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DIRECT PERCEPTION IN MATHEMATICS: A CASE FOR EPISTEMOLOGICAL PRIORITY

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> What is mathematics? What mathematicians do. What is a mathematician? ... More than just somebody who *does* mathematics. ... A mathematician is someone who sees opportunities for doing mathematics that the rest of us miss.
> (Stewart [1998], p. 1)

1. *Introduction*

Arguably under the persisting influence of Plato and Descartes, unearthly ontological and detached foundational issues are virtually all modern philosophy of mathematics has been concerned with until recently. More particularly, questions about existence of mathematical entities have been considered outside the context of mathematical practice and its historical development. Indeed, issues bearing upon the coming about of 'actual' mathematical knowledge have been regarded as matters of at best secondary philosophical relevance, and have been left for others, viz. empirical scientists, to sort out. In the light of this, it appears to us ironic that in recent treatments of mathematical knowledge, cognitive scientists seem to return to the very schemes of reification and internalisation that tied the Platonic-Cartesian philosophical tradition so closely to pure ontological concerns. In this paper, we briefly outline this circumstance, and then propose an alternative to this dominant cognitivist view, with the aim of revitalizing discussion in mathematical epistemology. Thus, in section 2, we consider some contemporary work in the field of cognitive science, as applied to mathematics. The central section 3 then develops the alternative proposal. We conclude our sketch of an alternative in section 4, pointing to some of its promising philosophical net gains. That is, first, that it will allow to take mathematical talk for granted, instead

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of trying to justify or bring down the abstract foundations of its functioning. Second, pure metaphysical speculation will loose its primacy, as the alternative view takes epistemology over ontology.

2. *Cognitive Science (dis)informing Philosophy*

Fuelled by ever improving brain scanning techniques, psychological investigation of mathematical activity has boosted the last couple of years. Its focus has been largely restricted, however, to elementary cognitive operations and their awakening in infants (and hominids), while no secret has been made of its philosophical *a priori*: to give as much plausibility as possible to the idea that (these) mathematical capacities are innate rather than learned.[1] We shall here only be indirectly occupied with calling into question this explicitly nativist strand. Instead, our line of criticism will be primarily aimed at the *internalism* inherent to these cognitive theories of mathematical ability, as reflected in the following line: "An organ specialized in the perception and representation of numerical qualities lies anchored in our brains" (Dehaene [1998], p. 86). We claim that prominent cognitive approaches should not apply this mechanistic picture, even if possibly plausible for some relatively unsophisticated mathematical abilities, across the board. Moreover, we interpret this mechanistic move as a determined effort to 'save' the objectivity of mathematical things before all else, thus bowing to the demands of the Platonic-Cartesian philosophical tradition.

Setting the stage to further discuss this, we remind that modern dismissals of mathematical Platonism are invariably indebted to the one expressed by *Paul Benacerraf* in his 1973 paper "Mathematical Truth". There, mainstream Platonism, which "assimilates the logical form of mathematical propositions to that of apparently similar empirical ones" (Benacerraf [2002], p. 103), a quality indeed praised by the author for reasons of scientific uniformity, is brought down. It is essentially unable to account for mathematical claims as constituting genuine knowledge, Benacerraf holds, for, in view of the utter abstractness of the objects under consideration, it cannot properly meet the requirement of a *causal theory of reference*. Put otherwise, this argument, broadly accepted by naturalistically aimed philosophers, says that the relation between a referring sign and its content must rely on a natural causal relation, impossible for Platonic objects to contribute to. From the stance taken by Benacerraf, it would then appear a refreshing and complementary idea to consider particular regions in the brain innately responsible

[1] Butterworth [2000], Dehaene [1998], Devlin [2001] and Lakoff/Núñez [2000] have all concerned themselves with carrying through (part of) this particular task.

for specific mathematical operations. This would avoid traditional references to an 'elusive' intuition putting us in contact with mathematical heaven.

One of the recently proposed cognitive theories of mathematics, that expounded in Lakoff/Núñez [2000], goes even considerably further than that, in adding a genealogical account of the mathematical mechanisms in terms of activities and practices, such as collecting, separating, etc. On the face of it, this strive to anchor mathematics in practice, or more specifically: to *embody* it in the practitioners, indeed brings about an appealing answer to the enigmatic question "where mathematics comes from". The answer is that it comes from mundane activities, which seems to free, or at least further distance, us from the tangle of ontological concerns of the philosophical tradition. However, in the theory of Lakoff and Núñez, initial emphasis on practice and embodiment is denied further on. More particularly, the concept of embodiment is subjected to a narrowly 'cognitivist' interpretation, taking the main mathematical constructs as resulting from internal mechanisms and representations rather than from practices performed in the world out there. That is, in the end, it is really not bodies or action that matter, but *representations* of bodies enabled by or equated to 'fossilized' cognitive mechanisms (dubbed 'metaphors' by Lakoff and Núñez).

Let us elaborate somewhat. Lakoff and Núñez explicitly call for and then claim to provide with an empirical account of mathematical activity. However, they do not give any independent arguments as to why their purported description of the cognitive mechanisms yielding mathematical knowledge is not itself arbitrary or infected by 'normative' prejudice. At the outset of chapter two, for example, they draw an analogy between mathematical and natural language use, e.g. concerning spatial behavior, claiming that "mathematical ideas [...] are often grounded in everyday experience [... as] ways of mathematizing ordinary ideas" (p. 29). They claim that "research in cognitive linguistics has shown that spatial relations in a given language decompose into conceptual primitives called *image schemas*, and these conceptual primitives appear to be universal" (p. 30). They go on to claim that some of these very concepts, particularly 'containment' and 'orientation', are of truly central importance to mathematics as to language in general. The upshot is "that the neural circuitry we have evolved for other purposes is an inherent part of mathematics, which suggests that embodied mathematics does not exist independently of other embodied concepts used in everyday life" (p. 33), as there are: "centrality, contact, closeness, balance, straightness, and many, many more" (p. 34). In other words: 1) Linguistic structure is grounded in experiences with concrete objects; 2) Practice can have this influence on language because it becomes a template for a 'cognitive structure' — ultimately a neurocognitive mechanism localised in the brain; 3) Mathematics is just another result of this transformation of concrete activities into cognitive mechanisms. Our capacities for arithmetic, for example, flow from neurocognitive mechanisms, which in their turn derive from the practical activities of collecting, grouping and separating concrete objects.

As their story develops, the authors hypothesize together a whole fabric of cognitive mechanisms to account for every possible aspect of mathematics in a speculative, or at least highly hypothetical way. Moreover, this technique is used on an *ad hoc* basis throughout the book, as new metaphorical schemes are introduced all the time to 'explain' the arisal of further, more complicated, mathematical operations and objects. For example, in order to arrive at the concept of zero, a species of "metaphors that introduce elements" (p. 64) are said to be at work. In our view, Lakoff and Núñez undertake but traditional idea analysis, offering rational reconstructions which are then ingeniously 'turned' into cognitive science, by simply relabeling all historical phenomena of idea development as newly emerging 'metaphors'. Next, they reify and 'mechanisize' these metaphors as 'neural connections'.[2] The pressing question is what is being gained with all this, in explanatory terms. Moreover, there is a good reason to think the whole approach might just be a setback from traditional idea analysis. For, by claiming that most of these cognitive mechanisms operate on an unconscious level,[3] Lakoff and Núñez hand us over to a cognitive level totally out of reach and control (a level which, nevertheless, the authors themselves *do* seem to have a theoretical grasp on). By equating cognition with hidden processes, consciousness is turned into a problematic epiphenomenon. For us, this is the world upside down. It remains an open question how such a view can account for the rationality of mathematical development, and especially for the fact that 'hard' thinking and argument seem to be involved, especially when there is change and novelty.

One interpretation of Lakoff and Núñez is that we are faced here, albeit in a different guise, with the same Platonic-Cartesian scheme of reification and that lies at the very origin of the problems struggled with by the tradition. Mathematics now becomes determined by a fixed realm of entities, no longer situated in Plato's heaven, but constituted by the mechanics of the

[2] A few illustrations. The complex image schemas that a nesting of basic metaphors gives rise to are considered to be both perceptual ánd conceptual, and to have, by virtue of their structure, "built-in spatial logics" (p. 31). Conceptual containers are said to be "part of the mind" (p. 32). "Such neural connections [viz. results of the correlations between counting and physical manipulation], we believe, *constitute a conceptual metaphor* at the neural level" (p. 55, our emphasis).

[3] E.g.: "Most of our everyday mathematical understanding takes place without our being able to explain exactly what we understood and how we understood it" (p. 28), or: "This metaphor [add and take away] is so deeply ingrained in our unconscious minds that we have to think twice to realize that numbers are not physical objects and do not literally have a size" (p. 56).

mind: mathematical structure has been moved from heaven into our heads. Humanity appears to loose deliberate control over mathematics to anonymous brain mechanisms. This means that in the end, we are out of touch with the world of mathematics, now not because it's up above in Plato's heaven, but instead because it is buried deep down in ourselves. Against all this, we shall contend that, for one committed to take mathematical practice seriously, mathematical objects and properties should find a place *out there*, in the *real* world. This enterprise can start we think, with picking up an idea of Penelope Maddy, that of *direct perception* in mathematics.

3. *Direct Perception: A Non-Internalist Alternative*

3.1. *Traces*

Penelope Maddy has at a point (in)famously defended that sets, i.e. what she holds to be the basic mathematical entities, are not abstract, but instead have spatio-temporal location, and can be directly perceived as such: "I intend to reject the traditional Platonist's characterization of mathematical objects; I will bring them into the world we know and into contact with our familiar cognitive apparatus" (Maddy [1990], p. 48). More specifically, Maddy has maintained that we literally see sets, thus dauntlessly holding on to mathematical realism, while rejecting abstractness. Enter Steve, an exemplary man searching his refrigerator for two eggs. "He opens the carton and sees, to his relief, three eggs there. My [Maddy's] claim is that Steve has perceived a set of three eggs. [... T]his requires that there be a set of three eggs in the carton, that Steve acquire perceptual beliefs about it, and that the set of eggs participate in the generation of these perceptual beliefs *in the same way* that my hand participates in the generation of my belief that there is a hand before me when I look at it in good light" (op. cit., p. 58, our emphasis).

In passing,[4] Maddy claims this account to be an extension to the field of mathematics of the perceptual theory called 'direct' or 'ecological' realism, originated by *J.J. Gibson*.[5] Quite to our surprise, she has presumed this theory to be mainstream among contemporary cognitive scientists, while, on the contrary, insisting on mental representations, and trying to link these to neural architecture (see the previous section), is in fact the more popular option.[6] The latter difficulty of assessment aside however, there is a more

[4] Maddy [1990], p.50n.

[5] Gibson [1979].

[6] A possible (partial) diagnosis for this confusion is that Maddy blends ecologism with Piaget's representational 'enrichment' theory of perceptional learning as discontinuous or

substantial issue. Namely, in order for sets to be really out there, for Maddy, perceivers alone do not seem to suffice. "As the amount we know about things by perception is very limited" (op. cit., p. 61), the existence of sets is said to be also dependent on *theoretical* considerations (partially) external to the perceiver. "The elementariness of the notion of set, its ease of manipulation, and the immense success of set theory, both as a foundation for other branches of mathematics and as a mathematical theory in its own right, all help to make the set of eggs the most attractive candidate for the role of number-bearer" (op. cit., p. 62). Indeed, this set-theoretic account has more recently been incorporated by Maddy in a full-fledged *foundational* program. It transfers the Quinean naturalist idea of seeing the whole of (natural) science as a going concern to the more limited case of (pure) mathematics. This means the latter's unconditional methodological independence as a discipline is defended, over any external limitations such as e.g. applicability, and over any philosophical 'interferences'. "Mathematical methodology is properly assessed and evaluated, defended or criticized, on *mathematical*, not philosophical (or any other extra-mathematical) grounds" (Maddy [1998], p. 164, our emphasis). The consequences for ontology are thus: "Take the [Quinean] indispensability arguments to provide good reasons to suppose that some mathematical things (e.g. the continuum) exist. Admit, however, that the history of the subject shows the best methods for pursuing the truth about these things are *mathematical* ones, not those of physical science" (Maddy [1997], p. 108, our emphasis). We spot the danger here of a subtle reintroduction, through foundationalism, of the priority of abstract ontology, something we believe could ultimately be to the disadvantage of practice.

That is, taken to its limits, an approach like Maddy's seems capable of leading us straight into to a re-opening of the gap between theory and practice. For what in this position prevents one to subscribe to the idea of a monolithic and unchangeable theory that determines ontology? And with this, we would indeed be back where we started from: stuck with a conception of ontology completely detached from actual practice, with pernicious philosophical consequences such as *fixedness*. As there will be no room left for any development of mathematical practice that is not mirrored in the 'real' ontology, an essential ingredient of genuine practice is simply discarded. Related to this problem is one of *skepticism*. When convinced that, in the end, there is just one correct ontological picture, then the possibility has to be considered that, in practice, all we have done up to now is wander in the dark, never mind the apparent successes of mathematics applied. Let

stage-wise concept formation, while Gibson's is a functionalist 'differentiation' theory of perceptual learning as refinement of the skills in responding to environmental stimuli. (Gibson/Pick [2000], ch. 1) it be clear that we are in complete agreement on both the possibility of direct perception and the idea that the content of perception can be determined by things outside the immediate grasp of the individual observer. Social norms and cultural practices are obvious candidates for such a determining role. Nevertheless, we think there is a danger in severing the firm link between the perceiver's capacities and the determinants of the content that he is perceiving. That is, the perceiver at some level has to be a participant in the social games played, or be involved in a particular culture or history. Without any such tangible link, the gate is swung clear open for the reentrance of pure decontextualized 'theory', which, as we argued, seems directly to lead to Plato's heaven. We are not claiming Maddy defends such an extreme case herself. We just want to point at the fact that nothing in her approach seems to prevent it from arising.

Another contemporary philosopher of mathematics following up upon Gibsonian ideas, but also in a highly implicit way, is *Philip Kitcher*. In his book *The Nature of Mathematical Knowledge* (Kitcher [1984]), he has set out to unfold a project of mathematical naturalism (or, alternatively, empiricism or constructivism), in effect embracing psychological, sociological and historical aspects as epistemologically informative. Kitcher equally challenges the *a priori* conception of what constitutes mathematical knowledge, showing that it requires the individual's temporal contact with both physical and social environment, and that mathematics concerns, in essence, manipulations of the world that are taught to us. The entire body of mathematics is then conceived of by him as a universal idealization of these elementary operations. At this level, the possible compatibility with a fixed ontology, e.g. of Maddian sets, is brought out again, however. Mathematics begins "with the apprehension of those structures that are instantiated in everyday physical phenomena. On the simplest version, we perceive the properties of small concrete sets (that is, sets whose members are physical objects). Mathematics proceeds by systematically investigating the abstract realm, to which our rudimentary perceptual experiences give us initial access" (Kitcher [1988], p.311). But then Kitcher confides that, in the end, "Platonists can simply take over my stories about rational interpractice transitions, regarding those transitions as issuing in the recognition of further aspects of the realm of abstract objects" (l.c.). What we shall be concerned with in the rest of the paper is to return with care to the original Gibsonian intuitions casually appealed to by both Maddy and Kitcher, and show that their straightforward application to the theory of mathematics in fact avoids any possible collapse into Platonism.

3.2. *Elaboration with examples*

The appeal of the Gibsonian approach as applied in the philosophy of mathematics seems to us to be double. First, the concept of direct perception provides the possibility of a robust, but non-abstract realism. Indeed, one of the central Gibsonian claims is that many different kinds of entities and properties, no matter how 'complex' they be from an absolutist perspective, can be immediately or directly perceived. Moreover, this immediate perception is grounded in the fact that the entity or property itself is present in reality *as such*. Thus, assuming its appropriateness as an example of direct perception, any of the sets that hungry Steve might perceive are structures *out there* in concrete reality. This first important Gibsonian idea appears to be fully endorsed by Maddy, and in fact motivates her account as the, or a, proper answer to Benacerraf's challenge about causal contact between man and mathematics. However, there is a second important aspect of the Gibsonian approach, one not so strongly present in Maddy's writings: the idea of the *relational* or *co-constituted* nature of what is perceived in direct perception. It is a fundamental Gibsonian insight that one should always consider the object of perception from a point of view that properly takes into account the capacities and specifics of the perceiver. Thus, though what is perceived is always some structure 'out there' in the world, it makes no sense, from the 'ecological' point of view defined and defended by Gibson, to think about what is 'out there' from an absolutist or nonrelational point of view, one that does not refer whatsoever to a perceiving creature and its capacities for picking up and acting upon these structures 'out there'. It is exactly this dimension of the theory which, when applied to mathematics, has the liberating effect of forbidding absolutist stances like Platonism. In the Gibsonian approach, the two aspects mentioned are the central components accommodating the pivotal notion of 'affordances'. The basic idea is that what is perceived are possibilities for action 'afforded' (made possible) precisely by that which is perceived. Affordances are relational by definition, depending at once on the structure of what is perceived, and on the practical actions the perceiver can bring to bear upon what causes the act of perception.

Examples of affordances can be found on many levels: from the 'drinkability' of liquids, or the 'eatability' of ripe fruits, to 'openability' afforded by doorknobs. An elaboration by means of a slightly less mundane example will, we hope, allow the reader to see the attractiveness of this concept for the philosophy of mathematics. The example comes from a (now rather famous) study by *William Warren* on the perception of (the affordance of) 'climbability' of stairs (Warren [1984]). In it, it was investigated whether people were able to 'directly perceive' whether or not particular stairways were 'climbable' with minimal energy expenditure (that is, without switching from climbing to clambering). The physics behind the story is that people can only climb stairs in a normal fashion if the proportion of the difference in height between stairs to the length of the climber's legs falls within a specific range. Once this proportion exceeds a certain threshold, it is simply no longer possible to maintain bipedal balance when climbing, and one has to start clambering. Warren now asked people (of various length) to assess the 'climbability' of different staircases with varying rising degrees, and found them very accurate indeed in correctly judging this relational property. In other words, people turned out to be able to directly perceive (the affordance of) 'climbability', a structure 'out there', defined by a specific numerical relation between stairway and personal leg length.

This example brings to the fore three important points. Firstly, it illustrates how direct perception is based on a robust realism, because the property detected is a property of reality. However, the example also shows, secondly, that the detected property is intrinsically related to the capacities of the perceivers, who are constrained by such things as locomotion or the having of a specific point of gravity. Outside this context, the detected relation, as a 'pure' proportion, becomes totally arbitrary, a needle in a haystack of a myriad of other contingent properties. Thirdly, affordances are (relatively) *complex*. The affordance defining relation cannot be thought of as an elementary property, but instead consists of an intricate environment/capacity attunement, which seems to necessitate quite some perceptual sophistication. This is conspicuous in our climbability example: its 'mathematical' profile, as a *proportion*, indicates this complexity. Still, perception can be said to be direct, that is, immediate and without the previous need of discursive inference or even much complicated ongoing cognitive activity. The complex nature of the property in the climbability example discourages (often encountered) *a priori* rejections of a Gibsonian approach to mathematics stressing the complex nature of mathematical properties.

Overall, the above discussion should properly suggest how the idea of affordances can be applied to (perception in) mathematics. That is, we propose to consider mathematical entities in terms of the sets of practices they 'afford'. E.g., numbers, in this sense, are what they are because they afford such operations as adding or multiplying. Just as with Gibsonian affordances in general, we would say that mathematical entities or properties can be *directly perceived* by those perceivers familiar with the constitutive operations connected with these entities or properties. Put otherwise, just as fruit of a specific colour might be perceived as 'ripe' by an animal with the appropriate visual system (as well as particular food preferences), numbers can be perceived by a someone possessing particular number-related skills.[7] Thus, in this way, a skilled mathematical perceiver can be said to 'directly perceive'

[7] Our idea is certainly not new. It is related to remarks by Wittgenstein, such as: "In the triangle I can see now *this* as apex, *that* as base. [...] 'Now he's seeing it like *this*, now like *that*' would only be said of someone *capable of making* certain applications of the figure quite

mathematical properties. He 'sees' a set by acknowledging that a number of set theoretical operations are applicable to the configuration at hand, and he 'sees' (or senses, which means 'acknowledges') numbers when applying or discerning the applicability of arithmetical operations. This account is robustly realistic in the sense explained above, without running the danger of collapsing into extreme externalism (i.e., radically separating theory and practice) or Platonism. Indeed the question of what kinds of mathematical properties 'really exist' in an absolute or noncontextual sense looses its urgency, because posing that question reveals a disregard for the fundamental relational nature of mathematical properties-considered-as-affordances. Moreover, this account allows to answer Benacerraf's concerns about causal (in)efficacy, because the effects of mathematical properties on those who perceive and perform activities with them belong to the relational core that *constitutes* the background against which the questions about the existence of these mathematical properties becomes meaningful. Of course mathematical entities can affect our practices, because they acquire ontological momentum *in virtue of* those practices. Finally, the Gibsonian approach has another fundamental advantage on offer. The appropriateness of questions about the existence of entities and properties are relativized with respect to a context of practical abilities, and as the potential for development of abilities is open-ended (to anyone's knowledge), this approach has an uneasy relationship with the idea of a 'definitive' ontology for mathematics. Again, this sets it radically apart from Platonism and any of the extremely externalist epistemologies accommodating it, viz. ultimately allowing (or even inviting) theory to be detached from practice. In the final section we will pick up this aspect of the Gibsonian approach again, and frame it as an endorsement of epistemological or methodological over ontological concerns.

To give further strength to this cognitive theory of open-ended mathematical practice, we first give an illustration of how it is in effect compatible with, indeed stands in support of, one of the most prominent theories in the foundational history of mathematics: that of the genetic method in arithmetic, by *Richard Dedekind* (1831–1916). This theory was provoked by an aporia prevalent in the nineteenth century, namely of how to make sense of the negatives and their roots. The practice of dealing with this type of integers, their 'discovery' if you will, does of course go way back, at least as far as medieval Hindu arithmetic (Kline [1990], ch. 9). What we are particularly considering here though is the systematic, foundational endeavour of *justifying* them as genuine numbers. Interestingly, and contrary to a century later, we should also note that philosophical preoccupation at that

freely. [...] *The substratum of this experience is the mastery of a technique*" (Wittgenstein [1953], part II ch. x, p. 208 in our version of the text).

time was of an epistemological rather than a logical kind, and that, accordingly, the idea of formal proof in the present, rigorous sense was still in the early stages of its development. Now, in a paper the importance of which is widely acknowledged by historians of foundational mathematics today, viz. his *Habilitationsrede* of 1854, Dedekind addressed the problem of the general progression of the numbers, and in response offered a (sketchy) 'rational reconstruction' of how the mathematical community had *de facto* succeeded to pass through all subsequent stages of numerical extension, up to the imaginaries. He evoked, e.g., "how the operation of adding positive numbers gives rise to the operation of subtraction, which in turn *engenders* the negative integers; how multiplication of integers then leads to the reciprocal operation of division, and division to the *creation* of the rational numbers" (Dedekind [1999], p. 755, emphasis added).[8] And so on.[9]

In our terms, what one sees happening here is the growth of ontology through the extension of practice. New mathematical entities acquire critical ontological weight because novel affordances enter the stage: the negatives and their further kin acquire their status as stable mathematical things through their allowing certain sets of operations to be applied upon them. Also, once these practices have gained foothold, it seems natural to us that, to someone familiar with those operations, negatives or imaginaries are considered as 'directly perceived'. Two more features of our approach can be illustrated by this example. The first one is that there is a structure to practices, to affordances, and thereby to mathematics. Mathematics grows as established practice starts transcending its boundaries. Therefore, it makes sense to speak of degrees of elementariness in mathematics. However, such distinctions are always context relative, as practice is without discernible beginnings and without discernible end. Moreover, such distinctions cannot be used to ground for example a distinction between what can be directly perceived (the elementary) versus what cannot (the complex), for, to the skilled perceiver, both are perceptually accessible in the same immediate

[8] The latter formulation is borrowed from the introduction by editor and translator William Ewald.

[9] "The successive repetition of the same multiplication — that is, the formation of a product out of a determinate number of identical factors (which are now rational numbers) — yields, when conceived as a single operation, the concept of exponentiation" (op. cit., p. 758). The division of an exponent, in its turn, "requires us to perform the unique inverse of the original operation of exponentiation, namely, to split a given number into an (also given) number of equal factors. This way of proceeding leads us to new number domains, because the previous domain no longer satisfies the demand for the general applicability of the arithmetical operations; one is thereby compelled to create the irrational numbers (with which the concept of limit appears) and finally also the imaginary numbers" (op. cit., p. 759).

sense. This brings us to the second aspect: open-endedness. The example seems to illustrate that no *a priori* limit can be imposed on the possible growth of practice, affordance and mathematics. Old operations can always form the basis for new operations. Old mathematics continually gives rise to new mathematics (in which it might be retained or transformed).

An additional example, including a further extension of the numerical realm, will allow us to further complicate (and refine) our account: the invention of *quaternions* by *William Rowan Hamilton* (1805–1865).[10] Although merely mentioned in Dedekind [1999], it is remarkable that this development did proceed along the very lines of the scheme set out there. That is, Hamilton literally tried to extend the complex numbers from two- to three-place systems (to finally arrive at his *four*-place quaternion system). According to Pickering [1995], this episode "marked an important turning point in the development of mathematics, involving as it did the first introduction of noncommuting quantities in the subject matter of the field, as well as the introduction of an exemplary *set of new entities and operations*" (p. 120, our emphasis). In the light of what was said above, that might not be very striking, yet on top of that, what does make this case particularly salient is the decisive methodological role played by the pursuit of a one-to-one association between algebra and geometry. Indeed, as Pickering explains, Hamilton was in constant search of proper equivalent geometrical representations of the algebraic hypotheses he came up with, and vice versa. Notably, this eagerness in the end proved vital for reaching his results, as it apparently allowed him to cope with specific problems (conceptual resistances, in Pickering's terminology), i.e. create for himself additional affordances accommodating his further research. This striving for a geometry-algebra correspondence by Hamilton should then be taken as another, higher-level type of cognitive 'handling' or operating upon theoretical configurations already at hand, extending their respective scopes.[11] So, not only did Hamilton act on the available number-types, he was also able to 'handle' larger bodies of mathematics and put them to his further theoretical advantage. The products of the operation-driven extension of mathematics become operative tools themselves, engendering yet further extension. Again, along the road from integers to quaternions, we have gradually passed from elementary, say daily, to abstract and *in se* complex scientific practices. Once more,

[10] For another example in this vein, viz. that of *Georg Cantor* (1845–1918) exploiting the then available language in order to refer to a transfinite ordinal as a number, see Kitcher [1984] (pp. 173–5).

[11] For a contemporary example of proceeding through this type of 'transpositions', see Visser [2001]. On the acknowledgement of their heuristic importance (as analogies), also see the pioneering work by Pólya [1973].

this illustrates quite well that affordances are spread throughout the simple-complex and concrete-abstract spectra, and that, consequently, their 'reality' is of all kinds, and not at all limited to the realm of 'ordinary' medium-sized material objects.

Before summing up the main philosophical merits of our cognitive account, we wish to explicate the affinities it exhibits with category theory, the major alternative (to set theory) in foundational research based on "axiomatizing not elements of sets but functions between sets" (Mac Lane [1985], p. 398).[12] The most important philosophical feature of category theory is its trading in of ontological for structural primacy, its basic axiom being that objects *and* functions together are formative of the basic entities, categories. The essential notion of function gives category theory a distinctly pragmatical dimension, and this is why it has been appealing to practicing mathematicians. In particular, it is *epistemologically* plausible (heuristically simple, analytically self-evident) and *methodologically* valuable (providing conceptual unification without the need for a reductive ontology). Category theory appears to have been vulnerable though to critical remarks about the cognitive underpinning of its epistemological plausibility, i.c. its relying on the elementary notion of operation, without properly explaining how this should be understood. It seems one could dismiss this criticism as unimportant (e.g. given that intuition is as bad a problem for Platonist accounts), or otherwise engage in the metaphysical debate. A forceful and independent line of defence, answering the call for naturalism, however, would be to connect with empirical/cognitive approaches to mathematical activity. As we have explained above, a broadly 'Gibsonian' approach, when applied to mathematics, is a good candidate for implementing such a strategy, outlining a philosophically less 'loaded' account in which a full-fledged empirical theory of mathematical operations could be moulded.[13]

4. *Philosophical Effects: Epistemological Priority*

In his famous essay on the effectiveness of mathematics,[14] *Eugene Wigner* holds that what strikes us as unreasonable or mystical about the applicability of mathematics becomes fairly straightforward if only looked at from the appropriate angle. That is, the perspective provided when one no longer assumes that mathematics fell (or was brought down) from the sky just like

[12] We thank Leon Horsten for first suggesting this link.

[13] This paragraph relies on Marquis [1995] to a considerable extent.

[14] Wigner [1960].

that, but on the contrary was created as an earthly science "of skillful operations and concepts" to begin with, i.e. *in order* to be useful. We share the profound pragmatic concern that is displayed here, and hope that the approach sketched in broad strokes above might be one way in which progress can be made on this path.

From the Platonist's point of view, no doubt, it will be remarked that our approach is *relativistic*, because it 'contextualizes' questions of ontology, and claims to settle them within the realm of practices and practitioners. For Platonism, the very subject matter of ontology is the absolute, i.e. essentially uncontextual, nature of things, which forbids any such move. Our approach, then, would not even start to address ontology. But is not this Platonist criticism incoherent to start with? For if indeed there is anything absolute about mathematics, one should find it in *any* context. Natural numbers might come to mind here (or, why not, sets). But then these aspects will be as discernible from our perspective as from any other! The difference between our approach and the Platonic one really is not about universality, but about whether or not it is possible *in principle* to consider ontological questions completely apart from considerations of practice. Take for example the question whether the complex numbers would exist if there never had been human beings, cognitive organisms or a material universe. To the Platonist such questions do make sense, while we would reject them, precisely because asking them relies on a complete disregard of the (essential) connection with practice.

In conclusion, we see ourselves as joining appeals that "a new *epistemology of mathematics* is needed before confronting ontological questions" (Echeverria [1996], p. 21, original emphasis). If we de-absolutize ontology and consider it to be truly open-ended, it will be, we believe, to the benefit of the further, unbiased exploration of how we come to know mathematically. In other words, to stop demanding any explanatory power from mathematical objects *on themselves* will have liberating effects on the philosophy of mathematics. Cognitive mechanicism is necessarily traded in for epistemological *pragmatism*, whereby the mind is considered as constitut*ed* and constitut*ive* at the same time. This would indeed and at last provide research concerned with development (see Piaget) and non-rigidity (see Lakatos) with proper philosophical room in order to be taken seriously, i.e. at face value.

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REFERENCES

Paul BENACERRAF (2002), "Mathematical Truth", in: Dale Jacquette, editor, *Philosophy of Mathematics. An Anthology*, Blackwell, Malden MA – London, pages 99–109. (revised version of the 1973 original published *The Journal of Philosophy* 70:661–80)

Brian BUTTERWORTH (2000), *The Mathematical Brain*, Papermac, London. (originally published in 1999 by Macmillan)

Richard DEDEKIND (1999), "On the Introduction of New Functions in Mathematics", in: William Ewald, editor, *From Kant to Hilbert. A Source Book in the Foundations of Mathematics Volume II*, Clarendon Press, Oxford, pages 754–62. (first paperback edition of the 1996 original)

Stanislas DEHAENE (1998), *The Number Sense. How the Mind Creates Mathematics*, Allen Lane – The Penguin Press, London – New York. (originally published in 1997 by Oxford University Press, New York)

Keith DEVLIN (2001), *The Maths Gene. Why Everyone Has It, But Most People Don't Use It*, Phoenix, London. (originally published in 2000 by Weidenfeld & Nicolson)

Javier ECHEVERRIA (1996), "Empirical Methods in Mathematics. A Case-Study: Goldbach's Conjecture", in: Gonzalo Munevar, editor, *Spanish Studies in the Philosophy of Science*, Kluwer, Dordrecht – Boston – London, pages 19–55.

J.J. GIBSON (1979), *The Ecological Approach to Visual Perception*, Houghton Mifflin, Boston.

Eleanor J. GIBSON and Anne D. PICK (2000), *An Ecological Approach to Perceptual Learning and Development*, Oxford University Press, New York – Oxford.

Philip KITCHER (1984), *The Nature of Mathematical Knowledge*, Oxford University Press, New York – Oxford.

Morris KLINE (1990), *Mathematical Thought from Ancient to Modern Times (3 Vols.)*, Oxford University Press, New York – Oxford. (reprint of the 1972 original)

Brendan P. LARVOR (2001), "What is Dialectical Philosophy of Mathematics?", *Philosophia Mathematica III 9*(2):212–29.

George LAKOFF and Rafael E. NUNEZ (2000), *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being*, Basic Books, New York.

Saunders MAC LANE (1985), *Mathematics. Form and Function*, Springer Verlag, New York – Berlin.

James J. MADDEN (2001), "Review of Lakoff/Núñez (2000)", *Notices of the AMS 48*(10):1182–8.

Penelope MADDY (1990), *Realism in Mathematics*, Clarendon Press, Oxford.

Penelope MADDY (1997), *Naturalism in Mathematics*, Clarendon Press, Oxford.

Penelope MADDY (1998), “How to Be a Naturalist about Mathematics?”, in: H.G. Dales, G. Oliveri, editors, *Truth in Mathematics*, Clarendon Press, Oxford, pages 161–80.

Jean-Pierre MARQUIS (1995), “Category Theory and the Foundations of Mathematics: Philosophical Excavations”, *Synthese 103*: 421–47.

Andrew PICKERING (1995), *The Mangle of Practice. Time, Agency, and Science*, The University of Chicago Press, Chicago – London.

George POLYA (1973), *Mathematics and Plausible Reasoning. Volume I: Induction and Analogy in Mathematics*, Princeton University Press, Princeton NJ. (eight print of the 1954 original)

Ian STEWART (1998), *The Magical Maze. Seeing the World Through Mathematical Eyes*, Weidenfeld & Nicolson, London. (third print of the 1997 original)

Robert THOMAS (2002), “Idea Analysis of Algebreic Groups. Review of Lakoff/Núñez (2000)”, *Philosophical Psychology 15*(2):185–95.

Jean Paul VAN BENDEGEM (1999), “The Creative Growth of Mathematics”, *Philosophica 63*(1):119–52.

Henk VISSER (2001), “Transpositional Tricks”, *ALP Newsletter 14*(3), available on line at http://www.cwi.nl/projects/alp/.

William WARREN (1984), “Perceiving affordances: Visual guidance of stair climbing” *Journal of Experimental Psychology: Human Perception and Performance 10*, 683–703.

Eugene WIGNER (1960), “The Unreasonable Effectiveness of Mathematics in the Natural Sciences”, *Communications in Pure and Applied Mathematics 13*(1):1–14.

Ludwig WITTGENSTEIN (1953), *Philosophical Investigations*, Basil Blackwell, Oxford. (translated from German by G.E.M. Anscombe)