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LEIBNIZ'S AND KANT'S PHILOSOPHICAL IDEAS AND THE DEVELOPMENT OF HILBERT'S PROGRAMME

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Abstract
The aim of this paper is to indicate some connections between Leibniz's and Kant's philosophical ideas on the one hand and Hilbert's and Gödel's philosophy of mathematics on the other. We shall be interested mainly in issues connected with Hilbert's programme and Gödel's incompleteness theorems.

Hilbert's programme — which gave rise to one of the important domains of mathematical logic, i.e., to proof theory, and to one of the three main theories in the contemporary philosophy of mathematics, i.e., to formalism — arose in a crisis situation in the foundations of mathematics on the turn of the nineteenth century.[1] Main controversy centered around the problem of the legitimacy of abstract objects, in particular of the actual infinity. Some paradoxes were discovered in Cantor's set theory but they could be removed by appropriate modifications of the latter. The really embarrassing contradiction was discovered a bit later by Russell in Frege's system of logic.

Hilbert's programme was on the one hand a protest against proposals of overcoming those difficulties and securing the edifice of mathematics by restricting the subject and methods of the latter (cf. Brouwer's intuitionism as well as proposals of L. Kronecker, H. Poincaré, H. Weyl) and on the other an attempt to justify the classical (infinite) mathematics and to save its integrity by showing that it is secure.[2] His attitude can be well characterized

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[1] It is not clear who introduced the name "crisis of the foundations of mathematics" (*Grundlagenkrise der Mathematik*) but it was Hermann Weyl who popularized it through his lecture "Über die neue Grundlagenkrise in der Mathematik" held in Zurich — cf. (Weyl, 1921).

[2] Detlefsen writes that "Hilbert did want to preserve classical mathematics, but this was not for him an end in itself. What he valued in classical mathematics was its efficiency (including its psychological naturalness) as a mean of locating the truths of real or finitary

by his famous (and often quoted) sentence from (1926): "Aus dem Paradies, das Cantor uns geschaffen hat, soll uns niemand vertreiben können" (No one should be able to drive us from the paradise that Cantor created for us).

The problem was stated for the first time by Hilbert in his lecture at the Second International Congress of Mathematicians held in Paris in 1900 (cf. Hilbert, 1901). Among twenty three main problems which should be solved he mentioned there as Problem 2 the task of proving the consistency of axioms of arithmetic (under which he meant number theory and analysis). He has been returning to the problem of justification of mathematics in his lectures and papers (especially in the twenties) where he proposed a method of solving it.[3] One should mention here his lecture from 1901 held at the meeting of Göttingen Mathematical Society in which he spoke about the problem of completeness and decidability. Hilbert asked there, in E. Husserl's formulation: "Would I have the right to say that every proposition dealing only with the positive integers must be either true or false on the basis of the axioms for positive integers?" (cf. Husserl, 1891, p. 445). In a series of lectures in the twenties Hilbert continued to make the problems more precise and simultaneously communicated partial results obtained by himself and his students and fellow researchers: Paul Bernays, Wilhelm Ackermann, Moses Schönfinkel, John von Neumann. One should mention here Hilbert's lectures in Zurich (1917), Hamburg (1922) (cf. Hilbert, 1922), Leipzig (1922), Münster (1925) (cf. Hilbert, 1926), second lecture in Hamburg (1927) (cf. Hilbert, 1927) and the lecture "Problems der Grundlegung der Mathematik" at the International Congress of Mathematicians held in Bologna (1928) (cf. Hilbert, 1929). In the latter Hilbert set out four open problems connected with the justification of classical mathematics which should be solved: (1) to give a (finitist) consistency proof of the basic parts of analysis (or second-order functional calculus), (2) to extend the proof for higher-order functional calculi, (3) to prove the completeness of the axiom systems for number theory and analysis, (4) to solve the problem of completeness of the system of logical rules (i.e., the first-order logic) in the sense that all (universally) valid sentences are provable. In this lecture Hilbert claimed also — wrongly, as

mathematics. Hence, any alternative to classical mathematics having the same benefits of efficiency would presumably have been equally welcome to Hilbert" (cf. Detlefsen, 1990, p. 374).

[3] A good account of the development of Hilbert's views can be found in (Smoryński, 1988); see also (Peckhaus, 1990) where detailed analysis of Hilbert's scientific activity in the field of the foundations of mathematics in the period 1899–1917 can be found, as well as (Detlefsen, 1986) and (Ketelsen, 1994).

it would turn out — that the consistency of number theory had been already proved.[4]

Hilbert's attempts to clarify and to make more precise the programme of justifying classical mathematics were accompanied by a philosophical reflection on mathematics. One can see here the turn to idealism,[5] in particular to Kant's philosophy. In fact Hilbert's programme was Kantian in character. It can be seen first of all in his paper "Über das Unendliche" (1926). He wrote there:

> Finally, we should recall our true theme and draw the net result of our reflections for the infinite. That net result is this: we find that the infinite is nowhere realized. It is neither present in nature nor admissible as a foundation in that part of our thought having to do with the understanding (*in unserem verstandesmäßigen Denken*) — a remarkable harmony between Being and Thought. We gain a conviction that runs counter to the earlier endeavors of Frege and Dedekind, the conviction that, if scientific knowledge is to be possible, certain intuitive conceptions (*Vorstellungen*) and insights are indispensable; logic alone does not suffice. The right to operate with the infinite can be secured only by means of the finite.
>
> The role which remains for the infinite is rather that of an idea — if, following Kant's terminology, one understands as an idea a concept of reason which transcends all experience and by means of which the concrete is to be completed into a totality [...][6]

In Kant's philosophy, ideas of reason, or transcendental ideas, are concepts which transcend the possibility of experience but on the other hand are an answer to a need in us to form our judgements into systems that are complete and unified. Therefore we form judgements concerning an external reality which are not uniquely determined by our cognition, judgements concerning things in themselves. To do that we need ideas of reason.

[4] Only after Gödel published his incompleteness theorems in 1931 did Hilbert come to realize that Ackermann's proof, which he meant here, did not establish the consistency of all of number theory. In fact, Ackermann showed in (1924–25) only the consistency of a fragment of number theory. Cf. also (Ackermann, 1940). Other results of that type were also obtained by J. von Neumann (1927) and J. Herbrand (1931).

[5] Detlefsen says in this context about instrumentalism — cf. his (1986).

[6] English translation after (Detlefsen, 1993).

Kant claimed that space and time as forms of intuition (*Formen der reinen Anschauung*) suffice to justify and to found the notion of potential infinity (the actual infinity was not considered by him). Hilbert indicated a mistake in this approach. Since the actual infinity cannot be justified by purely logical means, Hilbert treated it as an ideal element. Hence his solution to the problem of the actual infinity was in fact of a Kantian character.

In likening the infinite to a Kantian idea, Hilbert suggests that it is to be understood as a regulative rather than a descriptive device. Therefore sentences concerning the infinite, and generally expressions which Hilbert called ideal propositions, should not be taken as sentences describing externally existing entities. In fact they mean nothing in themselves, they have no truth-value and they cannot be the content of any genuine judgement. Their role is rather regulative than descriptive. But on the other hand they are necessary in our thinking. Hence their similarity to Kant's ideas of reason is evident — they play the similar cognitive role. We do not claim that the ideal elements by Hilbert are the same as (are identical with) ideas of reason by Kant. We claim only that they play the same role, i.e., they enable us to preserve the rules of reason (*Verstand*) in a simple, uniform and simultaneously general form. It is achieved just by extending the domain by ideal elements (ideas) (cf. Majer, 1993 and 1993a).

We use ideas of reason and ideal elements in our thinking because they allow us to retain the patterns of classical logic in our reasoning. But the operations of the classical logic can no longer be employed semantically as operations on meaningful propositions — there is nothing in the externally existing reality that would correspond to ideal elements and ideas of reason, in fact they are free creations of our reason and have only "symbolic" meaning. Their meaning can be determined only by an analogy — they cannot be understood as given by intuition (*Anschauung*). Therefore the operations of the classical logic should be understood only syntactically, as operations on signs and strings of sings. Hilbert wrote in (1926):

> We have introduced the ideal propositions to ensure that the customary laws of logic again hold one and all. But since the ideal propositions, namely, the formulas, insofar as they do not express finitary assertions, do not mean anything in themselves, the logical operations cannot be applied to them in a contentual way, as they are to the finitary propositions. Hence, it is necessary to formalize the logical operations and also the mathematical proofs themselves; this requires a transcription of the logical relations into formulas, so that to the mathematical signs we must still adjoin some logical signs, say

&	$\vee$	$\rightarrow$	—
and	or	implies	not

> and use, besides the mathematical variables, $a, b, c, \ldots$, also logical variables, namely variable propositions $A, B, C, \ldots$

Hence the abstracting from meaning of expressions is connected with Hilbert's attempt to preserve the laws of classical logic as laws governing mathematical thinking and reasoning. It is also connected with the distinction between real and ideal propositions according to which real propositions play the role of Kant's judgements of the understanding (*Verstand*) and the ideal propositions the part of his ideas of pure reason. In (1926) Hilbert wrote:

> Kant taught — and it is an integral part of his doctrine — that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure.
>
> As a further precondition for using logical deduction and carrying out logical operations, something must be given in conception, viz., certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else and which requires no reduction. This is the basic philosophy which I find necessary not just for mathematics, but for all scientific thinking, understanding and communicating. The subject mater of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable.

According to this Hilbert distinguished between the unproblematic, 'finitistic' part of mathematics and the 'infinitistic' part that needed justification. Finitistic mathematics deals with so called real propositions, which are completely meaningful because they refer only to given concrete objects. Infinitistic mathematics on the other hand deals with so called ideal propositions that contain reference to infinite totalities.

By Hilbert, analogously as it was by Kant, ideal propositions (and ideal elements) played an auxiliary role in our thinking, they were used to extend our system of real judgements. Hilbert believed that every true finitary proposition had a finitary proof. Infinitistic objects and methods enabled us to give easier, shorter and more elegant proofs but every such proof could be replaced by a finitary one. This is the reflection of Kant's views of the relationship between the ideas of reason and the judgements of the understanding (cf. Kant, 1787, p. 383).

Kant's and Hilbert's ideas on the nature of mathematics and the character of its propositions described above resemble also some ideas of Leibniz. Leibniz's ideas on the nature of mathematics can be characterized as simultaneously platonic and nominalistic. His nominalism was rather ontological than linguistical. He claimed that we can point out mathematical objects such as, for example, geometrical figures, only by our reason. They do not exist anywhere in things of the external world. The mathematics of the Nature does not apply to the substance, to what is ontologically primitive. To the latter applies mathematics of ideas and possibilities, i.e., the real God's mathematics.

Let us return to Hilbert's programme. According to that the infinitistic mathematics can be justified only by finitistic methods because only they can give it security (*Sicherheit*). Hilbert's proposal was to base mathematics on finitistic mathematics via proof theory (*Beweistheorie*).[7] It was planned as a new mathematical discipline in which one studies mathematical proofs by mathematical methods. Its main goal was to show that proofs which use ideal elements in order to prove results in the real part of mathematics always yield correct results. One can distinguish here two aspects: consistency problem and conservation problem.

The consistency problem consists in showing (by finitistic methods, of course) that the infinitistic mathematics is consistent;[8] the conservation problem consists in showing by finitistic methods that any real sentence which can be proved in the infinitistic part of mathematics can be proved also in the finitistic part, i.e., that infinitistic mathematics is conservative over finitistic mathematics with respect to real sentences and, even more, that there is a finitistic method of translating infinitistic proofs of real sentences into finitistic ones. Both those aspects are interconnected.

[7] Later Hilbert named it metamathematics (*Metamathematik*). This name was used by him for the first time in his lecture "Neubegründung der Mathematik" (1922). It is worth noting that the very term "Metamathematik", though in another meaning, appeared already in the nineteenth century in connection with discussions on non-Euclidean geometries. It was constructed in the analogy to the word "Metaphysik" (metaphysics) and had a pejorative meaning. For instance, F. Schultze in (1882) said about "die metamathematischen Spekulationen über den Raum" (metamathematical speculations about the space). B. Erdmann and H. von Helmholtz contributed to the change of the meaning of this term to a positive one.

[8] Note that consistency of the mathematical domain extended by ideal elements (which have only "symbolic" meaning given by analogy and not by intuition (*Anschauung*)) corresponds to the regulative role of the ideas of reason by Kant.

Hilbert's proposal to carry out this programme consisted of two steps. To be able to study seriously mathematics and mathematical proofs one should first of all define accurately the notion of a proof. In fact, the concept of a proof used in mathematical practice is intuitive, loose and vague, it has clearly a subjective character. This does not cause much trouble in practice. On the other hand if one wants to study mathematics as a science — as Hilbert did — then one needs a precise notion of proof. This was provided by mathematical logic. In works of G. Frege and B. Russell (who used ideas and achievements of G. Peano) one finds an idea (and its implementations) of a formalized system in which a mathematical proof is reduced to a series of very simple and elementary steps, each of which consisting of a purely formal transformation on the sentences which were previously proved. In this way the concept of mathematical proof was subjected to a process of formalization. Therefore the first step proposed by Hilbert in realization of his programme was to formalize mathematics, i.e., to reconstitute infinitistic mathematics as a big, elaborate formal system (containing classical logic, infinite set theory, arithmetic of natural numbers, analysis). An artificial symbolic language and rules of building well-formed formulas should be fixed. Next axioms and rules of inference (referring only to the form, to the shape of formulas and not to their sense or meaning) ought to be introduced. In such a way theorems of mathematics become those formulas of the formal language which have a formal proof based on a given set of axioms and given rules of inference. There was one condition put on the set of axioms (and rules of inference): they ought to be chosen in such a way that they suffice to solve any problem formulated in the language of the considered theory as a real sentence, i.e., they ought to form a complete set of axioms with respect to real sentences.

The second step of Hilbert's programme was to give a proof of the consistency and conservativeness of mathematics. Such a proof should be carried out by finitistic methods. This was possible since the formulas of the system of formalized mathematics are strings of symbols and proofs are strings of formulas, i.e., strings of strings of symbols. Hence they can be manipulated finitistically. To prove the consistency it suffices to show that there are not two sequences of formulas (two formal proofs) such that one of them has as its end element a formula φ and the other $\neg\varphi$ (the negation of the formula φ). To show conservativeness it should be proved that any proof of a real sentence can be transformed into a proof not referring to ideal objects.

Having formulated the programme of justification of the classical (infinite) mathematics Hilbert and his students set out to realize it. And they scored some successes — cf., e.g., (Ackermann, 1924–25 and 1940) or (von Neumann, 1927). But soon something was to happen that undermined Hilbert's programme.

In 1930 Kurt Gödel proved two theorems, called today Gödel's incompleteness theorems, which state that (1) arithmetic of natural numbers and all richer formal systems are essentially incomplete provided they are consistent and that (2) no consistent theory containing arithmetic of natural numbers proves its own consistency (cf. Gödel, 1931).

Gödel's results struck Hilbert's programme. We shall not consider here the problem whether they rejected it.[9] We shall rather ask what were the reactions and opinions of Hilbert and Gödel. It turns out that their proposals and solutions were strongly connected with some ideas of Leibniz.

Having learned about Gödel's result (i.e., the first incompleteness theorem) Hilbert "was angry at first, but was soon trying to find a way around it" (cf. Smoryński, 1988). He proposed to add to the rules of inference a simple form of the ω-rule:

$$\frac{\varphi(0), \varphi(1), \varphi(2), \ldots, \varphi(n), \ldots \qquad (n \in \mathsf{N})}{\forall x\, \varphi(x)}.$$

This rule allows the derivation of all true arithmetical sentences but, in contrast to all rules of the first-order logic, it has infinitely many premisses.[10] In Preface to the first volume of Hilbert and Bernays' monograph *Grundlagen der Mathematik* (1934/1939) Hilbert wrote:

> [...] the occasionally held opinion, that from the results of Gödel follows the non-executability of my Proof Theory, is shown to be erroneous. This result shows indeed only that for more advanced consistency proofs one must use the finite standpoint in a deeper way than is necessary for the consideration of elementary formalism.

To be able to indicate some connections between those ideas of Hilbert and some ideas of Leibniz, let us recall that according to Leibniz theorems are either primitive truths, i.e., axioms of a given theory, or propositions (called derived truths) which can be reduced to the primitive ones by means of a proof. A proof consists either of a finite number of steps (in this case one

[9] This problem is discussed, e.g., in (Murawski, 1994 and 1999a).

[10] In fact the rule proposed by Hilbert in his lecture in Hamburg in December 1930 (cf. Hilbert, 1931) had rather informal character (a system obtained by admitting it would be semi-formal). Hilbert proposed that whenever $A(z)$ is a quantifier-free formula for which it can be shown (finitarily) that $A(z)$ is correct (*richtig*) numerical formula for each particular numerical instance z, then its universal generalization $\forall x A(x)$ may be taken as a new premise (*Ausgangsformel*) in all further proofs.

says about finitistic truth or about finitely analytic sentences) or of an infinite number of steps (one deals then with infinitistic truths or with infinitely analytic sentences). Primitive truths are necessary and finitely (directly) analytic, derived truths are either necessary (though not identical), i.e., finitely analytic, or contingent, i.e., infinitely analytic. On the other hand Leibniz claimed that necessary truths possess finite proofs while contingent ones have only infinite proofs. So the difference between them is rather of a practical character and not of a substantial character. A being with unbounded calculating possibilities would be able to decide all the truths directly, hence any truth would be for him necessary. Here is the source and reason for Leibniz's idea of a real logical calculus, *ars combinatoria* (only its fragments are known to us in the form of finitistic systems of logic). This calculus must be infinitistic. So Leibniz allowed infinite elements in reasonings and proofs. Consequently Hilbert's proposal to allow the ω-rule is compatible with Leibniz's ideas and has in fact the Leibnizian character.

An explicit influence of Leibniz on Gödel's reactions on the new situation in logic and the foundations of mathematics after incompleteness results can be also seen. Before we present and discuss some details note that after 1945 Gödel's interests were concentrated almost exclusively on the philosophy of mathematics and on the philosophy in general. He studied works of Kant and Leibniz as well as phenomenological works of Husserl (especially in the fifties). In his *Nachlaß* several notes on the works of those philosophers and on their views were found. Gödel claimed that it was just Leibniz who had mostly influenced his own scientific thinking and activity (Gödel studied Leibniz's works already in the thirties). Hao Wang writes that "Gödel's major results and projects can be viewed as developments of Leibniz's conceptions along several directions" (cf. Wang Hao, 1987, p. 261). Gödel accepted main ideas of Leibniz's monadology, he was interested in a realization of a modified form of *characteristica universalis* (Gödel's incompleteness theorems indicated that the idea cannot be fully realized). Both Leibniz and Gödel were convinced of the meaning and significance of the axiomatic method. Gödel's results indicated the necessity of some changes and modifications in Leibniz's programme, though Gödel was still looking for axiomatic principles for metaphysics from which the whole of knowledge could be deduced (or which at least would be a base of any knowledge). He was searching among others for a method of analyzing concepts which would induce methods allowing to obtain new results. Add also that Gödel first got the idea of his proof of the existence of God in reading Leibniz (cf. Wang Hao, 1987, p. 195).

Having presented the connections between Leibniz and Gödel in general, let us turn now to problems related to the incompleteness theorems. Observe at the beginning that in his first philosophical paper "Russell's Mathematical Logic" (1944) Gödel has turned among others to the question whether (and in which sense) the axioms of *Principia Mathematica* can be considered to be analytic. And he answered that if analyticity is understood as reducibility by explicit or contextual definitions to instances of the law of identity then even arithmetic is not analytic because of its undecidability. He wrote in (1944, p. 150):

> [...] analyticity may be understood in two senses. First, it may have the purely formal sense that the terms occurring can be defined (either explicitly or by rules for eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of this law. In this sense even the theory of integers is demonstrably non-analytic, provided that one requires of the rules of elimination that they allow one actually to carry out the elimination in a finite number of steps in each case.[11]

The inspiration of Leibniz can be easily seen here (compare Leibniz's finitely analytic truths). On the other hand if infinite reduction, with intermediary sentences of infinite length, is allowed (as would be suggested by Leibniz's theory of contingent propositions) then all the axioms of *Principia* can be proved analytic, but the proof would require "the whole of mathematics". Gödel wrote in (1944, pp. 150–151):

> Leaving out this condition by admitting, e.g., sentences of infinite (and non-denumerable) length as intermediate steps of the process of reduction, all axioms of *Principia* (including the axioms of choice, infinity and reducibility) could be proved to be analytic for certain interpretations [...] But this observation is of doubtful value, because the whole of mathematics as applied to sentences of infinite length has to be presupposed in order to prove this analyticity, e.g., the axiom of choice can be proved to be analytic only if it is assumed to be true.

What concerns problems directly connected with the incompleteness results one should note that already in (1931) Gödel wrote explicitly:

> I wish to note expressly that Theorem XI (and the corresponding results for M and A) do not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary

[11] Because this would imply the existence of a decision procedure for all arithmetical propositions. Cf. *Turing 1937*. [Gödel's footnote]

> means of proof is used, and it is conceivable that there exist finitary proofs that cannot be expressed in the formalism of P (or M or A).[12]

And in the footnote 48^a (evidently an afterthought) to (1931) Gödel wrote:

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite [...] while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P). An analogous situation prevails for the axiom system of set theory.[13]

At the Vienna Circle meeting on 15th January 1931 Gödel argued that it is doubtful, "whether all intuitionistically correct proofs can be captured in a *single* formal system. That is the weak spot in Neumann's argumentation".[14]

Gödel suggested that Hilbert's programme may be continued by allowing two principles which can be treated as finitistic, namely (1) the principle of transfinite induction on certain primitive recursive well-orderings, and (2) a notion of computable functions of finite type (i.e., of computable functionals), to which the process of primitive recursion can be extended in a natural way.

The first principle (more exactly, the induction up to the ordinal ε_0) was later applied by Gerhard Gentzen to prove the consistency of the arithmetic of natural numbers (cf. Gentzen, 1936 and 1938). Later many Gentzen style proofs of the consistency of various fragments of analysis and set theory have been given — cf., e.g., the monographs (Schütte, 1960) and (Takeuti, 1975).

The second principle was applied by Gödel in (1958). He considered there the question of how far finitary reasoning might reach. The problem was

[12] Theorem XI states that if P (the system of arithmetic of natural numbers used by Gödel in (1931) and based on the system of *Principia*) is consistent then its consistency is not provable in P; M is set theory and A is classical analysis. English translation according to (Heijenoort, 1967, p. 615).

[13] English translation taken from (Heijenoort, 1967, p. 610). Note that as one of the confirmations of Gödel's thesis expressed here can serve the fact that if, e.g., T is an extension of Peano arithmetic PA and a predicate S of the language L(T) is a satisfaction predicate for the language L(PA) with the appropriate properties then T proves consistency of PA (cf., e.g., Murawski, 1997 and 1999).

[14] Quotation taken from (Sieg, 1988).

considered by him also in (1972) (this paper was a revised and expanded English version of (1958)). Gödel claimed here that concrete finitary methods are insufficient to prove the consistency of elementary number theory and some abstract concepts must be used in addition. He wrote (pp. 271–273):

> P. Bernays has pointed out [...] on several occasions that, in view of the fact that the consistency of a formal system cannot be proved by any deductive procedures available in the system itself, it is necessary to go beyond the framework of finitary mathematics in Hilbert's sense in order to prove the consistency of classical mathematics or even of classical number theory. Since finitary mathematics is defined [...] as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory. [...] By abstract concepts, in this context, are meant concepts which are essentially of the second or higher level, i.e., which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g., proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from a reflection upon the combinatorial (space-time) properties of the symbols representing them, but rather from a reflection upon the *meanings* involved.

And in the footnote b to (1972) Gödel added:

> What Hilbert means by 'Anschauung' is substantially Kant's space-time intuition confined, however, to configurations of a finite number of discrete objects. Note that it is Hilbert's insistence on *concrete* knowledge that makes finitary mathematics so surprisingly weak and excludes many things that are just as incontrovertibly evident to everybody as finitary number theory. E.g., while any primitive recursive definition is finitary, the general principle of primitive recursive definition is not a finitary proposition, because it contains the abstract concept of function. There is nothing in the term 'finitary' which would suggest a restriction to concrete knowledge. Only Hilbert's special interpretation of it makes this restriction.

However he was convinced that a precise definition of concrete finitary method would have to be given in order to establish with certainty the necessity of using abstract concepts. In (1946) he explicitly called for an effort to use progressively more powerful transfinite theories to derive new arithmetical theorems or new theorems of set theory. He wrote (p. 151):

> Let us consider, e.g., the concept of demonstrability. It is well known that, in whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident and justified as those with which you started, and that this process of extension can be iterated into the transfinite. So there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps (or at least all of them which give something new for the domain of propositions in which you are interested) could be described and collected together in some non-constructive way. In set theory, e.g., the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinational and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e., the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory (i.e., any proof involving the concept of truth which I just used) is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets.

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The above considerations indicated some connections between the philosophy of Leibniz and Kant on the one hand and ideas of Hilbert and Gödel concerning the philosophy of mathematics on the other. We showed that the essential assumptions of Hilbert's programme (i.e., the distinction between real and ideal propositions and the conception of the role and meaning of ideal elements in the mathematical knowledge) are connected with some distinctions made by Leibniz and Kant (ideas of reason). On the other hand

Hilbert's and Gödel's proposals how to continue the programme of justification of (infinitary) mathematics after the discovery of the incompleteness phenomenon indicate some influence of Leibniz, namely the admittance, at least in certain contexts, of some of the infinitistic methods and principles.

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REFERENCES

Ackermann, W.: 1924-25, 'Begründung des tertium non datur mittels der Hilbertschen Theorie der Widerspruchsfreiheit', *Mathematische Annalen* 93, 1–36.

Ackermann, W.: 1940, 'Zur Widerspruchsfreiheit der Zahlentheorie', *Mathematische Annalen* 117, 162–194.

Detlefsen, M.: 1986, *Hilbert's Program. An Essay on Mathematical Instrumentalism*, D. Reidel Publ. Comp., Dordrecht-Boston-Lancaster-Tokyo.

Detlefsen, M.: 1990, 'On the Alleged Refutation of Hilbert's Program Using Gödel's First Incompleteness Theorem', *Journal of Philosophical Logic* 19, 343–377.

Detlefsen, M.: 1993, 'The Kantian Character of Hilbert's Formalism'. In: Czermak, J. (Ed.) *Philosophy of Mathematics. Proceedings of the 15th International Wittgenstein-Symposium*, Part 1, Verlag Hölder-Pilcher-Tempsky, Wien 1993, 195–205.

Detlefsen, M.: 1993a, 'Hilbert's Formalism', *Revue Internationale de Philosophie* 47, 285–304.

Gentzen, G.: 1936, 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *Mathematische Annalen* 112, 493–565.

Gentzen, G.: 1938, 'Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie', *Forschung zur Logik und zur Grundlagen der exakten Wissenschaften, New Series* 4, 19–44.

Gödel, K.: 1931, 'Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme. I', *Monatshefte für Mathematik und Physik* 38, 173–198. Reprinted with English translation 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems' in: Gödel, K. *Collected Works*, vol. I, ed. by Feferman, S. *et al.*, Oxford University Press, New York and Clarendon Press, Oxford, 1986, pp. 144–195.

Gödel, K.: 1944, 'Russell's Mathematical Logic'. In: Schilpp, P.A. (Ed.) *The Philosophy of Bertrand Russell*, Northwestern University, Evanston, pp. 123–153. Reprinted in: Gödel, K. *Collected Works*, vol. II, ed. by

Feferman, S. *et al.*, Oxford University Press, New York and Oxford, 1990, pp. 119–141.

Gödel, K.: 1946, 'Remarks Before the Princeton Bicentennial Conference on Problems in Mathematics', 1–4; first published in: Davis, M. (Ed.) *The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable Problems, and Computable Functions*, Raven Press, Hewlett, N.Y., 1965, pp. 84–88. Reprinted in: Gödel, K. *Collected Works*, vol. II, ed. by Feferman, S. *et al.*, Oxford University Press, New York and Oxford, 1990, pp. 150–153.

Gödel, K.: 1958, 'Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes', *Dialectica* 12, 280–287. Reprinted with English translation 'On a Hitherto Unutilized Extension of the Finitary Standpoint' in: Gödel, K. *Collected Works*, vol. II, ed. by Feferman, S. *et al.*, Oxford University Press, New York and Oxford, 1990, pp. 240–251.

Gödel, K.: 1972, 'On an Extension of Finitary Mathematics Which Has Not Yet Been Used', revised and expanded English version of (Gödel, 1958), to have appeared in *Dialectica*, first published in: K. Gödel *Collected Works*, vol. II, ed. by Feferman, S. *et al.*, Oxford University Press, New York-Oxford 1990, pp. 271–280.

Herbrand, J.: 1931, 'Sur la non-contradiction de l'arithmétique', *Journal für die reine und angewandte Mathematik* 166, 1–8; English translation: 'On the Consistency of Arithmetic', in: Heijenoort, J. van (Ed.) *From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, Cambridge, Mass., 1967, pp. 618–628.

Heijenoort, J. van (Ed.): 1967, *From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, Cambridge, Mass.

Hilbert, D.: 1901, 'Mathematische Probleme', *Archiv der Mathematik und Physik* 1, 44–63 and 213–237. Reprinted in: Hilbert, D. *Gesammelte Abhandlungen*, Verlag von Julius Springer, Berlin, Bd. 3, pp. 290–329. English translation: 'Mathematical Problems', in: Browder, F. (Ed.) *Mathematical Developments Arising from Hilbert's Problems*, Proceedings of the Symposia in Pure Mathematics 28, American Mathematical Society, Providence, RI, 1976, pp. 1–34.

Hilbert, D.: 1919–20, *Natur und mathematisches Erkennen*, Vorlesungen, gehalten 1919–1920 in Göttingen; nach der Ausarbeitung von Paul Bernays, herausgegeben von D.E. Rowe, Birkhäuser, Basel-Boston-Berlin 1992.

Hilbert, D.: 1922, 'Neubegründung der Mathematik. Erste Mitteilung', *Abhandlungen aus dem Mathematischen Seminar der Hamburger Universität* 1, 157–177; reprinted in: Hilbert, D., *Gesammelte Abhandlungen*, Bd. 3, Verlag von Julius Springer, Berlin 1935, 157–177.

Hilbert, D.: 1922/23, *Wissen und mathematisches Denken*, Vorlesungsskript WS 1922/23, ausgearbeitet von W. Ackermann, Mathematisches Institut der Universität Göttingen, überarbeiteter Nachdruck 1988.

Hilbert, D.: 1923, 'Die logischen Grundlagen der Mathematik', *Mathematische Annalen* 88, 151–165; reprinted in: Hilbert, D., *Gesammelte Abhandlungen*, Bd. 3, Verlag von Julius Springer, Berlin 1935, 178–191.

Hilbert, D.: 1926, 'Über das Unendliche', *Mathematische Annalen* 95, 161–190; English translation: 'On the Infinite', in: Heijenoort, J. van (Ed.) *From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, Cambridge, Mass., 1967, pp. 367–392.

Hilbert, D.: 1927, 'Die Grundlagen der Mathematik', *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* 6, 65–85. Reprinted in: Hilbert, D. *Grundlagen der Geometrie*, 7th edition, Leipzig 1930. English translation: 'The Foundations of Mathematics', in: Heijenoort, J. van (Ed.) *From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931*, Harvard University Press, Cambridge, Mass., 1967, pp. 464–479.

Hilbert, D.: 1929, 'Problems der Grundlegung der Mathematik'. In: *Atti del Congresso Internationale dei Mathematici, Bologna 3–10 September 1928*, Bologna, vol. I, pp. 135–141. Also in: *Mathematische Annalen* 102, 1–9.

Hilbert, D.: 1930, 'Naturerkennen und Logik', *Die Naturwissenschaften* 18, 959–963; reprinted in: Hilbert, D., *Gesammelte Abhandlungen*, Bd. 3, Verlag von Julius Springer, Berlin 1935, 378–387.

Hilbert, D.: 1931, 'Die Grundlegung der elementaren Zahlentheorie', *Mathematische Annalen* 104, 485–494.

Hilbert, D. and P. Bernays (1934/1939): *Grundlagen der Mathematik*, Springer–Verlag, Berlin, Bd. I 1934, Bd. II 1939.

Husserl, E.: 1891, *Philosophie der Arithmetik*, Husserliana, 1970.

Kant, I.: 1787, *Kritik der reinen Vernunft*. J.H. Hartknoch, Riga (second edition).

Ketelsen, Ch.: 1994, *Die Gödelschen Unvollständigkeitssätze. Zur Geschichte ihrer Entstehung und Rezeption*, Franz Steiner Verlag, Stuttgart.

Majer, U.: 1993, 'Hilberts Methode der Idealen Elemente und Kants regulativer Gebrauch der Ideen', *Kant-Studien. Philosophische Zeitschrift der Kant-Gesellschaft* 84, 51–77.

Majer, U.: 1993a, 'Das Unendliche – eine blosse Idee?', *Revue Internationale de Philosophie* 47, 319–341.

Murawski, R.: 1994, 'Hilbert's Programme: Incompleteness Theorems vs. Partial Realizations'. In: Woleński, J. (Ed.) *Philosophical Logic in Poland*, Kluwer Academic Publishers, Dordrecht-Boston-London, pp. 103–127.

Murawski, R.: 1997, 'Satisfaction Classes – A Survey'. In: Murawski, R. and J. Pogonowski (Eds.) *Euphony and Logos*, Edition Rodopi, Amsterdam–Atlanta, GA, 259–281.

Murawski, R.: 1999, 'Undefinability vs. Definability of Satisfaction and Truth'. In: Woleński, J. and E. Köhler (Eds.), *Alfred Tarski and the Vienna Circle. Austro-Polish Connections in Logical Empiricism*, Kluwer Academic Publishers, Dordrecht-Boston-London, 203–215.

Murawski, R.: 1999a, *Recursive Functions and Metamathematics*, Kluwer Academic Publishers, Dordrecht-Boston-London.

Neumann von, J.: 1927, 'Zur Hilbertschen Beweistheorie', *Mathematische Zeitschrift* 26, 1–46. Reprinted in: *Collected Works*, vol. I: *Logic, Theory of Sets, and Quantum Mechanics*, ed. by A.H. Taub, Pergamon Press, Oxford 1961, pp. 256–300.

Peckhaus, V.: 1990, *Hilbertprogramm und Kritische Philosophie. Das Göttinger Model interdisziplinärer Zusammenarbeit zwischen Mathematik und Philosophie*, Vandenhoeck & Ruprecht, Göttingen.

Schultze, F.: 1882, *Philosophie der Naturwissenschaften*.

Schütte, K.: 1960, *Beweistheorie*, Springer-Verlag, Berlin; English translation: *Proof Theory*, Springer-Verlag, Berlin-Heidelberg-New York 1977.

Sieg, W.: 1988, 'Hilbert's Program Sixty Years Later', *Journal of Symbolic Logic* 53, 338–348.

Smoryński, C.: 1988, 'Hilbert's Programme', *CWI Quarterly* 1, 3–59.

Takeuti, G.: 1975, *Proof Theory*, North-Holland Publ. Comp., Amsterdam.

Turing, A.: 1937, 'On Computable Numbers, with an Application to the Entscheidungsproblem', *Proceedings of the London Mathematical Society* (2) 42, 230–265. Reprinted in: Davis, M. (Ed.) *The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable Problems, and Computable Functions*, Raven Press, Hewlett, N.Y., 1965, pp. 116–154.

Wang, Hao: 1987, *Reflections on Kurt Gödel*, M.I.T. Press, Cambridge, Mass.

Wang, Hao: 1996, *A Logical Journey. From Gödel to Philosophy*, M.I.T. Press, Cambridge, Mass. – London, England.

Weyl, H.: 1921, 'Über die neue Grundlagenkrise der Mathematik', *Mathematische Zeitschrift* 10, 39–79. Reprinted in: Weyl, H. *Gesammelte Abhandlungen*, vol. 2, Springer Verlag, Berlin-New York 1968, 143–180.