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IMPLICATIONAL CONVERSES

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Abstract
We look for, and in several cases find, informative characterizations of the deductive relations between formulas φ and ψ necessary and sufficient for them to be respectively equivalent to an implication and its converse in various logics.

1. *Characterizing Converses in Intuitionistic and Classical Logic*

Suppose we have a consequence relation $\vdash$ over some language supporting a binary connective $\to$ we call implication. Then formulas of this language φ and ψ are *explicit implicational converses* if there are formulas χ_0, χ_1, for which φ is the formula $\chi_0 \to \chi_1$ and ψ is the formula $\chi_1 \to \chi_0$; we say that φ and ψ are *implicit implicational converses according to* $\vdash$ when φ and ψ are (according to $\vdash$) equivalent respectively to formulas which are explicit implicational converses. We usually drop the words "implicit" and "implicational". Thus for φ and ψ to be converses according to $\vdash$ is for there to exist formulas χ_0, χ_1, in the language of $\vdash$ for which $\varphi \dashv\vdash \chi_0 \to \chi_1$ and $\psi \dashv\vdash \chi_1 \to \chi_0$. (Here, "$\varphi \dashv\vdash \chi_0 \to \chi_1$" means $\{\varphi\} \vdash \chi_0 \to \chi_1$ and $\{\chi_0 \to \chi_1\} \vdash \varphi$. Similar notational liberties such as those illustrated here will be taken without special mention below.)[1]

We will look for simple conditions which do not involve the existential quantification over formulas (χ_0, χ_1 above) of this definition but are necessary and sufficient for formulas φ and ψ to be implicational converses according to various logics, here identified with consequence relations. In particular, we treat in this section the cases of intuitionistic and classical

[1] If, as is frequently done, we identify the proposition expressed by φ (relative to some fixed $\vdash$) as the set of all φ' such that $\varphi' \dashv\vdash \varphi$, then because the various converses (according to $\vdash$) are not in general equivalent ($\dashv\vdash$) to each other, there is no such thing as "the" converse of the proposition expressed by $\chi_0 \to \chi_1$. This point is well known in philosophical circles — *cf.* note 3 on p. 268 of [Ur]. We might call the occasionally encountered assumption of uniqueness here the Converse Proposition Fallacy.

logic, taken as the consequence relations (assumed familiar) $\vdash_{\mathrm{IL}}$ and $\vdash_{\mathrm{CL}}$ usually associated with those logics. For the latter case, we know that since $\vdash_{\mathrm{CL}} (\chi_0 \to \chi_1) \vee (\chi_1 \to \chi_0)$, the disjunction of any implicational converses will be classically provable (a consequence of the empty set, that is). As a corollary to our treatment of the former case, we shall see that this condition is not only necessary but also sufficient also for a pair of formulas to be implicational converses. Many further such characterizations of special relations between formulas may be found in [Hu4], and some also in [Hu2]. The notion of a "special relation" will be defined in Section 2 below, where we also give (following Proposition 2.2) a general definition of what we have in mind under the heading of "characterizing consequences", after we have familiarized ourselves with some examples.

Proposition 1.1: *Formulas φ and ψ are implicational converses according to $\vdash_{\mathrm{IL}}$ if and only if $\varphi \to \psi \vdash_{\mathrm{IL}} \psi$ and $\psi \to \varphi \vdash_{\mathrm{IL}} \varphi$.*

Proof. 'If': Here we write $\vdash_{\mathrm{IL}}$ as $\vdash$. Suppose $\varphi \to \psi \vdash \psi$ and $\psi \to \varphi \vdash \varphi$. Then by well known properties of intuitionistic logic we have $\varphi \to \psi \dashv\vdash \psi$ and $\psi \to \varphi \dashv\vdash \varphi$, so φ and ψ are implicational converses, taking χ_0, χ_1, as φ, ψ. 'Only if': Suppose that there are χ_0, χ_1, with $\varphi \dashv\vdash \chi_0 \to \chi_1$ and $\psi \dashv\vdash \chi_1 \to \chi_0$. Using the fact that (regardless of the choice of χ_0, χ_1) we have (i) $(\chi_0 \to \chi_1) \to (\chi_1 \to \chi_0) \dashv\vdash \chi_1 \to \chi_0$ and (ii) $(\chi_1 \to \chi_0) \to (\chi_0 \to \chi_1) \dashv\vdash \chi_0 \to \chi_1$ we get $\varphi \to \psi \vdash_{\mathrm{IL}} \psi$ and $\psi \to \varphi \vdash_{\mathrm{IL}} \varphi$.

In the terminology of [Hu2], the condition given here for φ and ψ to be implicational converses in intuitionistic logic would be expressed by saying that each of these formulas 'anticipates' the other (according to $\vdash_{\mathrm{IL}}$).

Since the properties of intuitionistic logic alluded to or stated in the above proof are all properties of classical logic, we get

Corollary 1.2: *Formulas φ and ψ are implicational converses according to $\vdash_{\mathrm{CL}}$ if and only if $\vdash_{\mathrm{CL}} \varphi \vee \psi$.*

Proof. Directly copying the proof of Proposition 1.1, we get as a necessary and sufficient condition that $\varphi \to \psi \vdash_{\mathrm{CL}} \psi$ and $\psi \to \varphi \vdash_{\mathrm{CL}} \varphi$, but the two parts here are equivalent to each other and to the condition mentioned in the Corollary (that $\vdash_{\mathrm{CL}} \varphi \vee \psi$).

Letting $\vdash_{\mathrm{LC}}$ be the consequence relation of the intermediate logic LC (see [Du]) we have the same characterization here also, which is not surprising in view of the fact that not only for $\vdash = \vdash_{\mathrm{CL}}$ but also for $\vdash = \vdash_{\mathrm{LC}}$, we have $\vdash (\chi_0 \to \chi_1) \vee (\chi_1 \to \chi_0)$.

Corollary 1.3: *Formulas φ and ψ are implicational converses according to $\vdash_{\mathrm{LC}}$ if and only if $\vdash_{\mathrm{LC}} \varphi \vee \psi$.*

Proof. The argument given for Corollary 1.2 works equally well here, in view of the LC-equivalence of $\varphi \vee \psi$ with $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

Corollary 1.4: *For any $\vdash \supseteq \vdash_{\mathrm{IL}}$, for any formulas φ, ψ, we have: ψ is an implicational converse of an implicational converse of φ, according to $\vdash$.*

Proof. We use the notation $\top$ here for a truth constant, which, if not primitive, may be taken as abbreviating $p \to p$ (p some sentence letter). Since $\top \to \varphi \vdash_{\mathrm{IL}} \varphi$ and $\varphi \to \top \vdash_{\mathrm{IL}} \top$, $\top$ is an implicational converse of φ. Similarly in the case of ψ.

As the proof shows, writing $\mathrm{C}(\varphi, \psi)$ for "φ is an implicational converse of ψ", we have not just $\forall\varphi\forall\psi\exists\delta(\mathrm{C}(\varphi, \delta) \wedge \mathrm{C}(\delta, \psi))$ or even just the $\forall\exists\forall$ version of this, but the still stronger $\exists\forall\forall$ form (taking $\top$ as the promised δ once and for all).

Some aspects of the discussion to this point echo observations in [MM]. Specifically, the $\vdash$-direction of the equivalence

$$(\chi_0 \to \chi_1) \to (\chi_1 \to \chi_0) \dashv\vdash \chi_1 \to \chi_0$$

appearing as (i) in the proof of the 'only if' half of Proposition 1.1 attracts Meyer and Martin's critical attention, as an example of an intuitively implausible principle in an implicational logic, and they note its presence in intuitionistic and classical logic. We need not go into their objections to the principle, which are of some interest but are not pertinent to our own rather different theme — that of finding necessary and sufficient conditions stated in terms of the logical relations that φ and ψ must exhibit for them to be implicational converses. However, two points made in [MM] bear directly on the further development of this theme. Specifically, they note that the implicational formula encoding the $\vdash$-direction of the above equivalence:

$L \qquad ((q \to p) \to (p \to q)) \to (p \to q)$

is respectively provable and unprovable in the logics RM and R.[2] (L was used in Łukasiewicz's axiomatization of his infinite-valued logic and appears under the name (L) in such places as [BF1], [BF2], *q.v.* for further details — details which permit one to think of "L" as standing for "linear"

[2] See §8.15 and §29.3 (by Meyer) of [AB] for the details of RM.

— i.e. total — ordering, rather than as a corrupted version of "Ł".) Let us write "$\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{R}} \psi$" to mean that the implication $\varphi_1 \to (\varphi_2 \to \ldots \to (\varphi_n \to \psi) \ldots)$ is provable in R, with an analogous understanding in the case of RM. We do so for visual convenience, notwithstanding the fact that, so defined, $\vdash_{\mathsf{R}}$ and $\vdash_{\mathsf{RM}}$ are not consequence relations. In the case of $\vdash_{\mathsf{R}}$, this is because the correctness of a claim to the effect that, $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{R}} \psi$, does not guarantee that of the claim that $\chi_1, \ldots, \chi_m \vdash_{\mathsf{R}} \psi$ whenever $\{\varphi_1, \ldots, \varphi_n\} = \{\chi_1, \ldots, \chi_m\}$: so here we are using the "$\vdash$" notation for something which is not even a relation between sets of formulas and formulas. (As is well known, we need to think of the left-hand side here as representing not a set but a multiset of formulas.) This point does not arise in the case of $\vdash_{\mathsf{RM}}$, which is still not a consequence relation, however, since, $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{RM}} \psi$ does not imply $\varphi_1, \ldots, \varphi_n, \varphi_{n+1} \vdash_{\mathsf{RM}} \psi$. Now the RM-provability of L may seem to take us not very far in the direction of replicating Proposition 1.1 for this logic, since we also need its converse, which is a special case (proper substitution instance) of the RM-unprovable

$K \qquad p \to (q \to p)$

As is remarked several times in [AB], as well as in [MM], however, this special case is already provable even in R:

Special K $\qquad (p \to q) \to ((q \to p) \to (p \to q))$

so in fact the same reasoning as given in the proof of Proposition 1.1 establishes

Proposition 1.5: *Formulas φ and ψ are implicational converses according to* RM *if and only if $\varphi \to \psi \vdash_{\mathsf{RM}} \psi$ and $\psi \to \varphi \vdash_{\mathsf{RM}} \varphi$.*

We recall that the implicational fragments of intuitionistic and relevant logic are, in C.A. Meredith's terminology, *BCKW* logic and *BCIW* logic, and that logics lacking either the 'thinning' principle K (as above) or the contraction schema W (below) are known as substructural logics.

$W \qquad (p \to (p \to q)) \to (p \to q)$

Although we have characterized converses for one logic — RM — which (or whose implicational fragment) is substructural in this sense, we devote a separate section to a consideration of some other such logics. Since the Meredith labels indicate axiom sets for there logics when the rules of Uniform Substitution and Modus Ponens are employed as rules — and all logics to be considered here are assumed to be closed under these rules, as well as to contain the formula I below — to keep the discussion relatively self-contained we list the remaining candidate axioms here also:

B	$(q \to r) \to ((p \to q) \to (p \to r))$
C	$(p \to (q \to r)) \to (q \to (p \to r))$
I	$p \to p$

2. *Characterizing Converses in Some Substructural Logics*

Returning to the case of R, in which we did not have L provable, and so could not duplicate the proof of Proposition 1.1 for that case, let us begin by backing up and looking at a more general version of the strategy of proof. Let α be any formula constructed from $p \to q$ and $q \to p$ with the aid of $\to$, and let α^* be the result of interchanging $p \to q$ and $q \to p$ in α (equivalently: the result of simultaneously substituting p for q and q for p in α). Then if a logic $\vdash$ provides any equivalence of the form $\alpha \to \alpha^* \dashv\vdash p \to q$ we can pursue the style of proof employed for Proposition 1.1 to obtain a characterization of converses according to $\vdash$. We illustrate this with a representative example. (In the case of Proposition 1.1 α and α^* were respectively just $p \to q$ and $q \to p$.) Suppose that for a given $\vdash$ we have α as the following (= the formula Special K above):

$$(p \to q) \to ((q \to p) \to (p \to q)). \tag{1}$$

Then the hypothesis concerning $\alpha \to \alpha^*$ takes the following form, in which to save space we have omitted all but the main "$\to$" in favour of simple concatenation:

$$((pq)((qp)(pq))) \to ((qp)((pq)(qp))) \dashv\vdash p \to q \tag{2}$$

We can then say:

Any formulas φ, ψ, are converses according to $\vdash$ if and only if (again omitting subordinate $\to$'s)

$$(\varphi(\psi\varphi)) \to (\psi(\varphi\psi)) \dashv\vdash \varphi \textit{ and } (\psi(\varphi\psi)) \to (\varphi(\psi\varphi)) \dashv\vdash \psi \tag{3}$$

The "if" part of this claim is clear, since (3) exhibits φ and ψ as equivalent to formulas (on the left hand sides) which are explicit converses. For the "only if" part, suppose that φ and ψ are converses according to $\vdash$, i.e., that there exist χ_0, χ_1, with $\varphi \dashv\vdash \chi_0 \to \chi_1$ and $\psi \dashv\vdash \chi_1 \to \chi_0$. Then substituting χ_0, χ_1, for p, q, respectively, and then for q, p, respectively, in (2), we get

(4) and then (5):

$$((\chi_0\chi_1)((\chi_1\chi_0)(\chi_0\chi_1))) \to ((\chi_1\chi_0)((\chi_0\chi_1)(\chi_1\chi_0))) \dashv\vdash \chi_0 \to \chi_1 \quad (4)$$
$$((\chi_1\chi_0)((\chi_0\chi_1)(\chi_1\chi_0))) \to ((\chi_0\chi_1)((\chi_1\chi_0)(\chi_0\chi_1))) \dashv\vdash \chi_1 \to \chi_0 \quad (5)$$

which is to say, given the equivalences $\varphi \dashv\vdash \chi_0 \to \chi_1$ and $\psi \dashv\vdash \chi_1 \to \chi_0$, that we have (3), as was to be shown.

The general situation illustrated by this example and by Proposition 1.1 is best described by isolating a formula of which α and α^* are both substitution instances. We shall call such a formula β. That is, β is any implicational formula constructed from two propositional variables, and α is the result of substituting $p \to q$ and $q \to p$ for those two variables, while α^* is the result of substituting for them (in the same order) $q \to p$ and $p \to q$. Thus β in the our representative example just worked through may be taken as $r \to (s \to r)$, for example or indeed as $p \to (q \to p))$. For definiteness, let us stipulate that the variables in β are to be r and s, and we write $\beta(\varphi, \psi)$ for the result of substituting φ for r and ψ for s in β. Then the general method we have been describing can be summarised as follows, and the proof is simply a matter of following the steps we have just gone through using schematic rather than specific formulas.

Proposition 2.1: *Given* $\vdash$ *and* β *a two-variable formula as above, with* $\alpha = \beta(p \to q, q \to p)$ *and* $\alpha^* = \beta(q \to p, p \to q)$*, suppose that* $\alpha \to \alpha^* \dashv\vdash p \to q$*. Then a necessary and sufficient condition for arbitrary formulas* φ *and* ψ *to be converses according to* $\vdash$ *is that* $\beta(\varphi, \psi) \dashv\vdash \varphi$ *and* $\beta(\psi, \varphi) \dashv\vdash \psi$*.*

We are now in a position to ask about the existence of an implication of the form $\alpha \to \alpha^*$ of the above form, provably equivalent in R to $p \to q$. By Proposition 2.1, such an implication would yield a characterization of converses for R. Meyer shows in [M2], however, that no such implication can be found. The proof uses the four-element matrix for the implicational fragment of R appearing on the left of the figure on p. 450 of [M1]; here we simply state the result:

Proposition 2.2: (Meyer) *For no formula* α *constructed from* $p \to q$ *and* $q \to p$ *using* $\to$*, with* α^* *constructed similarly from* $q \to p$ *and* $p \to q$*, do we have* $\alpha \to \alpha^*$ *and* $p \to q$ *provably equivalent in* R.

Since the implicational fragment of R is *BCIW* logic, we note that *a fortiori* the still weaker *BCI* logic certainly affords no α constructed from $p \to q$,

$q \to p$, with $\alpha \to \alpha^*$ provably equivalent to $p \to q$. In fact, using a different consideration, we can see that no such $\alpha \to \alpha^*$ provably implies or is provably implied by $p \to q$ in *BCI* logic, since the provable formulas of the latter logic all become two-valued tautologies when the $\to$ is interpreted as $\leftrightarrow$, whereas this is not so for either of

$$(p \to q) \to (\alpha \to \alpha^*) \text{ or } (\alpha \to \alpha^*) \to (p \to q)$$

since in view of the way α and α^* are constructed, $\alpha \to \alpha^*$ has an even number of occurrences of p in it and an even number of occurrences of q, giving the above formulas altogether an odd number of occurrences of these variables and so violating — twice over — a well-known condition (due to Leśniewski) of provability for purely equivalential formulas of classical logic. (See [AB], p. 84.) This last fact shows that Proposition 2.2 by itself does not tell us that there is no characterization of converses available for R, or by the same reasoning there would be no such characterization for classical equivalential logic. The talk of there being no characterization of converses available for a given logic S is meant simply as denying the existence of a set of formulas $\Sigma(p, q)$ in at least the two variables indicated, for which it holds that for all formulas φ, ψ : φ and ψ are converses according to S iff $\Sigma(\varphi, \psi) \subseteq \mathsf{S}$, where $\Sigma(\varphi, \psi)$ is the set of formulas resulting from substituting φ and ψ respectively for p and q in the formulas in $\Sigma(p, q)$. (Thus in Proposition 1.1, where S is intuitionistic logic, converses are characterized by supplying $\Sigma(p, q)$ as $\{(p \to q) \to q, (q \to p) \to p\}$. We are here retrospectively interpreting the "$\vdash$" notation with formulas on the left as purely abbreviatory; if one does not wish to do this, $\Sigma(p, q)$ should be taken as a collection of sequents rather than of formulas — and likewise for the logics themselves.) In the equivalential fragment of classical logic (though we continue to write $\to$ for the sole binary connective) such a characterization is available, in effect supplying $\Sigma(p, q)$ as $\{p \to q, q \to p\}$, or indeed more simply as just $\{p \to q\}$, as we see in Proposition 2.3 below. For this logic the abbreviative use of the "$\vdash$" notation is potentially confusing, since it clashes with the use of this notation for the consequence relation of classical equivalential logic, so we avoid it here. The clash arises because on the abbreviative use of "$\vdash$", the claim that $\varphi \vdash \psi$ (abbreviating "$\vdash \varphi \to \psi$", understood as "$\vdash \varphi \leftrightarrow \psi$") implies that $\psi \vdash \varphi$, whereas on the classical equivalential consequence relation understanding, $\varphi \vdash \psi$ implies only that either $\psi \vdash \varphi$ or else $\vdash \psi$. (More generally — as shown in [Hu1] — for this understanding, whenever $\varphi_1, \ldots, \varphi_n \vdash \psi$, we have either $\varphi_1, \ldots, \varphi_{n-1} \vdash \psi$ or else $\varphi_1, \ldots, \varphi_{n-1}, \psi \vdash \varphi_n$.[3])

[3] Added in press: since this was written, the author has discovered that this result appeared already in §11 of [Su].

Proposition 2.3: *Where $\vdash$ is the consequence relation of classical equivalential logic, formulas φ and ψ are converses according to $\vdash$ if and only if $\vdash \varphi \to \psi$.*

Proof. If φ and ψ are converses, there are χ_0, χ_1, with $\vdash \varphi \to (\chi_0 \to \chi_1)$ as well as conversely, and $\vdash \psi \to (\chi_1 \to \chi_0)$, as well as conversely. Since in the present case, for any formulas playing the roles of χ_0, χ_1, we have $\vdash (\chi_0 \to \chi_1) \to (\chi_1 \to \chi_0)$, we get $\vdash \varphi \to \psi$. Conversely, if $\vdash \varphi \to \psi$, then we can write φ in the equivalent form $\psi \to (\psi \to \psi)$ (recalling that "$\to$" here would usually appear as "$\leftrightarrow$"), whose converse is equivalent to ψ.

Thus the inapplicability of Proposition 2.1 in the case of classical equivalential logic does not rule out the existence of a characterization of converses for that case. The following problem remains open, then, in spite of Proposition 2.2:

PROBLEM *Is there a set of formulas $\Sigma(p, q)$ such that for arbitrary formulas $\varphi, \psi : \Sigma(\varphi, \psi) \subseteq \mathsf{R}$ if and only if φ and ψ are converses according to R?*

The implicit converse relation is certainly a *special relation* in R in the sense of [Hu4], meaning that $\Sigma(p, q)$ can be found for which (i) whenever φ and ψ are converses according to R, $\Sigma(\varphi, \psi) \subseteq \mathsf{R}$ and (ii) it is not the case that for arbitrary φ, ψ, we have $\Sigma(\varphi, \psi) \subseteq \mathsf{R}$. We know this already from Section 1 and the formula *Special K* there mentioned. This gives $\Sigma(p, q) = \{p \to (q \to p)\}$ satisfying (i) and (ii). What would be needed for an affirmative solution to the above Problem would be a strengthening of (i) so that we had all *and only* pairs of converses, rather than simply *all* pairs of converses, yield theorems when substituted for p and q in $\Sigma(p, q)$.[4] This is clearly not satisfied for the choice of Σ just mentioned, in view of Proposition 1.1 and the fact that *BCKW* logic is stronger than *BCIW* logic — the respective implicational fragments of intuitionistic logic and R.

We shall discuss that most famous of all substructural logics, *BCK* logic, in the following section, along with Abelian *BCI* logic, which is intermediate between *BCI* logic and the extension of the latter to the equivalential fragment of classical logic touched on in Proposition 2.3. We close the present

[4] In the version of this paper accepted for publication, there appears at this point the following parenthetical remark: "This is clearly not satisfied for the choice of Σ just mentioned, in view of Proposition 1.1 and the fact that *BCKW* logic is stronger than *BCIW* logic — the respective implicational fragments of intuitionistic logic and R." However, while the claim that the given Σ cannot work (i.e. characterize converses) for the *BCKW* case is straightforward enough, the author cannot now reconstruct any justification for the suggestion that this implies it cannot work for the weaker *BCIW* case.

section with the observation that although the pure equivalential fragment of intuitionistic logic is very different from that of classical logic (see the discussion and references provided by [KW]), Proposition 2.3 and its proof transfer to this weaker setting — the only logic considered here which is not an extension of *BCI* logic[5]:

Proposition 2.4: *Where* $\vdash$ *is the consequence relation of intuitionistic equivalential logic, formulas* φ *and* ψ *are converses according to* $\vdash$ *if and only if* $\vdash \varphi \rightarrow \psi$.

Proof. The proof of Proposition 2.3 applies verbatim here.

3. *Converses in Abelian Logic and in BCK Logic*

The "Abelian Logic" of [MS] adds to the *BCI* axioms the following

$A \qquad ((p \rightarrow q) \rightarrow q) \rightarrow p$

We shall call the resulting pure implicational logic Abelian *BCI* logic. Meyer and Slaney consider also the extension of this logic by suitable principles governing conjunction, negation, disjunction and other connectives. The pure implicational fragment is presented in algebraic form in [Da] — i.e., as a variety of *BCI*-algebras term equivalent to the variety of Abelian groups (see below).[6] The implicational fragment behaves very differently, as [MS] notes, from the full logic, especially as regards converses: whenever a formula is provable in the fragment, so is its converse — though this is not so for implicational formulas in the full logic. (Note that we do not say *pure* implicational formulas.) We will not exploit any such features in our discussion, so nothing hangs on the fact that we are discussing the implicational fragment. However, this fact does immediately yield the "$\vdash$" direction of the

[5] The formula C, for example, rewritten here with "$\rightarrow$" replaced by "$\leftrightarrow$", as $(p \leftrightarrow (q \leftrightarrow r)) \leftrightarrow (q \leftrightarrow (p \leftrightarrow r))$, is not intuitionistically provable, as one sees most easily by substituting $\bot$ for p and r, giving a left-hand side equivalent to $\neg\neg q$ and right-hand side equivalent to q.

[6] This logic was first studied by Meredith in the 1950s; see [MP], p. 221. That discussion is followed up in [Kal1], [Kal2] — see the sections on the left-subtraction connective L. [Kab3] discusses a consequence operation he calls C_{BCII} (the second "I" being intended to recall Roman Suszko's identity connective) whose set of 'tautologies' (i.e. $C_{BCII}(\emptyset)$) coincides with the set of Abelian *BCI*-theorems, and rediscovers the connection with Abelian groups. In 2000 the present author learnt (from Su Rogerson and Bob Meyer) that Branden Fitelson had shown the axiomatization of this logic with B, C, I, and A (using *Modus Ponens*) was not independent, C and I being derivable from B and A.

following lemma from the "⊣" direction, which we have preferred to argue from scratch:

Lemma 3.1: *With* $\vdash$ *indicating provability in Abelian BCI logic, we have*

$$(p \to q) \to (q \to q) \dashv\vdash q \to p$$

Proof. The "⊣" direction holds even for $\vdash = \vdash_{BCI}$. For the "⊢" direction we reason as follows, using some self-explanatory annotations (*cf.* [AB]):

(i)	$(p \to q) \to q \vdash p$	by A
(ii)	$q \to ((p \to q) \to q) \vdash q \to p$	from (i), prefixing
(iii)	$(p \to q) \to (q \to q) \vdash q \to p$	permuting antecedents on the left

This is quite promising for the project of characterizing converses in Abelian *BCI* logic, since we will be able to think of converses φ, ψ, respectively as $\chi_0 \to \chi_1$ and $\chi_1 \to \chi_0$, for which by this Lemma we have $(\chi_0 \to \chi_1) \to (\chi_1 \to \chi_1) \dashv\vdash \chi_1 \to \chi_0$. The trouble is that we want to obtain such an equivalence just using $\chi_0 \to \chi_1$ and its (explicit) converse, so that we can write the equivalence in terms of φ and ψ, whereas here the $\chi_1 \to \chi_1$ is neither of these. However, as Meyer and Slaney remark ([MS], p. 256), in contrast with the case of *BCIW* (and *a fortiori BCI*) logic, in Abelian *BCI* logic any two formulas of the form $\chi \to \chi$ are equivalent. (Recall that for logics not extending *BCK* logic, i.e. — approximately[7] — *relevant* logics, the provability of two formulas does not suffice for their provable equivalence — though the later does suffice for their interreplaceability *salva provabilitate* in all contexts. In fact in Abelian *BCI* logic we have not only all "self-implications" thus equivalent, but all provable formulas.[8] Though we do not need this for our discussion, it is worth noting that it means that the mismatch between *BCI* logic and the class of *BCI*-algebras mentioned in note 14 below disappears when we come to consider Abelian *BCI* logic and the class of Abelian *BCI*-algebras, as defined in note 10.). In case this point

[7] This is only approximate if one's notion of relevance is given by Belnap's criterion, *viz.*: that no implicational should be provable (at least in the absence of sentential constants) unless the antecedent and consequent share a variable — for, as is noted in [MS], Lemma 3.2 below shows this condition to be violated in the case of Abelian *BCI* logic, even though this is not an extension of *BCK* logic.

[8] Clearly it suffices to show that whenever φ and ψ are provable (in Abelian *BCI* logic) so is $\varphi \to \psi$. But the provability of φ yields that of $(\varphi \to \psi) \to \psi$ (whether or not ψ is provable), and so — this time appealing to the fact that for the current logic the converse of any provable implication is also provable — $\psi \to (\varphi \to \psi)$ is provable, whence by *Modus Ponens* given the provability of ψ, we have $\varphi \to \psi$ as promised.

of Meyer and Slaney may seem to depend on a detour through the part of Abelian logic involving the boolean connectives — or even the Ackermann-style truth constant t[9] — we give here a simple syntactic proof.

Lemma 3.2: *With* $\vdash$ *indicating provability in Abelian BCI logic, we have* $p \to p \dashv\vdash q \to q$.

Proof. Clearly it suffices to do the $\vdash$ direction.

(i)	$p \to p \vdash q \to ((q \to p) \to p)$	Already in *BCI* logic
(ii)	$(q \to p) \to p \vdash q$	A
(iii)	$q \to ((q \to p) \to p) \vdash q \to q$	(ii), prefixing
(iv)	$p \to p \vdash q \to q$	(i), (iii), transitivity

Thus the “$q \to q$” in Lemma 3.1 can be replaced by any other $\chi \to \chi$ formula (and we could now safely introduce t to abbreviate any such formula if we wanted); for our characterization we have chosen $\varphi \to \varphi$:

Proposition 3.3: *Formulas* φ *and* ψ *are implicational converses according to Abelian BCI logic, here indicated by* $\vdash$, *if and only if* $\varphi \to (\varphi \to \varphi) \dashv\vdash \psi$.

Proof. “Only if”: Suppose φ and ψ are converses according to $\vdash$. Then by Lemma 3.1, taking φ and ψ are respectively equivalent to $\chi_0 \to \chi_1$ and its converse, we have

$$\varphi \to (\chi_1 \to \chi_1) \dashv\vdash \psi$$

and so by the equivalence of different “self-implications” provided by Lemma 3.2:

$$\varphi \to (\varphi \to \varphi) \dashv\vdash \psi.$$

“If”: Suppose that $\varphi \to (\varphi \to \varphi) \dashv\vdash \psi$. Then to show that φ and ψ are converses, it suffices to show that $(\varphi \to \varphi) \to \varphi \dashv\vdash \varphi$. But we have this equivalence for any φ, even in *BCI* logic.

As one may glean from [MS], or more directly (though the notation has to be dualized) from [Da], the element $a \to (a \to a)$ (or $a \to 1$) in an Abelian group is the group-theoretic inverse of the element a, where the group multiplication is obtained from $\to$, the binary (fundamental) operation

[9] Recall that the simple-minded $\top$ of intuitionistic and classical logic gives way in a substructural context to a distinction between (and here we use the notation of [AB]) t, a left identity for $\to$, and T (a two-sided identity element for $\wedge$ — or, to put it in terms of $\to$: a formula provably implied by every formula), sometimes called the Ackermann and Church truth-constants, respectively.

in an Abelian *BCI*-algebra,[10] by setting

$$a \cdot b = (a \to 1) \to b.$$

Thus the appearance in Proposition 3.3 of "$\varphi \to (\varphi \to \varphi)$" (which, as already remarked, could be written as "$\varphi \to t$") can be thought of as denoting the 'inverse' of the formula φ, and Proposition 3.3 itself can be loosely summarised as saying that in this logic, converses are inverses. (Going in the other direction, any Abelian group gives rise to an Abelian *BCI*-algebra by putting $a \to b = a^{-1} \cdot b$, and these two procedures applied in the appropriate order to an Abelian *BCI*-algebra or an Abelian group lead us back to the original structure.)

Abelian *BCI* logic presents in an interesting form a phenomenon we shall call recoverability of converses. We say that a logic $\vdash$ *has recoverable converses* when the language of $\vdash$ provides a formula $\gamma(p)$ in one propositional variable (as indicated) with $\gamma(\varphi \to \psi) \dashv\vdash \psi \to \varphi$. Here $\gamma(\varphi \to \psi)$ is the result of substituting $\varphi \to \psi$ for p uniformly in $\gamma(p)$. Thus $\gamma(\cdot)$ can be thought of as the *definiens* for a defined 1-ary connective of the language of $\vdash$ which connective, when applied to an implicational formula, recovers for us the converse of that formula (to within equivalence). Abelian *BCI* logic provides recoverable converses in this sense because we may choose $\gamma(p)$ as $p \to (p \to p)$ (or indeed, if the constant t is present, as $p \to t$). Converses are also recoverable, though in a less interesting way, in classical and intuitionistic equivalential logics (notated, as above, with $\to$), since here we may simply select for $\gamma(p)$ the formula p itself. We return to this contrast between less and more interesting manifestations of the recoverability of converses below. First, we note another respect in which Abelian *BCI* logic distinguishes itself, namely as the weakest extension of *BCI* logic to have recoverable converses. For the proof, we need to allude to the following formula, named after the way the variables p and q "pivot" around the fixed r:

(*Pivot*) $\quad ((p \to r) \to q) \to ((q \to r) \to p)$

and the following rule:

[10] Assuming known the notion of a *BCI*-algebra, the Abelian such algebras are those satisfying the ("Abelian") identity $(x \to y) \to y \approx x$; they constitute a variety of *BCI*-algebras because we can "de-quasify" the quasi-identity $x \to y \approx y \to x \approx 1 \Rightarrow x \approx y$ by the following reasoning: $x \to y \approx y \to x$ implies $(x \to y) \to y \approx (y \to x) \to y$, whose l.h.s. reduces to x (by the Abelian identity) and whose r.h.s. reduces, since $y \to x \approx 1$, via $1 \to y$ to y. This makes the Abelian identity what would be called by analogy of the talk of *BCK*-varietizing identities in [Hu3], a *BCI*-varietizing identity. (It is well known that neither the class of *BCI*-algebras nor the class of *BCK*-algebras is itself a variety.)

(*Converse Recovery Rule*) From $\varphi_1 \to \psi_1 \dashv\vdash \varphi_2 \to \psi_2$ to $\psi_1 \to \varphi_1 \dashv\vdash \psi_2 \to \varphi_2$.

Although we isolate this last rule here for application in the proof of Proposition 3.4 below, we pause to notice that for $\vdash$ closed under the rule, the 'Converse Proposition Fallacy' of note 1 above is not a fallacy after all.

Proposition 3.4: *Abelian BCI logic is the smallest logic with recoverable converses extending BCI logic.*

Proof. We note without proof that instead of using A as above to axiomatize Abelian *BCI* logic as an addendum to the axioms B, C, I, we could equally well have used Pivot. Next, observe that any logic with recoverable converses is closed under the Converse Recovery Rule — since we may apply $\gamma(\cdot)$ to both sides of the "premiss". By C, for $\vdash$ as *BCI* logic, we have $p \to (q \to r) \dashv\vdash q \to (p \to r)$, from which that rule delivers Pivot.

The Converse Recovery Rule is suggestive of the following quasi-identity which may or may not be satisfied by a binary operation, here symbolized by juxtaposition (and under which some set is presumed to be closed, over whose elements the variables range):

(*Proto-commutativity*) $\quad xy \approx uz \Rightarrow yx \approx zu.$

We call an operation satisfying this condition proto-commutative. The condition has many interesting properties. For example, it suffices to deliver left-cancellativity (for the operation concerned) from right-cancellativity, or vice versa.[11] Any proto-commutative semigroup (i.e., whose fundamental binary operation is proto-commutative) with a left- or right-identity element is commutative.[12] On the other hand, it is clear that commutativity implies proto-commutativity since the left-hand sides of the antecedent and consequent of the latter condition are equal by commutativity, as are the right-hand sides. The sense in which the equivalential logics mentioned provide recoverable converses in a less interesting manner than Abelian *BCI* logic does is simply that the corresponding classes of algebras in the former case are

[11] Suppose we have right cancellation, and $ab = ac$. For left cancellation, we want $b = c$. We reason: given $ab = ac$, we have $ba = ca$ by proto-commutativity, so $b = c$ as desired: left cancellation. (*Mutatis mutandis* for the converse direction.)

[12] Proof: Suppose $\mathsf{A} = (A, \cdot, e_{\mathrm{L}})$ is a semigroup with a left-identity element e_{L} (i.e. $e_{\mathrm{L}}a = a$, all $a \in A$), and A satisfies Proto-commutativity. Since for all $a, b, c \in A$, we have $(ab)c = a(bc)$, proto-commutativity gives $c(ab) = (bc)a$. Choose $c = e_{\mathrm{L}}$. Then $ab = e_{\mathrm{L}}(ab) = (be_{\mathrm{L}})a = b(e_{\mathrm{L}}a) = ba$. (A similar argument works in the presence of a right-identity.)

commutative and therefore automatically proto-commutative, while in the latter case we have proto-commutativity without commutativity. Part (i) of the following gives a fund of proto-commutative though not in general commutative operations, subsuming the Abelian logic case via the connection with groups noted above, though in fact we do not exploit much of the hypothesis that we are dealing with a group — Part (i) works for any involuted groupoid[13], and (ii) for any groupoid with a two-sided identity element:

Proposition 3.5: *Let* $\mathsf{A} = (A, \cdot, ^{-1}, 1)$ *be a group. Then (i) the operations* $/$ *and* $\backslash$ *defined by:* $a/b = a \cdot b^{-1}$ *and* $a\backslash b = a^{-1} \cdot b$*, are proto-commutative, and (ii) the operation* $\cdot$ *is proto-commutative iff it is commutative (i.e.,* A *is abelian).*

Proof. (*i*) Suppose $a/b = c/d$, i.e. (suppressing the "$\cdot$") $ab^{-1} = cd^{-1}$. Then $(ab^{-1})^{-1} = (cd^{-1})^{-1}$, and thus $ba^{-1} = dc^{-1}$, i.e. $b/a = d/c$. Similarly in the case of $\backslash$.

(*ii*) "If": as already noted, commutative operations are always proto-commutative. "Only if": We are supposing that A satisfies the quasi-identity $xy \approx uz \Rightarrow yx \approx zu$, so when 1 is taken as the value of z, we get $xy \approx u \Rightarrow yx \approx u$, which is equivalent to the commutative law since we can substitute xy for u.

Before returning to our sentential logic theme, we pause to note that the relation between commutativity and proto-commutativity is a special case of something more general. Given any identity (now in the sense of *equation*) between terms t and u, we can consider the corresponding "proto" version, namely the pair of quasi-identities

$$\mathsf{t} \approx \mathsf{t}' \Rightarrow \mathsf{u} \approx \mathsf{u}', \mathsf{u} \approx \mathsf{u}' \Rightarrow \mathsf{t} \approx \mathsf{t}'$$

where terms u and u' are obtained from t and u by relettering the variables in these terms to a disjoint set of variables. More precisely, if the variables occurring in t are $x_1, \ldots, x_m$ and those in u are $y_1, \ldots, y_n$, (where we allow $\{x_1, \ldots, x_m\} \cap \{y_1, \ldots, y_n\} \neq \emptyset$), so that the original identity is $\mathsf{t}(x_1, \ldots, x_m) \approx \mathsf{u}(y_1, \ldots, y_n)$, then t' and u' are $\mathsf{t}(x'_1, \ldots, x'_n)$ and $\mathsf{u}(y'_1, \ldots, y'_n)$, where this time we require that the variables $x'_1, \ldots, x'_m$ are disjoint from $x_1, \ldots, x_m$ and likewise in the case of the y_i and y'_i. Thus, for

[13] By this is meant the class of algebras one obtains by removing the condition of associativity from the definition of an involuted semigroup in (e.g.) [Sch].

example, proto-associativity is given by

$$x(yz) \approx x'(y'z') \Rightarrow (xy)z \approx (x'y')z'$$
$$(xy)z \approx (x'y')z' \Rightarrow x(yz) \approx x'(y'z').$$

The "proto" quasi-identities all follow from the corresponding identities as in the commutativity case, and for the same reason, but the author has little general information on the subject, which would appear to merit further investigation. The case of commutativity allowed the further simplification that we only gave the single quasi-identity (re-written here with the "′" notation) $x_1x_2 \approx x'_1x'_2 \Rightarrow x_2x_1 \approx x'_2x'_1$, since the converse quasi-identity follows by re-lettering. A similar simplification is possible for a different reason in the case of proto-idempotence, where the identity $xx \approx x$ (and we omit any numerical subscripting here) gives rise to the two quasi-identities $xx \approx x'x' \Rightarrow x \approx x'$ and $x \approx x' \Rightarrow xx \approx x'x'$, the second of which is redundant since it is a free gift of equational logic.

We turn now to the case of *BCK* logic. It will be helpful to bear in mind the following relation between *BCK* logic and *BCK*-algebras, which we represent here in the "→" notation (as in [BF1], [BF2]), rather than the dual notation used in (e.g.) [Pa]: a formula α is provable in *BCK* logic just in case the identity $\mathsf{t}_\alpha \approx 1$ holds in all *BCK*-algebras, where t_α is the term corresponding to the formula α, i.e., resulting from it by replacing propositional variables by individual variables.[14] As Corollary 2 in [Pa], with notation adjusted, we find:

Proposition 3.6: (Pałasińska) *Let* s, t, *be terms built up from variables* x, y, *and the constant 1, by means of the operation symbol* $\rightarrow$, *and* s', t', *be the corresponding terms resulting from the substitution of* $x \rightarrow y$, $y \rightarrow x$ *for* x, y *in* s *and* t *respectively. Then the identity* $\mathsf{s} \approx \mathsf{t}$ *holds in all BCK algebras if and only if the identity* $\mathsf{s}' \approx \mathsf{t}'$ *holds in all BCK-algebras.*

[14] As was pointed out in [Bu] and [Kab1], there is no such correspondence in the case of *BCI*-algebras and *BCI* logic; the correspondence here noted in the *BCK* case falls far short of the claim that *BCK* algebras constitute an equivalent quasivariety semantics for *BCK* logic in the sense of [BP]. See further, on this point, [Kab2]. As to how to respond to the mismatch between *BCI* logic and the quasivariety of *BCI* algebras, Bunder (in [Bu]) suggests strengthening the logic, while Meyer and Ono (in [MO]) suggest a different meaning for the term '*BCI* algebras'. (However, it should be noted that *BCI* algebras in the newly proposed sense are not strictly algebras, carrying an undefined partial order in their similarity types, and nor — in the sketched completeness proof on p. 108 — is any indication given as to what the "Lindenbaum algebra" of *BCI* logic is supposed to be: no equivalence class of purely implicational formulas is specified as the element 1 of this (so-called) algebra, and no relation is defined as the partial order. This is not meant as questioning the completeness result — just the proposed proof.)

The non-trivial (i.e., "if") half of this result gives us a negative solution to the problem of characterizing converses in *BCK* logic; we restate the content for use in sentential logic:

Lemma 3.7: *Let $\beta(p, q)$ be a formula constructed from the variables p, q with the aid of $\rightarrow$, and $\beta(p \rightarrow q, q \rightarrow p)$ be the result of substituting $p \rightarrow q$, $q \rightarrow p$, respectively, for p, q, in $\beta(p, q)$. Then if $\beta(p \rightarrow q, q \rightarrow p)$ is provable in BCK logic, so is $\beta(p, q)$.*

Proof. Let t and t' be the *BCK*-algebraic terms corresponding to $\beta(p, q)$ and $\beta(p \rightarrow q, q \rightarrow p)$. Then if $\beta(p \rightarrow q, q \rightarrow p)$ is provable in *BCK* logic, we have (by the correspondence mentioned) that $t' \approx 1$ holds in all *BCK*-algebras, whence by Proposition 3.6, the identity $t \approx 1$ holds in all such algebras, so $\beta(p, q)$ is provable in *BCK* logic.

Notice that the principle we called *Special K* in Section 1 shows that *BCIW* logic lacks Pałasińska's property (in the sense that Lemma 3.7 does not hold for that logic: *cf.* the notion of special relation defined in the preceding section).

Proposition 3.8: *There is no set of formulas $\Sigma(p, q)$ such that for arbitrary formulas $\varphi, \psi : \Sigma(\varphi, \psi) \subseteq$ BCK if and only if φ and ψ are converses according to BCK logic.*

Proof. Suppose there exists such a $\Sigma(p, q)$. Then since $p \rightarrow q$ and $q \rightarrow p$ are converses according to *BCK* logic (as are any explicit implicational converses), we should have $\Sigma(p \rightarrow q, q \rightarrow p)(= \{\sigma(p \rightarrow q, q \rightarrow p) \mid \sigma(p, q) \in \Sigma(p, q)\}) \subseteq$ *BCK*. By Lemma 3.7, this implies that $\Sigma(p, q) \subseteq$ *BCK*, which is impossible since p and q are not even converses according to (the stronger) intuitionistic logic or classical logic (by Proposition 1.1 and Corollary 1.2).

This concludes our sample exploration of the properties of implicational converses in various logics. Perhaps the cases covered will serve as a stimulus for a more systematic study of the topic. Many cases of interest have not been considered, such as "commutative" *BCK* logic, which adds the axiom (here given the name — after Tanaka — its equational analogue customarily receives in the *BCK*-algebraic tradition):

$T \qquad ((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)$

The connective on which commutativity is conferred by T is of course not $\rightarrow$ but rather the derived binary connective taking α and β to the formula $(\alpha \rightarrow \beta) \rightarrow \beta$, which then offers a fair simulation of the disjunction of α with β. The basic Łukasiewicz implicational logic is given by adding as a further axiom the principle L from Section 1 above, which given the presence of T amounts to the disjunction of $p \to q$ with $q \to p$. (See [Ko] for the justification of the adjective "basic" here.[15]) Thus by the method of proof of Proposition 1.1, as summed up in Proposition 2.1, we have the same characterization of converses for Łukasiewicz's logic(s) as for intuitionistic logic. We have not investigated the case of the weaker commutative *BCK* logic, however. The same goes for many of the numerous relevant logics treated in [AB] — for entailment, ticket entailment, etc. And of course when we did consider the question of characterizing converses for the relevant logic R, we left it open (with the Problem in Section 2). Another interesting class of cases worthy of consideration would be the various strict implicational logics, amongst with the strict implication fragment of S5 seems especially interesting in view of its bearing on converses of universally quantified implications — e.g., in the case of classical predicate logic, can a characterization of the type sought above be found for the relation between formulas φ and ψ which are "class inclusion" converses in the sense of being equivalent respectively to formulas $\forall x(\chi_0(x) \to \chi_1(x))$ and $\forall x(\chi_1(x) \to \chi_0(x))$ for some formulas χ_0, χ_1, in which at most the variable x is free? The same question obviously arises in the more general case also in which instead of a single initial $\forall$ we allow several (whose variables may appear in the χ_i). We do not at present have any information on this topic, however.

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[15] Further background is supplied in [BF2], esp. p. 241*f*.

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