from m to m such that $\pi \circ \pi^* = \pi^* \circ \pi = \langle m : x_1, \dots, x_m\rangle$. So $\pi^*\Box\pi A \to \Box\pi^*\pi A$ by $S^\Box$, $\pi^*\Box\pi A \to \Box A$ by S^i, $\pi\pi^*\Box\pi A \to \pi\Box A$ by SV, $\Box\pi A \to \pi\Box A$ by S^i.

and

$$CD^* \qquad \langle m : x_{i_1}, \ldots, x_{i_n}\rangle \Box A \rightarrow \Box\langle m : x_{i_1}, \ldots, x_{i_n}\rangle A$$

where $\langle m : x_{i_1}, \ldots, x_{i_n}\rangle$ is a selection.

CD^* is $\mathcal{L}$-valid since it is a particular instance of $S^\Box$, whereas D^* can be obtained from D and S^i.[12]

As a consequence, the equivalence:

$$DS \qquad \Box\langle m : x_{i_1}, \ldots, x_{i_n}\rangle A \leftrightarrow \langle m : x_{i_1}, \ldots, x_{i_n}\rangle \Box A$$

is valid on all $\mathcal{L}$-frames where $\mathfrak{C}$ is u-totally defined. Validity of DS is going to play a major role in what follows.

The strategy we are going to make use of in the sequel is a refinement of the one introduced in [5] to relate classical logic formalized in a usual first order language to classical logic formalized in a typed language.

Let E be either a term or a formula of L. Define

$$\phi[E] = max_k(x_k \text{ occurs in } E)$$

ϕ counts both the free and the bound variables of E. For any wff A of L and $n \geq \phi(A)$ we define a formula $\tau_n[A]$[13] of L^τ of type n as follows:

$$\begin{array}{lcll}
\tau_n[P^m(x_{i_1}, \ldots, x_{i_m})] & = & P^m(n : x_{i_1}, \ldots, x_{i_m}) & \\
\tau_n[\neg A] & = & \neg\tau_n[A] & \\
\tau_n[\Box A] & = & \Box\tau_n[A] & \\
\tau_n[A * B] & = & \tau_n[A] * \tau_n[B] & * \in \{\vee, \wedge, \rightarrow\} \\
\tau_n[\exists x_i A] & = & \exists x_{n+1}(\langle n{+}1 : x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n\rangle \tau_n[A]) & \\
\tau_n[\forall x_i A] & = & \forall x_{n+1}(\langle n{+}1 : x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n\rangle \tau_n[A]) &
\end{array}$$

For simplicity's sake, we will often write $\langle n+1 : x_1, \ldots, x_{n+1}/_i, \ldots, x_n\rangle$ instead of $\langle n+1 : x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n\rangle$.

By $A(x_{i_1}/x_{k_1}, \ldots, x_{i_n}/x_{k_n})$ we denote the formula obtained by simultaneously substituting x_{i_h} for the free occurrences of x_{k_h}, $i \leq h \leq n$, in A. We use the notation $A(x_{i_1}, \ldots, x_{i_n})$ to denote the formula obtained from A (whose free variables are all among $x_1, \ldots, x_n$) by simultaneously substituting x_{i_1} for $x_1, \ldots, x_{i_n}$ for x_n.

[12] If $\langle m : x_{i_1}, \ldots, x_{i_n}\rangle$ is a selection, there are $x_{i_{n+1}}, \ldots, x_{i_m}$ such that $\langle m : x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}, \ldots, x_{i_m}\rangle$ is a permutation. Since $\langle m : x_{i_1}, \ldots, x_{i_n}\rangle = \langle m : x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}, \ldots, x_{i_m}\rangle \circ \langle m : x_1, \ldots, x_{m-1}\rangle \circ \langle m{-}1 : x_1, \ldots, x_{m-2}\rangle \circ \cdots \circ \langle n{+}1 : x_1, \ldots, x_n\rangle$, D^* obtains.

[13] This definition is taken from [5].

Let A be a wff whose free variables are *among* $x_1, \ldots, x_n$. We say that $(x_{i_1}, \ldots, x_{i_n})$ is an *A-suitable substitution* iff
(i) $(x_{i_1}, \ldots, x_{i_n})$ is a selection,
(ii) if x_k has a bound occurrence in A, then either x_k is different from any of $x_{i_1}, \ldots, x_{i_n}$ or $x_k = x_{i_k}$.[14]

For example, if $\phi[A] \leq n$, then $(x_1, \ldots, x_n)$ is an A-suitable substitution.

Given (i) and (ii), $x_{i_1}, \ldots, x_{i_n}$ are free for $x_1, \ldots, x_n$ in A. Moreover $(x_{i_1}, \ldots, x_{i_n})$ is a B-suitable substitution for any subformula B of A.

In brief, now our aim is to show that model V is "a model for $Q^\circ.B + BF$" in the following sense: for any theorem A of $Q^\circ.B + BF$ and for any $n \geq \phi[A]$, we show that $\tau_n[A]$ is valid on all $\mathcal{L}$-frames where R is symmetric and $\mathfrak{C}$ is u-totally defined and surjective, so $\tau_n[A]$ is valid on model V. Since $\tau_1[\Box \forall x_1 P(x_1) \rightarrow \forall x_1 \Box P(x_1)]$ is not valid on model V, as shown at the beginning of this section, CBF is not a theorem of $Q^\circ.B + BF$.

In the next lemmas we will use the symbol " $\models$ " to denote validity on all $\mathcal{L}$-frames where $\mathfrak{C}$ is u-totally defined, if not otherwise specified.

Lemma *4.1*: *Let A be a wff of L. If $n \geq \phi[A]$, $m \geq n$, $m \geq \phi(x_{i_1}), \ldots, m \geq \phi(x_{i_n})$, and $(x_{i_1}, \ldots, x_{i_n})$ is an A-suitable substitution, then*

$$\models \quad \tau_m[A(x_{i_1}, \ldots, x_{i_n})] \leftrightarrow \langle m : x_{i_1}, \ldots, x_{i_n} \rangle \tau_n[A]$$

Proof. By induction on A. $\models \tau_m[P^n(x_{i_1}, \ldots, x_{i_n})]$ iff $\models P^n(m : x_{i_1}, \ldots, x_{i_n})$ iff $\models \langle m : x_{i_1}, \ldots, x_{i_n} \rangle P^n(n : x_1, \ldots, x_n)]$ iff $\models \langle m : x_{i_1}, \ldots, x_{i_n} \rangle \tau_n[P^n(x_1, \ldots, x_n)]$.

$A = \Box B$.
$\models \tau_m[(\Box B)(x_{i_1}, \ldots, x_{i_n})]$ iff $\models \tau_m[\Box(B(x_{i_1}, \ldots, x_{i_n}))]$ iff $\models \Box \tau_m[(B (x_{i_1}, \ldots, x_{i_n}))]$ iff by induction hyp. $\models \Box \langle m : x_{i_1}, \ldots, x_{i_n} \rangle \tau_n[B]$ iff by DS ($\langle m : x_{i_1}, \ldots, x_{i_n} \rangle$ is a selection, since it is a B-suitable substitution) $\models \langle m : x_{i_1}, \ldots, x_{i_n} \rangle \Box \tau_n[B]$ iff $\models \langle m : x_{i_1}, \ldots, x_{i_n} \rangle \tau_n[\Box B]$.

$A = \exists x_k B$.
$\models \tau_m[(\exists x_k B)(x_{i_1}, \ldots, x_{i_n})]$ iff $\models \tau_m[\exists x_k (B(x_{i_1}, \ldots, x_k/x_k, \ldots, x_{i_n}))]$ iff $\models \exists x_{m+1} \langle m+1 : x_1, \ldots, x_{m+1}/_k, \ldots, x_m \rangle \tau_m[B(x_{i_1}, \ldots, x_k/x_k, \ldots, x_{i_n})]$ iff by induction hyp. $\models \exists x_{m+1} \langle m+1 : x_1, \ldots, x_{m+1}/_k, \ldots, x_m \rangle$

[14] Condition (ii) guarantees that if a selection is expanded with an identical substitution, say x_k/x_k (x_k bound in A), then it remains a selection. For example (x_2) is not a suitable substitution for $(P(x_1) \wedge \exists x_2 Q(x_2))$ because $(\exists x_2 Q(x_2))(x_2)$ is going to be equal to $\exists x_2(Q(x_2)(x_2, x_2))$ and (x_2, x_2) is not a selection any more.

$\langle m : x_{i_1}, \ldots, x_k/_k, \ldots, x_{i_n}\rangle\tau_n[B]$ iff $\models \exists x_{m+1}\langle m{+}1 : x_{i_1}, \ldots, x_{m+1}/_k, \ldots, x_{i_n}\rangle\tau_n[B]$.

$\models \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[\exists x_k B]$ iff
$\models \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\exists x_{n+1}\ (\langle n+1 : x_1, \ldots, x_{n+1}/_k, \ldots, x_n\rangle\tau_n[B])$ iff by $S^{\exists}$
$\models \exists x_{m+1}\langle m+1 : x_{i_1}, \ldots, x_{i_n}, x_{m+1}\rangle(\langle n+1 : x_1, \ldots, x_{n+1}/_k, \ldots, x_n\rangle\tau_n [B])$ iff $\models \exists x_{m+1}\langle m+1 : x_{i_1}, \ldots, x_{m+1}/_k, \ldots, x_{i_n}\rangle\tau_n[B]$.

Whence
$\models \tau_m[(\exists x_k B)(x_{i_1}, \ldots, x_{i_n})] \leftrightarrow \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[\exists x_k B]$.

Corollary *4.2*: *Let A be a pure wff. If $n \geq \phi[A]$, then for any $p \geq n$,*

$$(a) \qquad \models \tau_p[A] \leftrightarrow \langle p : x_1, \ldots, x_n\rangle\tau_n[A]$$

If $n \geq 1$, then

$$(b) \qquad \models \tau_p[A] \quad only\ if \quad \models \tau_n[A]$$

Proof. (a)
$\models \quad \tau_p[A] \leftrightarrow \tau_p[A(x_1, \ldots, x_n)]$
$\models \quad \tau_p[A] \leftrightarrow \langle p : x_1, \ldots, x_n\rangle\tau_n[A]$ by lemma 4.1

(b)
$\models \quad \tau_p[A]$
$\models \quad \tau_p[A]$
$\models \quad \langle p : x_1, \ldots, x_n\rangle\tau_n[A]$ by (a)
$\models \quad \langle n : x_1, \ldots, x_n, \underbrace{x_1, \ldots, x_1}_{(p-n)-times}\rangle\langle p : x_1, \ldots, x_n\rangle\tau_n[A]$
by SV, the proviso that $n \geq 1$ is used
$\models \quad \langle n : x_1, \ldots, x_n\rangle\tau_n[A]$
$\models \quad \tau_n[A]$

Lemma *4.3*: *Let A be any wff. If $n \geq \phi[A]$, $m \geq n$, $m \geq \phi(x_{i_1}), \ldots, m \geq \phi(x_{i_n})$, and $x_{i_1}, \ldots, x_{i_n}$ are free for $x_1, \ldots, x_n$ in A, then*

$$\models \quad \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[A] \rightarrow \tau_m[A(x_{i_1}, \ldots, x_{i_n})]$$

Proof. By induction on A. Analogous to the proof of the previous lemma.

$A = \Box B$. When $\langle m : x_{i_1}, \ldots, x_{i_n}\rangle$ is not a selection, DS doesn't hold anymore, and by using $S^{\Box}$ we can only prove that
$\models \quad \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[\Box B] \rightarrow \tau_m[\Box B(x_{i_1}, \ldots, x_{i_n})]$. To wit,

$\models \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[\Box B]$ iff $\models \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\Box\tau_n[B]$ only if by $S^\Box$ $\models \Box\langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[B]$. Then by induction hypothesis, $\models \Box\tau_m[B(x_{i_1}, \ldots, x_{i_n})]$ therefore $\models \tau_m[\Box B(x_{i_1}, \ldots, x_{i_n})]$.

$A = \exists x_k B$. By inspecting the last paragraph of the proof of lemma 4.1 we soon realize that $\models \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[\exists x_k B]$ iff $\models \exists x_{m+1}\langle m+1 : x_{i_1}, \ldots, x_{m+1}/_k, \ldots, x_{i_n}\rangle\tau_n[B]$. This holds iff, by composition,
$\models \exists x_{m+1}\langle m{+}1 : x_1, \ldots, x_{m+1}/_k, \ldots, x_m\rangle\langle m : x_{i_1}, \ldots, x_k/_k, \ldots, x_{i_n}\rangle\tau_n$ $[B]$, therefore, by induction hypothesis,
$\models \exists x_{m+1}\langle m+1 : x_1, \ldots, x_{m+1}/_k, \ldots, x_m\rangle\tau_m[B(x_{i_1}, \ldots, x_k/x_k, \ldots,$ $x_{i_n})]$, so
$\models \tau_m[\exists x_k[B(x_{i_1}, \ldots, x_k/x_k, \ldots, x_{i_n})]$ whence
$\models \tau_m[(\exists x_k B)(x_{i_1}, \ldots, x_{i_n})]$. Consequently
$\models \langle m : x_{i_1}, \ldots, x_{i_n}\rangle\tau_n[\exists x_k B] \rightarrow \tau_m[(\exists x_k B)(x_{i_1}, \ldots, x_{i_n})]$.

Lemma *4.4*: *Let A be a wff of L and $n = max(1, \phi[A])$.*

If $Q^\circ.K \vdash A$ then $\mathcal{F} \models \tau_n[A]$

for any $\mathcal{L}$-frame $\mathcal{F}$ where $\mathfrak{C}$ is u-totally defined.

If $Q^\circ.B + BF \vdash A$ then $\mathcal{F} \models \tau_n[A]$

for any symmetric $\mathcal{L}$-frame $\mathcal{F}$ where $\mathfrak{C}$ is surjective and u-totally defined.

Proof. The lemma holds also for any $n \geq max(1, \phi[A])$ and the proof is the same. The condition that $n \geq 1$ entitles us to make use of corollary 4.2(b). By induction on the length of the proof of A. Consider axiom UI°: $\forall x_j(\forall x_i A \rightarrow A(x_j/x_i))$ which is the same as $\forall x_j(\forall x_i A \rightarrow A(x_1, \ldots, x_j/x_i, \ldots, x_n))$, where for each $k \neq i$, x_k is replaced by itself.

$\models \tau_n[\forall x_j(\forall x_i A \rightarrow A(x_1, \ldots, x_j/x_i, \ldots, x_n))]$ iff
$\models \forall x_{n+1}\langle n+1 : x_1, \ldots, x_{n+1}/_j, \ldots, x_n\rangle(\tau_n[\forall x_i A] \rightarrow \tau_n[A(x_1, \ldots,$ $x_j/x_i, \ldots, x_n)])$. Let $\sigma = \langle n+1 : x_1, \ldots, x_{n+1}/_j, \ldots, x_n\rangle$, then
$\models \forall x_{n+1}\sigma(\tau_n[\forall x_i A] \rightarrow \tau_n[A(x_1, \ldots, x_j/x_i, \ldots, x_n)])$,
iff
$\models \forall x_{n+1}\,\sigma\,(\forall x_{n+1}\langle n{+}1 : x_1, \ldots, x_{n+1}/_i, \ldots, x_n\rangle\tau_n[A] \rightarrow \tau_n[A(x_1, \ldots,$ $x_j/x_i, \ldots, x_n)])$, IF, by lemma 4.3,
$\models \forall x_{n+1}\,\sigma\,(\forall x_{n+1}\langle n+1 : x_1, \ldots, x_{n+1}/_i, \ldots, x_n\rangle\tau_n[A] \rightarrow \langle n : x_1, \ldots,$ $x_j/_i, \ldots, x_n\rangle\tau_n[A])$ iff
$\models \forall x_{n+1}\,\sigma\,(\forall x_{n+1}\,\langle n+1 : x_1, \ldots, x_{n+1}/_i, \ldots, x_n\rangle\tau_n[A] \rightarrow$
$\langle n+1 : x_1, \ldots, x_n, x_j\rangle\langle n+1 : x_1, \ldots, x_{n+1}/_j, \ldots, x_n\rangle\tau_n[A])$

Let $C = \langle n+1 : x_1, \ldots, x_{n+1}/_j, \ldots, x_n\rangle\tau_n[A]$, then

$\models \forall x_{n+1}\,\sigma\,(\forall x_{n+1}C \rightarrow \langle n : x_1, \ldots, x_n, x_j\rangle C)$ iff
$\models \forall x_{n+1}\,(\sigma\,\forall x_{n+1}C \rightarrow \sigma \circ \langle n : x_1, \ldots, x_n, x_j\rangle C)$

$\models\ \forall x_{n+1}(\langle n+1 : x_1,\ldots,x_{n+1}/_j,\ldots,x_n\rangle \forall x_{n+1}C \rightarrow$
$\langle n+1 : x_1,\ldots,x_{n+1}/_j,\ldots,x_n\rangle\langle n : x_1,\ldots,x_n,x_j\rangle C)$ iff
$\models\ \forall x_{n+1}(\forall x_{n+2}\langle n+2 : x_1,\ldots,x_{n+1}/_j,\ldots,x_n,x_{n+2}\rangle C \rightarrow$
$\langle n+1 : x_1,\ldots,x_{n+1}/_j,\ldots,x_n,x_{n+1}\rangle C)$.

But this formula is $\mathcal{L}$-valid, hence $\tau_n[\forall x_j(\forall x_i A \rightarrow A(x_j/x_i))]$ is valid on all $\mathcal{L}$-frames where $\mathfrak{C}$ is u-totally defined.

Consider axiom $A \rightarrow \forall x_i A$, where x_i does not occur in A.
$\models \tau_n[A] \rightarrow \tau_n[\forall x_i A]$ iff $\models \tau_n[A] \rightarrow \forall x_{n+1}\langle n+1 : x_1,\ldots,x_{n+1}/_i,\ldots, x_n\rangle\tau_n[A]$, iff by lemma 4.1, $\models \tau_n[A] \rightarrow \forall x_{n+1}\tau_{n+1}[A(x_1,\ldots,x_{n+1}/x_i, \ldots,x_n)]$ iff since x_i does not occur in A, $\models \tau_n[A] \rightarrow \forall x_{n+1}\tau_{n+1}[A(x_1, \ldots,x_n)]$ iff by lemma 4.1, $\models \tau_n[A] \rightarrow \forall x_{n+1}\langle n+1 : x_1,\ldots,x_n\rangle\tau_n[A]$, and this formula is $\mathcal{L}$-valid.

Consider $\forall x_i(A \rightarrow B) \rightarrow (\forall x_i A \rightarrow \forall x_i B)$. Let $\sigma = \langle n+1 : x_1,\ldots, x_{n+1}/x_i,\ldots,x_n\rangle$, then $\models \tau_n[\forall x_i(A \rightarrow B)] \rightarrow (\tau_n[\forall x_i A] \rightarrow \tau_n[\forall x_i B])$ iff $\models \forall x_{n+1}\sigma\tau_n[A \rightarrow B] \rightarrow (\forall x_{n+1}\sigma\tau_n[A] \rightarrow \forall x_{n+1}\sigma\tau_n[B])$ iff $\models \forall x_{n+1}\tau_{n+1}\ [A(\sigma) \rightarrow B(\sigma)] \rightarrow (\forall x_{n+1}\tau_{n+1}[A(\sigma)] \rightarrow \forall x_{n+1}\tau_{n+1}[B(\sigma)])$ iff $\models \forall x_{n+1}(\tau_{n+1}[A(\sigma)] \rightarrow \tau_{n+1}[B(\sigma)]) \rightarrow (\forall x_{n+1}\tau_{n+1}[A(\sigma)] \rightarrow \forall x_{n+1} \tau_{n+1}\ [B(\sigma)])$, and this formula is $\mathcal{L}$-valid.

Consider axiom BF: $\forall x_i\Box A \rightarrow \Box\forall x_i A$. $\models \tau_n[\forall x_i\Box A] \rightarrow \tau_n[\Box\forall x_i A]$ iff
$\models \forall x_{n+1}\langle n+1 : x_1,\ldots,x_{n+1}/_i,\ldots,x_n\rangle\tau_n[\Box A] \rightarrow \Box\tau_n[\forall x_i A]$ iff
$\models \forall x_{n+1}\langle n+1 : x_1,\ldots,x_{n+1}/_i,\ldots,x_n\rangle\Box\tau_n[A] \rightarrow$
$\Box\forall x_{n+1}\langle n+1 : x_1,\ldots,x_{n+1}/_i,\ldots,x_n\rangle\tau_n[A]$ iff by DS,
$\models \forall x_{n+1}\Box\langle n+1 : x_1,\ldots,x_{n+1}/_i,\ldots,x_n\rangle\tau_n[A] \rightarrow$
$\Box\forall x_{n+1}\langle n+1 : x_1,\ldots,x_{n+1},\ldots,x_n\rangle\tau_n[A]$, and this formula is valid on all $\mathcal{L}$-frames where $\mathfrak{C}$ is surjective.

As to the rule of Modus Ponens, assume by induction hypothesis that $\models \tau_n[A]$ and $\models \tau_m[A \rightarrow B]$, where $m \geq n$. So $\models \langle m : x_1,\ldots,x_n\rangle\tau_n[A]$ by SV and $\models \tau_m[A]$ by corollary 4.2(a). Moreover
$\models \tau_m[A] \rightarrow \tau_m[B]$, so $\models \tau_m[B]$, and by corollary 4.2(a)
$\models \langle m : x_1,\ldots,x_q\rangle\tau_q[B]$, since $m \geq q$, where $q = max(1,\phi[B])$, then $\models \tau_q[B]$, by corollary 4.2(b).

As to the generalization rule, assume by induction hypothesis that
$\models \tau_n[A]$. By the rule of substitution for the free variables, $\models \langle n+1 : x_1,\ldots,x_{n+1},\ldots,x_n\rangle\tau_n[A]$, so $\models \forall x_{n+1}\langle n+1 : x_1,\ldots,x_{n+1}/_i,\ldots,x_n\rangle \tau_n[A]$, therefore $\models \tau_n\forall x_i[A]$.

5. $Q^\circ.B + BF$ *and* $Q^\circ_=.B + BF$

A consequence of the $\mathcal{K}$-incompleteness of $Q^\circ.B + BF$ is that $Q^\circ.B + BF$ with identity, $Q^\circ_=.B + BF$, is not a conservative extension of $Q^\circ.B + BF$ since CBF is a theorem of $Q^\circ_=.B + BF$.

$Q^\circ_=.B + BF$ is $Q^\circ.B + BF$ plus

REF	$x = x$
$SUBS$	$x = y \to (A(x//z) \to A(y//z))$.

Here are some auxiliary lemmas:

1.	$\vdash_{Q^\circ_=.K}$	$\forall x \exists y(y = x)$		
2.	$\vdash_{Q^\circ_=.K}$	$\exists x A(x) \to \exists x[\exists y(y = x) \wedge A(x)]$		
3.	$\vdash_{Q^\circ_=.K}$	$x = y \to \Box(x = y)$	(Necessity of Identity)	NI
4.	$\vdash_{Q^\circ_=.B}$	$x \neq y \to \Box(x \neq y)$	(Necessity of Distinction)	ND
5.	$\vdash_{Q^\circ_=.B+BF}$	$\exists y(y = x) \to \Box\exists y(y = x)$	(Necessity of Existence)	NE

Proof of 2.:

$\vdash_{Q^\circ_=.K}$	$\forall x \exists y(y = x)$	1.
"	$\exists x A(x) \to \forall x \exists y(y = x) \wedge \neg\forall x \neg A(x)$	
"	$\exists x A(x) \to \neg[\forall x \exists y(y = x) \to \forall x \neg A(x)]$	
"	$\exists x A(x) \to \neg\forall x[\exists y(y = x) \to \neg A(x)]$	
"	$\exists x A(x) \to \exists x[\exists y(y = x) \wedge A(x)]$	

Proof of 4.:

$\vdash_{Q^\circ_=.B}$	$x = y \to \Box(x = y)$	NI
"	$\Diamond(x \neq y) \to (x \neq y)$	
"	$\Box\Diamond(x \neq y) \to \Box(x \neq y)$	via B
"	$(x \neq y) \to \Box(x \neq y)$	

Proof of 5.:

$\vdash_{Q^\circ_=.B+BF}$	$x = y \to \Box(x = y)$	NI
"	$\Diamond(x = y) \to (x = y)$	via B
"	$\exists y \Diamond(x = y) \to \exists y(x = y)$,	
"	$\Diamond\exists y(x = y) \to \exists y(x = y)$	via BF
"	$\Box\Diamond\exists y(x = y) \to \Box\exists y(x = y)$,	
"	$\exists y(x = y) \to \Box\exists y(x = y)$	via B.

Let $A = \Diamond A(x) \wedge \exists y(y = x)$ and $B = \Box\forall y \neg A(y)$.

$\vdash_{Q^\circ_=.B+BF}$	$A \wedge B \to \Diamond A(x) \wedge \exists y(y = x) \wedge \Box\forall y \neg A(y)$	
"	$A \wedge B \to \Diamond A(x) \wedge \Box\exists y(y = x) \wedge \Box\forall y \neg A(y)$	via NE
"	$A \wedge B \to \Diamond[A(x) \wedge \exists y(y = x) \wedge \forall y \neg A(y)]$	
"	$A \wedge B \to \Diamond[A(x) \wedge \neg A(x)]$	

"	$A \wedge B \to \Diamond\bot$	
"	$A \wedge B \to \bot$	
"	$A \to \neg B$	
"	$A \to \Diamond\exists y A(y)$	
"	$\Diamond A(x) \wedge \exists y(y = x) \to \Diamond\exists y A(y)$	
"	$\exists x[\Diamond A(x) \wedge \exists y(y = x)] \to \Diamond\exists y A(y)$	
"	$\exists y \Diamond A(y) \to \Diamond\exists y A(y)$	via 2.

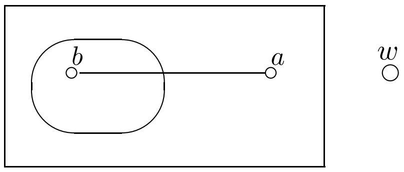
So CBF is a theorem of $Q^{\circ}_{=}.B + BF$.

6. *$\mathcal{K}$-incompleteness of extensions of $Q^{\circ}.B + BF$*

It follows from lemma 4.4 that all the logics $Q^{\circ}.L + BF$, where L is any propositional modal logic valid on the frame of model V are Kripke incomplete. Moreover all the logics $Q^{\circ}.L + BF$ where L is any propositional modal logic such that K + $(\Box A \leftrightarrow A) \supseteq L \supseteq K + T + B$ are Kripke incomplete, in fact model VI below is a model for any such logic and still it falsifies CBF. T is the axiom $\Box A \to A$ corresponding to reflexivity. In particular, $Q^{\circ}.S5 + BF$ is $\mathcal{K}$-incomplete.

The frame of model VI consists of a single reflexive point w, D_w consists of the single individual b and U_w of both individuals a and b. The counterpart relation is reflexive, symmetric (and transitive), u-totally defined and surjective. Therefore model VI is a model for any (consistent) free quantified extension of $K + T + B + BF$.

Model VI $\mathcal{F} \not\models CBF$



$b \in \hat{P}$ $a \notin \hat{P}$

$\langle\rangle \models_w \forall x_1 P(x_1)$ $\langle b\rangle \not\models_w \Box P(x_1)$

$\langle\rangle \not\models_w \Box\forall x_1 P(x_1) \to \forall x_1 \Box P(x_1)$

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REFERENCES

[1] Barcan Marcus, Ruth C., "A Functional Calculus of First Order Based on Strict Implication", *Journal of Symbolic Logic*, 11, 1946, pp. 1–16.

[2] Corsi, Giovanna, "Counterparts and possible worlds. A study on quantified modal logic", *Preprint, Università di Bologna, Dipartimento di Filosofia*, 21, 2001, pp. 1–61.

[3] Corsi, Giovanna, "A unified completeness theorem for quantified modal logics", *The Journal of Symbolic Logic*, 67, 2002, pp. 1483–1510.

[4] Corsi, Giovanna and Ghilardi, Silvio, "Semantical Aspects of Quantified Modal Logic", in C. Bicchieri & M.L. Dalla Chiara (eds), *Knowledge, Belief and Strategic Interaction*, Cambridge UP, 1992, pp. 167–196.

[5] Ghilardi, Silvio, *Modalità e Categorie*, 1990, PhD Thesis, Università degli Studi di Milano.

[6] Ghilardi, Silvio, "Substitution, Quantifiers and Identity in Modal Logic", in E. Morscher & A. Hieke (eds), *New Essays in Free Logic. In Honour of Karl Lambert.*, Kluwer A P, 2001, pp. 87–115.

[7] Ghilardi, Silvio and Meloni, Giancarlo, "Modal and Tense Predicate Logic: Models in Presheaves and Categorical Conceptualization", in *Categorical algebra and its applications*, Springer LNM 1348, 1988, pp. 130–142.

[8] Hughes, George and Cresswell, Max, *A New Introduction to Modal Logic,* Routledge, London, 1996.

[9] Kripke, Saul, "A Completeness Theorem in Modal Logic", *Journal of Symbolic Logic*, 24, 1959, pp. 1–14.

[10] Lewis, David, 'Counterpart Theory and Quantified Modal Logic', *Journal of Philosophy*, 1968, pp. 113–126 [Reprinted in Loux, 1979, pp. 110–128].

[11] Stalnaker, R.C. "The interaction of Modality with Quantification and Identity", in W. Sinnott-Armstrong (ed.), *Modality, Morality and Belief*, Cambridge UP, 1955.