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IMAGINARY LOGIC-2: FORMAL RECONSTRUCTION OF THE UNNOTICED NIKOLAI VASILIEV'S LOGICAL SYSTEM*

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1. *Introduction*

One of the founders of non-classical logic Nikolai Vasiliev made the first attempt in the history of logic to set out a consistent logical system related to a contradictory ontology. That is why he is often considered to be a predecessor of paraconsistent logic. Vasiliev named his system 'imaginary non-Aristotelian logic' because it was conceived as an alternative to Aristotelian logic.

Since the pioneer V.A. Smirnov's paper [4] and D. Comey's review on it in *The Journal of Symbolic Logic* [1], logical ideas of Nikolai Vasiliev have been in focus of attention in contemporary logic. They have become a source of some new non-classical systems. Among the significant samples are propositional paraconsistent systems of A. Arruda and V.A. Smirnov's [5],[6] combined logic of propositions and situations with two levels: metalogical level of Assertion and internal level of Predication. At the same time genuine Vasiliev's logical systems were not investigated properly. Only V.A. Smirnov [7] made the first attempt to reconstruct them.

Originally, Vasiliev himself proposed the ideas of several logical systems. In his first article "About Particular Statements, Triangle of Oppositions and the Law of Excluded Forth", he presented the logic of syllogistic type founded on the ground of three kinds of propositions: universal affirmative, universal negative and the so called accidental (that is definite particular) propositions 'Only some (not every) S is P'. This system was formalized by V.A. Smirnov [7].

The most talked about Vasiliev's system is his 'imaginary non-Aristotelian logic', which contains not only affirmative and negative statements but also

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contradictory ('indifferent') statements with the copula 'is and isn't simultaneously'[1]. According to Vasiliev, these indifferent statements are false in 'our' world but can be true in some imaginary world, while the logic of this world — imaginary logic — will be a consistent theory. Vasiliev investigated the main version of imaginary logic in detail as well as gave a brief sketch of other three variations on the same theme (Vasiliev called them 'interpretations of imaginary logic'). The aim of this paper is to present a reconstruction of certain variant of imaginary logic which we call 'Logic of Concepts' or 'Imaginary Logic-2'.

2. *'Logic of Concepts': A Brief History*

In the final part of his paper "Imaginary (non-Aristotelian) logic" Vasiliev compares imaginary logic with non-Euclidean geometry of N. Lobachevsky, and raises a question about possible interpretation of imaginary logic in terms of 'our', 'terrestrial' logic:

> "We can propose a real interpretation of non-Euclidean geometry, we can find in our Euclidean space the essences with non-Euclidean geometry ... A real interpretation of Lobachevsky's geometry is a geometry of a surface with constant negative curvature, of so called pseudo-sphere ... In exactly the same way it is possible to find in our world the essences with the logic analogous to imaginary logic" [8, p. 81].

Vasiliev proposed three 'terrestrial' interpretations of imaginary logic.

According to the first interpretation [8, pp. 81–82], categorical statements of his system are treated as modal: affirmative as containing modality of necessary inherence of a property to an individual, negative as containing modality of necessary lack (of impossibility for an individual to have a property), and indifferent as containing modality of contingency[2].

The second interpretation [8, p. 87] is based on the idea to explicate affirmative, negative and indifferent statements via relations of absolute likeness (coincidence), absolute difference, and partial likeness and difference between phenomena, correspondingly.

[1] A complete axiomatization of this theory as well as an adequate semantics was proposed by T.P. Kostyuk and V.I. Markin in [2].

[2] The formal explication of modal interpretation of imaginary logic by means of logic with *de re* modalities was proposed by V.I. Markin in [3].

The core idea of the third interpretation of imaginary logic [8, p. 87–88] is to associate with each term of a categorical statement not a set of individuals but a concept considered as a set of characters and to treat syllogistic constants as denoting intensional relations between concepts. According to this approach, 'Every S is P' means that S contains all characters from P. The interpretation of two other universal statements is based on the difference between absolute (strong) negation and weak negation: 'Every S is not (absolutely) P' means that, for an arbitrary character from P, the concept S contains contradictory one, 'Every S is not (in a weak sense) P' means that S contains some characters from P as well as characters which contradict to some others.

Vasiliev considered the weak negation to be close to ordinary negative copula of Aristotelian syllogistic. Indeed, if two concepts contain at least two contradictory characters their extensions have no common elements. Nevertheless, it is natural to interpret statements with absolute negation as analogous to negative statements of imaginary logic. Both of them presuppose exclusively negative predication. Then indifferent statements, which combine assertion with negation, should be treated as statements with weak negation.

Vasiliev emphasized that the logic of concepts differs from the main version of imaginary logic as well as from the standard syllogistic. For example, some first figure syllogisms with minor negative (in a strong sense) premise are valid: 'Every M is P. Every S is not M. Hence, every S is not P'.

So, Vasiliev not only was one of the founders of non-classical logic, but as well he showed the manifold of non-classical logical systems, which are formulated in the same language and differ from each other in sets of laws. This is his indubitable contribution to logic, which as far as we know stays unnoticed up to now.

3. *Semantics for Imaginary Logic-2*

We present a semantics of logic of concepts based on the preformal intuitions underlying Vasiliev's system. Let L be a set of *literals* — positive and negative characters — p_1, $\sim p_1$, p_2, $\sim p_2$, ... Then a *concept* α is, by definition, an arbitrary non-empty and consistent subset of L, i.e. α satisfies conditions

$$\text{(i) } \alpha \neq \emptyset; \quad \text{(ii) for every literal } p_i \in \mathsf{L},\ p_i \notin \alpha \text{ or } \sim p_i \notin \alpha.$$

Let M be the set of all concepts. We define a function $*$ on concepts, which assigns to every concept α a *contrary* concept $\alpha*$:

$$p_i \in \alpha* \Leftrightarrow \sim p_i \in \alpha \text{ and } \sim p_i \in \alpha* \Leftrightarrow p_i \in \alpha.$$

Vasiliev writes about this operation:

> "If the concept A consists of characters p, q, r, s, ..., then the concept non-A must consist of characters non-p, non-q, non-r, non-s, and so on." [Vasiliev 1989, p. 88].

It can be easily shown that function $*$ has the following properties:

$$\text{(a) } \alpha \cap \alpha* = \emptyset, \quad \text{(b) } \alpha ** = \alpha, \quad \text{(c) } \alpha \subseteq \beta \Rightarrow \alpha* \subseteq \beta* .$$

Let d be a function assigning arbitrary concepts to terms: $\mathsf{d}(P) \in \mathsf{M}$. Vasiliev himself proposed semantical definitions only for universal statements. Let A_1SP be the form of universal affirmative statements 'Every S is P', A_2SP — the form of universal absolutely negative statements 'Every S is not P', and A_3SP — the form of universal indifferent (weakly negative) statements 'Every S is and is not P'.

Define a valuation $||^{\mathsf{d}}$ associated with d:

$| \mathsf{A}_1SP |^{\mathsf{d}} = 1$ iff $\mathsf{d}(P) \subseteq \mathsf{d}(S)$;
$| \mathsf{A}_2SP |^{\mathsf{d}} = 1$ iff $\mathsf{d}(P)* \subseteq \mathsf{d}(S)$;
$| \mathsf{A}_3SP |^{\mathsf{d}} = 1$ iff $\mathsf{d}(P) \cap \mathsf{d}(S) \neq \emptyset$ and $\mathsf{d}(P) * \cap \mathsf{d}(S) \neq \emptyset$.

However to formulate complete system of imaginary logic one needs more then just universal statements. In main version of this logic Vasiliev uses as well particular statements:

– *definite-particular statements* 'Some S is P, and each of the rest S is not P' (let it be denoted as T_1SP), 'Some S is P, and each of the rest S is and is not P' (T_2SP), 'Some S is not P, and each of the rest S is and is not P' (T_3SP), 'Some S is P, another S is not P, and each of the rest S is and is not P' (T_4SP);

– *indefinite particular statements* "Some S is P" (I_1SP), "Some S is not P" (I_2SP), "Some S is and is not P" (I_3SP).

All kinds of definite particular statements can be expressed with the help of indefinite particular ones:

$$\mathsf{T}_1SP \Leftrightarrow \mathsf{I}_1SP \,\&\, \mathsf{I}_2SP \,\&\, \neg\mathsf{I}_3SP, \quad \mathsf{T}_2SP \Leftrightarrow \mathsf{I}_1SP \,\&\, \neg\mathsf{I}_2SP \,\&\, \mathsf{I}_3SP,$$
$$\mathsf{T}_3SP \Leftrightarrow \neg\mathsf{I}_1SP \,\&\, \mathsf{I}_2SP \,\&\, \mathsf{I}_3SP, \quad \mathsf{T}_4SP \Leftrightarrow \mathsf{I}_1SP \,\&\, \mathsf{I}_2SP \,\&\, \mathsf{I}_3SP.$$

To set out appropriate truth definitions for particular statements it is possible to reformulate the semantics of general statements as follows:

$| \mathsf{A}_1SP |^{\mathsf{d}} = 1$ iff $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P) \subseteq \alpha]$;

$|\ \mathsf{A}_2 SP\ |^{\mathsf{d}} = 1$ iff $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P)* \subseteq \alpha]$;
$|\ \mathsf{A}_3 SP\ |^{\mathsf{d}} = 1$ iff $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P) \cap \alpha \neq \emptyset$ and $\mathsf{d}(P) * \cap\alpha \neq \emptyset]$.

Proposition 1. *The conditions* $\mathsf{d}(P) \subseteq \mathsf{d}(S)$ *and* $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P) \subseteq \alpha]$ *are equivalent.*

Proof. The implication from left to right is straightforward using transitivity of $\subseteq$.

In order to prove the statement from right to the left one should assume $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P) \subseteq \alpha]$ and the negation of $\mathsf{d}(P) \subseteq \mathsf{d}(S)$. Then there is a literal in $\mathsf{d}(P)$, which does not belong to $\mathsf{d}(S)$.

Let it be a positive literal p_i: $p_i \in \mathsf{d}(P)$ and $p_i \notin \mathsf{d}(S)$. Consider the literal $\sim p_i$, contradicting to p_i. It is evident that $\sim p_i \in \mathsf{d}(S)$ or $\sim p_i \notin \mathsf{d}(S)$. If $\sim p_i \in \mathsf{d}(S)$ then applying $\forall_e$ to initial assumption one gets $\mathsf{d}(S) \subseteq \mathsf{d}(S) \Rightarrow \mathsf{d}(P) \subseteq \mathsf{d}(S)$, and hence $\mathsf{d}(P) \subseteq \mathsf{d}(S)$, which leads to a contradiction. If $\sim p_i \notin \mathsf{d}(S)$ then the set $\mathsf{d}(S) \cup \{\sim p_i\}$ satisfies conditions (i) and (ii), which makes it possible to eliminate the universal quantifier in the following way: $\mathsf{d}(S) \subseteq \mathsf{d}(S) \cup \{\sim p_i\} \Rightarrow \mathsf{d}(P) \subseteq \mathsf{d}(S) \cup \{\sim p_i\}$. So, $\mathsf{d}(P) \subseteq \mathsf{d}(S) \cup \{\sim p_i\}$, and since $p_i \in \mathsf{d}(P)$, the concept $\mathsf{d}(S) \cup \{\sim p_i\}$ contains contradictory literals $\sim p_i$ and p_i, but this is impossible.

The case with the negative literal is similar. ∎

Proposition 2. *The conditions* $\mathsf{d}(P)* \subseteq \mathsf{d}(S)$ *and* $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P)* \subseteq \alpha]$ *are equivalent.*

The proof is analogous to the previous one.

Proposition 3. *The conditions* $[\mathsf{d}(P) \cap \mathsf{d}(S) \neq \emptyset$ and $\mathsf{d}(P) * \cap\mathsf{d}(S) \neq \emptyset]$ *and* $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P)\cap\alpha \neq \emptyset$ and $\mathsf{d}(P)*\cap\alpha \neq \emptyset]$ *are equivalent.*

Proof. From left to the right the proof is trivial by the properties of $\subseteq$ and $\cap$.

To prove the converse assume $\forall\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha \Rightarrow \mathsf{d}(P) \cap \alpha \neq \emptyset$ and $\mathsf{d}(P) * \cap\alpha \neq \emptyset]$, as well as $\mathsf{d}(P) \cap \mathsf{d}(S) = \emptyset$ or $\mathsf{d}(P) * \cap\mathsf{d}(S) = \emptyset$.

If $\mathsf{d}(P) \cap \mathsf{d}(S) = \emptyset$, then the set $\mathsf{d}(P) * \cup\mathsf{d}(S)$ is a concept, and an application of $\forall_e$ to initial assumption gives $\mathsf{d}(S) \subseteq \mathsf{d}(P)*\cup\mathsf{d}(S) \Rightarrow \mathsf{d}(P)\cap(\mathsf{d}(P) * \cup\mathsf{d}(S)) \neq \emptyset$ and $\mathsf{d}(P) * \cap(\mathsf{d}(P) * \cup\mathsf{d}(S)) \neq \emptyset$. It leads to $(\mathsf{d}(P) \cap \mathsf{d}(P)*) \cup (\mathsf{d}(P) \cap \mathsf{d}(S)) \neq \emptyset$. By the property (a) of $*$-function, $\mathsf{d}(P) \cap \mathsf{d}(S) \neq \emptyset$, which contradicts the assumption.

If $\mathsf{d}(P) * \cap\mathsf{d}(S) = \emptyset$ then the set $\mathsf{d}(P) \cup \mathsf{d}(S)$ is a concept, and the universal quantifier can be eliminated as follows: $\mathsf{d}(S) \subseteq \mathsf{d}(P) \cup \mathsf{d}(S) \Rightarrow \mathsf{d}(P) \cap (\mathsf{d}(P) \cup \mathsf{d}(S)) \neq \emptyset$ and $\mathsf{d}(P) * \cap(\mathsf{d}(P) \cup \mathsf{d}(S)) \neq \emptyset$. And the second member of this conjunction gives $\mathsf{d}(P) * \cap\mathsf{d}(S) \neq \emptyset$, i.e. leads to the contradiction again. ∎

Proceeding from the modified interpretation of universal statements it is natural to treat indefinite particular statements as follows:

$\mid \mathsf{I}_1 SP \mid^{\mathsf{d}} = 1$ iff $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \subseteq \alpha]$,
$\mid \mathsf{I}_2 SP \mid^{\mathsf{d}} = 1$ iff $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P)* \subseteq \alpha]$,
$\mid \mathsf{I}_3 SP \mid^{\mathsf{d}} = 1$ iff $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \cap \alpha \neq \emptyset$ and $\mathsf{d}(P) * \cap\alpha \neq \emptyset]$.

These definitions in return can be simplified:

$\mid \mathsf{I}_1 SP \mid^{\mathsf{d}} = 1$ iff $\mathsf{d}(P) * \cap \mathsf{d}(S) = \emptyset$;
$\mid \mathsf{I}_2 SP \mid^{\mathsf{d}} = 1$ iff $\mathsf{d}(P) \cap \mathsf{d}(S) = \emptyset$;
$\mid \mathsf{I}_3 SP \mid^{\mathsf{d}} = 1$ iff $\mathsf{d}(P) \backslash \mathsf{d}(S) \neq \emptyset$ and $\mathsf{d}(P) * \backslash \mathsf{d}(S) \neq \emptyset$.

Proposition 4. *The conditions* $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \subseteq \alpha]$ *and* $\mathsf{d}(P) * \cap \mathsf{d}(S) = \emptyset$ *are equivalent.*

Proof. Assume $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \subseteq \alpha]$, and $\mathsf{d}(P) * \cap \mathsf{d}(S) \neq \emptyset$. The latter means that there is a literal, which belongs to both $\mathsf{d}(P)*$ and $\mathsf{d}(S)$. Let it be a positive literal p_i (the case with negative literal is similar): $p_i \in \mathsf{d}(P)*$ and $p_i \in \mathsf{d}(S)$. By the definition of $*$, it implies that $\sim p_i \in \mathsf{d}(P)$. From this and the initial assumption we infer the existence of a concept α containing p_i and $\sim p_i$, that is impossible.

Now assume $\mathsf{d}(P) * \cap \mathsf{d}(S) = \emptyset$. It means that $\mathsf{d}(P) \cup \mathsf{d}(S)$ is a concept, and $\mathsf{d}(S)$ and $\mathsf{d}(P)$ are subsets of it. Hence, $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \subseteq \alpha]$. ■

Proposition 5. *The conditions* $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P)* \subseteq \alpha]$ *and* $\mathsf{d}(P) \cap \mathsf{d}(S) = \emptyset$ *are equivalent.*

The proof is analogous to the previous one.

Proposition 6. *The conditions* $\exists\alpha \in \mathsf{M}[\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \cap \alpha \neq \emptyset$ and $\mathsf{d}(P) * \cap\alpha \neq \emptyset]$ *and* $[\mathsf{d}(P) \backslash \mathsf{d}(S) \neq \emptyset$ and $\mathsf{d}(P) * \backslash \mathsf{d}(S) \neq \emptyset]$ *are equivalent.*

Proof. Assume there is a concept α such that $\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \cap \alpha \neq \emptyset$ and $\mathsf{d}(P) * \cap\alpha \neq \emptyset$. To get a contradiction assume $\mathsf{d}(P) \backslash \mathsf{d}(S) = \emptyset$ or $\mathsf{d}(P) * \backslash \mathsf{d}(S) = \emptyset$.

If $\mathsf{d}(P) \backslash \mathsf{d}(S) = \emptyset$ then $\mathsf{d}(P) \subseteq \mathsf{d}(S)$, and (by the transitivity of $\subseteq$) $\mathsf{d}(P) \subseteq \alpha$. From this and $\mathsf{d}(P) * \cap\alpha \neq \emptyset$ one can derive that α contains contradictory literals, that is impossible. If $\mathsf{d}(P) * \backslash \mathsf{d}(S) = \emptyset$, then $\mathsf{d}(P)* \subseteq \mathsf{d}(S)$, and hence, $\mathsf{d}(P)* \subseteq \alpha$. Taking into account $\mathsf{d}(P) \cap \alpha \neq \emptyset$, it leads to a contradiction again.

Now assume $\mathsf{d}(P) \backslash \mathsf{d}(S) \neq \emptyset$ and $\mathsf{d}(P) * \backslash \mathsf{d}(S) \neq \emptyset$. The former presupposes the existence of a literal l_1 (positive or negative), which belongs to

$\mathsf{d}(P)$ and does not belong to $\mathsf{d}(S)$. The latter means that there is a literal l_2 (positive or negative), which is present in $\mathsf{d}(P)*$ and absent in $\mathsf{d}(S)$. Let l'_1 and l'_2 represent literals contradicting to l_1 and l_2, correspondingly. Neither l_1, nor l_2 belong to $\mathsf{d}(S)$, so the set $\mathsf{d}(S) \cup \{l'_1, l'_2\}$ turns out to be a concept α, which satisfies the condition: $\mathsf{d}(S) \subseteq \alpha$ and $\mathsf{d}(P) \cap \alpha \neq \emptyset$ (where l'_2 is common for $\mathsf{d}(P)$ and α) and $\mathsf{d}(P) * \cap \alpha \neq \emptyset$ (where l'_1 belongs to both $\mathsf{d}(P)*$ and α). ■

It should be stated that Vasiliev realized his innovative ideas in traditional old-fashioned form. He presented all his systems as the *syllogistic type* theories. So, in order to reconstruct Vasiliev's systems we will follow Łukasiewicz in his formalization of Aristotelian syllogistic and choose classical propositional logic to be the ground for the reconstruction below presented. In so doing, the standard truth definitions for complex formulas will hold.

A formula is valid (in imaginary logic-2) iff it takes value "1" under any assignment d.

4. *Axiomatization of Imaginary Logic-2*

The set of valid formulas is axiomatized by the calculus IL2 containing propositional tautologies, and axiom schemes:

A1. $(\mathsf{A}_1 MP \,\&\, \mathsf{A}_1 SM) \supset \mathsf{A}_1 SP$,
A2. $(\mathsf{A}_1 MP \,\&\, \mathsf{A}_2 SM) \supset \mathsf{A}_2 SP$,
A3. $(\mathsf{A}_2 MP \,\&\, \mathsf{A}_1 SM) \supset \mathsf{A}_2 SP$,
A4. $(\mathsf{A}_2 MP \,\&\, \mathsf{A}_2 SM) \supset \mathsf{A}_1 SP$,
A5. $(\mathsf{A}_1 MP \,\&\, \mathsf{I}_1 SM) \supset \mathsf{I}_1 SP$,
A6. $(\mathsf{A}_1 MP \,\&\, \mathsf{I}_2 SM) \supset \mathsf{I}_2 SP$,
A7. $(\mathsf{A}_2 MP \,\&\, \mathsf{I}_1 SM) \supset \mathsf{I}_2 SP$,
A8. $(\mathsf{A}_2 MP \,\&\, \mathsf{I}_2 SM) \supset \mathsf{I}_1 SP$,
A9. $\mathsf{A}_1 SS$,
A10. $\neg(\mathsf{A}_1 SP \,\&\, \mathsf{I}_2 SP)$,
A11. $\neg(\mathsf{A}_2 SP \,\&\, \mathsf{I}_1 SP)$,
A12. $\mathsf{I}_1 SP \supset \mathsf{I}_1 PS$,
A13. $\mathsf{I}_2 SP \supset \mathsf{I}_2 PS$,
A14. $\mathsf{A}_1 SP \supset \mathsf{I}_1 SP$,
A15. $\mathsf{A}_2 SP \supset \mathsf{I}_2 SP$,
A16. $\mathsf{A}_3 SP \equiv \neg\mathsf{I}_1 SP \,\&\, \neg\mathsf{I}_2 SP$,
A17. $\mathsf{I}_3 SP \equiv \neg\mathsf{A}_1 SP \,\&\, \neg\mathsf{A}_2 SP$.

The only rule is *modus ponens*.

There is a huge number of strange theorems in IL2. They are neither laws of Aristotelian syllogistic, nor laws of the imaginary logic main version. For instance, scheme A2 is a formal equivalent to above mentioned first figure syllogism with minor negative premise. A similar deductive principle is expressed by A6. Schemes A4 and A8 are also non-trivial examples of first figure syllogisms with two absolutely negative premises and affirmative conclusion.

The conversion principles, formalized by IL2, also differ from standard syllogistic ones. The simple conversion of the universal absolutely negative statements is not provable in IL2, there is only restricted conversion for them (the law $A_2SP \supset I_2PS$ is derivable from A15 and A13). Particular absolutely negative statements convert simply (scheme A13). The simple conversion law for the universal indifferent (weekly negative) statements — $A_3SP \supset A_3PS$ — is derivable from A16, A12 and A13, but there is no such a law for the particular indifferent statements. It should be mentioned that in the main version of imaginary logic neither negative, nor indifferent statements convert into the statements of standard types.[3]

In IL2 unlike in imaginary logic particular indifferent statements can be expressed via universal (affirmative and absolutely negative), the fact is established by A17.

Observe some IL2 theorems, which will be used later.

T1. $\neg I_2SS$
Proof is by A9 and A10.

T2. $\neg(A_2SP \;\&\; A_1SP)$
Proof is by A10 and A15.

T3. $\neg(A_1MP \;\&\; A_1MS \;\&\; \neg I_1SP)$

1. $A_1MS \supset I_1MS$	A14
2. $I_1MS \supset I_1SM$	A12
3. $(A_1MP \;\&\; I_1SM) \supset I_1SP$	A5
4. $(A_1MP \;\&\; A_1MS) \supset I_1SP$	1, 2, 3
5. $\neg(A_1MP \;\&\; A_1MS \;\&\; \neg I_1SP)$	4

T4. $\neg(A_2MP \;\&\; A_2MS \;\&\; \neg I_1SP)$
Proof is analogous to T3 using A15, A13, and A8.

T5. $\neg(A_2MP \;\&\; A_1MS \;\&\; \neg I_2SP)$
Proof is analogous to T3 using A14, A12, and A7.

T6. $\neg(A_1MP \;\&\; A_2MS \;\&\; \neg I_2SP)$
Proof is analogous to T3 using A15, A13, and A6.

T7. $\neg(A_1PM \;\&\; A_2SM \;\&\; I_1SP)$

[3] In the system IL [2] formalizing the main version of imaginary logic only the following "quasi-conversion" principles for negative and indifferent statements are provable: $A_2SP \supset \neg I_1PS$, $I_2SP \supset \neg A_1PS$, $A_3SP \supset \neg I_1PS$, $I_3SP \supset \neg A_1PS$.

1. $(\mathsf{A}_1 PM \;\&\; \mathsf{I}_1 SP) \supset \mathsf{I}_1 SM$	A5
2. $\neg(\mathsf{A}_2 SM \;\&\; \mathsf{I}_1 SM)$	A11
3. $\neg(\mathsf{A}_1 PM \;\&\; \mathsf{A}_2 SM \;\&\; \mathsf{I}_1 SP)$	1,2

T8. $\neg(\mathsf{A}_2 PM \;\&\; \mathsf{A}_1 SM \;\&\; \mathsf{I}_1 SP)$
Proof is analogous to T7 using A7 and A10.

T9. $\neg(\mathsf{A}_1 PM \;\&\; \mathsf{A}_1 SM \;\&\; \mathsf{I}_2 SP)$
Proof is analogous to T7 using A6 and A10.

T10. $\neg(\mathsf{A}_2 PM \;\&\; \mathsf{A}_2 SM \;\&\; \mathsf{I}_2 SP)$
Proof is analogous to T7 using A8 and A11.

T11. $\neg(\neg\mathsf{I}_1 QR \;\&\; \mathsf{A}_1 PQ \;\&\; \mathsf{A}_1 SR \;\&\; \mathsf{I}_1 SP)$

1. $(\mathsf{A}_1 PQ \;\&\; \mathsf{I}_1 SP) \supset \mathsf{I}_1 SQ$	A5
2. $\mathsf{I}_1 SQ \supset \mathsf{I}_1 QS$	A12
3. $(\mathsf{A}_1 SR \;\&\; \mathsf{I}_1 QS) \supset \mathsf{I}_1 QR$	A5
4. $(\mathsf{A}_1 PQ \;\&\; \mathsf{A}_1 SR \;\&\; \mathsf{I}_1 SP) \supset \mathsf{I}_1 QR$	1, 2, 3
5. $\neg(\neg\mathsf{I}_1 QR \;\&\; \mathsf{A}_1 PQ \;\&\; \mathsf{A}_1 SR \;\&\; \mathsf{I}_1 SP)$	4

T12. $\neg(\neg\mathsf{I}_1 QR \;\&\; \mathsf{A}_1 PR \;\&\; \mathsf{A}_1 SQ \;\&\; \mathsf{I}_1 SP)$
Derived from T11 using A12.

T13. $\neg(\neg\mathsf{I}_1 QR \;\&\; \mathsf{A}_2 PQ \;\&\; \mathsf{A}_2 SR \;\&\; \mathsf{I}_1 SP)$
Proof is analogous to T11 using A7, A13, and A8.

T14. $\neg(\neg\mathsf{I}_1 QR \;\&\; \mathsf{A}_2 PR \;\&\; \mathsf{A}_2 SQ \;\&\; \mathsf{I}_1 SP)$
Derived from T13 using A12.

T15. $\neg(\neg\mathsf{I}_2 QR \;\&\; \mathsf{A}_2 PQ \;\&\; \mathsf{A}_1 SR \;\&\; \mathsf{I}_1 SP)$
Proof is analogous to T11 using A7, A13, and A6.

T16. $\neg(\neg\mathsf{I}_2 QR \;\&\; \mathsf{A}_2 PR \;\&\; \mathsf{A}_1 SQ \;\&\; \mathsf{I}_1 SP)$
Derived from T15 using A13.

T17. $\neg(\neg\mathsf{I}_2 QR \;\&\; \mathsf{A}_1 PQ \;\&\; \mathsf{A}_2 SR \;\&\; \mathsf{I}_1 SP)$
Proof is analogous to T11 using A5, A12, and A7.

T18. $\neg(\neg\mathsf{I}_2 QR \;\&\; \mathsf{A}_1 PR \;\&\; \mathsf{A}_2 SQ \;\&\; \mathsf{I}_1 SP)$
Derived from T17 using A13.

T19. $\neg(\neg \mathsf{I}_2 QR$ & $\mathsf{A}_1 PQ$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_2 SP)$
Proof is analogous to T11 using A6, A13, and A6.

T20. $\neg(\neg \mathsf{I}_2 QR$ & $\mathsf{A}_1 PR$ & $\mathsf{A}_1 SQ$ & $\mathsf{I}_2 SP)$
Derived from T19 using A13.

T21. $\neg(\neg \mathsf{I}_2 QR$ & $\mathsf{A}_2 PQ$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_2 SP)$
Proof is analogous to T11 using A8, A12, and A7.

T22. $\neg(\neg \mathsf{I}_2 QR$ & $\mathsf{A}_2 PR$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_2 SP)$
Derived from T21 using A13.

T23. $\neg(\neg \mathsf{I}_1 QR$ & $\mathsf{A}_1 PQ$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_2 SP)$
Proof is analogous to T11 using A6, A13, and A8.

T24. $\neg(\neg \mathsf{I}_1 QR$ & $\mathsf{A}_1 PR$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_2 SP)$
Derived from T23 using A12.

T25. $\neg(\neg \mathsf{I}_1 QR$ & $\mathsf{A}_2 PQ$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_2 SP)$
Proof is analogous to T11 using A8, A12, and A5.

T26. $\neg(\neg \mathsf{I}_1 QR$ & $\mathsf{A}_2 PR$ & $\mathsf{A}_1 SQ$ & $\mathsf{I}_2 SP)$
Derived from T25 using A12.

5. *Soundness and Completeness*

The aim of this section is to demonstrate that the calculus IL2 is an adequate formalization of the semantics of Section 3.

It can be easily shown that all axioms of IL2 are valid and *modus ponens* preserves validity, it allows to assert IL2 soundness:

Soundness Theorem. *Every* IL2*-provable formula is valid.*

The converse metatheorem (Completeness theorem) will be proved in a Henkin-style manner.

A set Γ of formulas of the language of IL2 is IL2-*consistent* iff there are no formulas B_1, B_2, ..., B_k such that the formula $\neg(B_1$ & B_2 & $\ldots$ & $B_k)$ be provable in IL2. A set Δ is IL2-*maximal* iff it is IL2-consistent and for any formula A of the language of IL2, $A \in \Delta$ or $\neg A \in \Delta$.

There is a number of important characteristics of IL2-maximal set Δ:

(m1) Δ contains all theorems of IL2,

(m2) Δ is closed under *modus ponens*,
(m3) $\neg A \in \Delta$, iff $A \notin \Delta$,
(m4) A & $B \in \Delta$, iff $A \in \Delta$ and $B \in \Delta$,
(m5) $A \vee B \in \Delta$, iff $A \in \Delta$ or $B \in \Delta$,
(m6) $A \supset B \in \Delta$, iff $A \in \Delta \Rightarrow B \in \Delta$.

Extension Lemma. *An arbitrary* IL2*-consistent set has at least one* IL2*-maximal extension.*
Proof is in a usual way.

Now we need some preliminaries to associate *canonical assignment* d_Δ with every IL2-maximal set Δ. Consider an ordered list C_1, C_2, ... of all the formulas from Δ which are of the form $\neg \mathsf{I}_1 QR$ or $\neg \mathsf{I}_2 QR$. This list is non-empty, because all the formulas of the type $\neg \mathsf{I}_2 SS$ are the theorems (T1) of IL2, and hence they belong to every IL2-maximal set (m1).

A canonical assignment d_Δ can be defined in a following way:
(1) if C_i is $\neg \mathsf{I}_1 QR$, then
– the literal p_i is included in $\mathsf{d}_\Delta(T)$, for any term T such that $\mathsf{A}_1 TQ \in \Delta$ or $\mathsf{A}_2 TR \in \Delta$,
– the literal $\sim p_i$ is included in $\mathsf{d}_\Delta(T)$, for any term T such that $\mathsf{A}_1 TR \in \Delta$ or $\mathsf{A}_2 TQ \in \Delta$;
(2) if C_i is $\neg \mathsf{I}_2 QR$, then
– the literal p_i is included in $\mathsf{d}_\Delta(T)$, for any term T such that $\mathsf{A}_1 TQ \in \Delta$ or $\mathsf{A}_1 TR \in \Delta$,
– the literal $\sim p_i$ is included in $\mathsf{d}_\Delta(T)$, for any term T such that $\mathsf{A}_2 TR \in \Delta$ or $\mathsf{A}_2 TQ \in \Delta$.
Thus,

$$\begin{aligned} \mathsf{d}_\Delta(T) = \; & \{p_i : C_i \text{ is } \neg \mathsf{I}_1 QR \text{ and } [\mathsf{A}_1 TQ \in \Delta \text{ or } \mathsf{A}_2 TR \in \Delta]\} \cup \\ & \{\sim p_i : C_i \text{ is } \neg \mathsf{I}_1 QR \text{ and } [\mathsf{A}_1 TR \in \Delta \text{ or } \mathsf{A}_2 TQ \in \Delta]\} \cup \\ & \{p_i : C_i \text{ is } \neg \mathsf{I}_2 QR \text{ and } [\mathsf{A}_1 TQ \in \Delta \text{ or } \mathsf{A}_1 TR \in \Delta]\} \cup \\ & \{\sim p_i : C_i \text{ is } \neg \mathsf{I}_2 QR \text{ and } [\mathsf{A}_2 TQ \in \Delta \text{ or } \mathsf{A}_2 TR \in \Delta]\}. \end{aligned}$$

Now it is necessary to show that, for arbitrary term T, $\mathsf{d}_\Delta(T)$ is a concept, i.e. it satisfies conditions (i) and (ii).
(i). $\neg \mathsf{I}_2 TT$ is a theorem of IL2 (T1), and hence belongs to Δ and to the list C_1, C_2, ... Let it be *i*-th formula in the list. Then, by the definition of d_Δ, there is the literal p_i in $\mathsf{d}_\Delta(M)$, for all M such that $\mathsf{A}_1 MT \in \Delta$. At the same time $\mathsf{A}_1 TT \in \Delta$, because $\mathsf{A}_1 TT$ is an axiom (A9) and by (m1). Thus $p_i \in \mathsf{d}_\Delta(T)$, i.e. $\mathsf{d}_\Delta(T) \neq \emptyset$.
(ii). Assume, on the contrary, there is a p_i such that $p_i \in \mathsf{d}_\Delta(T)$ and $\sim p_i \in \mathsf{d}_\Delta(T)$. Then, one needs to consider two cases: (ii-1) when C_i is

$\neg I_1QR$, and (ii-2) when C_i is $\neg I_2QR$.
(ii-1). Assumption and definition of d_Δ implies $\neg I_1QR \in \Delta$, $A_1TQ \in \Delta$ or $A_2TR \in \Delta$, $A_1TR \in \Delta$ or $A_2TQ \in \Delta$. By characteristic (m4) of IL2-maximal set it means that Δ contains at least one of these conjunctions: $(A_1TR\,\&\,A_1TQ\,\&\,\neg I_1QR)$, $(A_2TQ\,\&\,A_1TQ)$, $(A_2TR\,\&\,A_1TR)$, $(A_2TR\,\&\,A_2TQ\,\&\,\neg I_1QR)$. It is impossible, because the negations of all of them are IL2-provable (T3, T2, T2, T4, correspondingly), i.e. Δ contains them, that contradicts to IL2-consistency.
(ii-2). In this case we have $\neg I_2QR \in \Delta$, $A_1TQ \in \Delta$ or $A_1TR \in \Delta$, $A_2TQ \in \Delta$ or $A_2TR \in \Delta$. Then Δ contains at least one of the following conjunctions: $(A_2TR\,\&\,A_1TQ\,\&\,\neg I_2QR)$, $(A_2TQ\,\&\,A_1TQ)$, $(A_2TR\,\&\,A_1TR)$, $(A_1TR\,\&\,A_2TQ\,\&\,\neg I_2QR)$. It is also impossible. The negations of all these formulas are theorems of IL2 (T5, T2, T2, T6).

Therefore, $d_\Delta(T)$ satisfies the condition (ii).

Now we are in a position to prove the main lemma by induction on the length of a formula A:

Main Lemma. *For arbitrary* IL2*-maximal set* Δ *and arbitrary formula* A, $A \in \Delta$ *iff* $\mid A \mid= 1$ under d_Δ.

Proof. Basis presupposes six cases.

(1). A is A_1SP.

First prove that $A_1SP \in \Delta \Rightarrow\mid A_1SP \mid= 1$ under d_Δ.

Assume $A_1SP \in \Delta$. Required is that, for every literal (positive or negative) from $d_\Delta(P)$, it is contained in $d_\Delta(S)$.

Consider an arbitrary positive literal $p_i \in d_\Delta(P)$. By the definition of d_Δ, it means that either C_i is $\neg I_1QR$ and [$A_1PQ \in \Delta$ or $A_2PR \in \Delta$], or C_i is $\neg I_2QR$ and [$A_1PQ \in \Delta$ or $A_1PR \in \Delta$]. Using axioms $(A_1PQ\,\&\,A_1SP) \supset A_1SQ$ (A1), $(A_2PR\,\&\,A_1SP) \supset A_2SR$ (A3) and $(A_1PR\,\&\,A_1SP) \supset A_1SR$ (A1), and by (m2), it follows that either C_i is $\neg I_1QR$ and [$A_1SQ \in \Delta$ or $A_2SR \in \Delta$], or C_i is $\neg I_2QR$ and [$A_1SQ \in \Delta$ or $A_1SR \in \Delta$]. According to the definition of d_Δ it means that $p_i \in d_\Delta(S)$.

For an arbitrary negative literal $\sim pi \in d_\Delta(P)$, either C_i is $\neg I_1QR$ and [$A_1PR \in \Delta$ or $A_2PQ \in \Delta$], or C_i is $\neg I_2QR$ and [$A_2PQ \in \Delta$ or $A_2PR \in \Delta$]. By (m1), (m2) and axioms $(A_1PR\,\&\,A_1SP) \supset A_1SR$ (A1), $(A_2PQ\,\&\,A_1SP) \supset A_2SQ$ (A3) and $(A_2PR\,\&\,A_1SP) \supset A_2SR$ (A3), it leads to either C_i is $\neg I_1QR$ and [$A_1SR \in \Delta$ or $A_2SQ \in \Delta$], or C_i is $\neg I_2QR$ and [$A_2SQ \in \Delta$ or $A_2SR \in \Delta$], that is $\sim p_i \in d_\Delta(S)$.

To prove the converse, assume $\mid A_1SP \mid= 1$ under d_Δ, i.e. $d_\Delta(P) \subseteq d_\Delta(S)$. Then formula $\neg I_2PP$ (T1) is provable in IL2, and by (m1) belongs to Δ. Therefore it is a member of the list C_1, C_2, ..., and let its number be i. There is also an axiom A_1PP (A9) in Δ. So C_i is $\neg I_2PP$ and $A_1PP \in \Delta$. Hence, by the definition of d_Δ, $p_i \in d_\Delta(P)$. Thus, $p_i \in d_\Delta(S)$. From the fact that C_i is $\neg\mathsf{I}_2PP$ and by the definition of d_Δ we come to the desideratum: $\mathsf{A}_1SP \in \Delta$.

(2) A is A_2SP.

First prove that $\mathsf{A}_2SP \in \Delta \Rightarrow \mid \mathsf{A}_2SP \mid = 1$ under d_Δ.

Assume $\mathsf{A}_2SP \in \Delta$ and show, for arbitrary positive (negative) literal p_i ($\sim p_i$) from $\mathsf{d}_\Delta(P)*$, $p_i \in \mathsf{d}_\Delta(S)(\sim p_i \in \mathsf{d}_\Delta(S))$.

Consider an arbitrary positive literal $p_i \in \mathsf{d}_\Delta(P)*$. By the definition of $*$, $\sim p_i \in \mathsf{d}_\Delta(P)$. The latter is possible only if either C_i is $\neg\mathsf{I}_1QR$ and $[\mathsf{A}_1PR \in \Delta$ or $\mathsf{A}_2PQ \in \Delta]$, or C_i is $\neg\mathsf{I}_2QR$ and $[\mathsf{A}_2PQ \in \Delta$ or $\mathsf{A}_2PR \in \Delta]$. Using axioms $(\mathsf{A}_1PR \,\&\, \mathsf{A}_2SP) \supset \mathsf{A}_2SR$ (A2), $(\mathsf{A}_2PQ \,\&\, \mathsf{A}_2SP) \supset \mathsf{A}_1SQ$ (A4) and $(\mathsf{A}_2PR \,\&\, \mathsf{A}_2SP) \supset \mathsf{A}_1SR$ (A4), and by (m2), we conclude that either C_i is $\neg\mathsf{I}_1QR$ and $[\mathsf{A}_2SR \in \Delta$ or $\mathsf{A}_1SQ \in \Delta]$, or C_i is $\neg\mathsf{I}_2QR$ and $[\mathsf{A}_1SQ \in \Delta$ or $\mathsf{A}_1SR \in \Delta]$. By the definition of d_Δ, it means that $p_i \in \mathsf{d}_\Delta(S)$.

For an arbitrary negative literal $\sim p_i \in \mathsf{d}_\Delta(P)*$, $p_i \in \mathsf{d}_\Delta(P)$. Then either C_i is $\neg\mathsf{I}_1QR$ and $[\mathsf{A}_1PQ \in \Delta$ or $\mathsf{A}_2PR \in \Delta]$, or C_i is $\neg\mathsf{I}_2QR$ and $[\mathsf{A}_1PQ \in \Delta$ or $\mathsf{A}_1PR \in \Delta]$. By (m1) and (m2), using $(\mathsf{A}_1PQ \,\&\, \mathsf{A}_2SP) \supset \mathsf{A}_2SQ$ (A2), $(\mathsf{A}_2PR \,\&\, \mathsf{A}_2SP) \supset \mathsf{A}_1SR$ (A4) and $(\mathsf{A}_1PR \,\&\, \mathsf{A}_2SP) \supset \mathsf{A}_2SR$ (A2), one gets either C_i is $\neg\mathsf{I}_1QR$ and $[\mathsf{A}_2SQ \in \Delta$ or $\mathsf{A}_1SR \in \Delta]$, or C_i is $\neg\mathsf{I}_2QR$ and $[\mathsf{A}_2SQ \in \Delta$ or $\mathsf{A}_2SR \in \Delta]$, i.e. $\sim p_i \in \mathsf{d}_\Delta(S)$.

To prove this case in the opposite direction, assume $\mid \mathsf{A}_2SP \mid = 1$ under d_Δ, that is $\mathsf{d}_\Delta(P)* \subseteq \mathsf{d}_\Delta(S)$. The theorem $\neg\mathsf{I}_2PP$ (T1) belongs to Δ, and therefore, it is a member of the ordered list (with a number i). Δ contains A_1PP (A9), so C_i is $\neg\mathsf{I}_2PP$ and $\mathsf{A}_1PP \in \Delta$. Then $p_i \in \mathsf{d}_\Delta(P)$, and $\sim p_i \in \mathsf{d}_\Delta(P)*$. Therefore, because $\mathsf{d}_\Delta(P)* \subseteq \mathsf{d}_\Delta(S)$, $\sim p_i \in \mathsf{d}_\Delta(S)$. Thus, by the definition of d_Δ, and from C_i, being $\neg\mathsf{I}_2PP$, $\mathsf{A}_2SP \in \Delta$.

(3) A is I_1SP.

$\mathsf{I}_1SP \in \Delta \Rightarrow \mid \mathsf{I}_1SP \mid = 1$ under d_Δ.

Assume $\mathsf{I}_1SP \in \Delta$, but $\mid \mathsf{I}_1SP \mid = 0$ under d_Δ. Since the latter means that $\mathsf{d}(P) * \cap \mathsf{d}(S) \neq \emptyset$, there is a literal (positive or negative), which belongs to both $\mathsf{d}(P)*$ and $\mathsf{d}(S)$.

Let's treat the case with positive literal: $p_i \in \mathsf{d}_\Delta(P)*$ and $p_i \in \mathsf{d}_\Delta(S)$. By the definition of $*$, $\sim p_i \in \mathsf{d}_\Delta(P)$. By the definition of d_Δ, firstly, either C_i is $\neg\mathsf{I}_1QR$ and $[\mathsf{A}_1PR \in \Delta$ or $\mathsf{A}_2PQ \in \Delta]$, or C_i is $\neg\mathsf{I}_2QR$ and $[\mathsf{A}_2PQ \in \Delta$ or $\mathsf{A}_2PR \in \Delta]$, secondly, either C_i is $\neg\mathsf{I}_1QR$ and $[\mathsf{A}_2SR \in \Delta$ or $\mathsf{A}_1SQ \in \Delta]$, or C_i is $\neg\mathsf{I}_2QR$ and $[\mathsf{A}_1SQ \in \Delta$ or $\mathsf{A}_1SR \in \Delta]$. Then, by (m4) and from the assumption $\mathsf{I}_1SP \in \Delta$, it follows that Δ contains at least one of the conjunctions: $\mathsf{A}_1PR \,\&\, \mathsf{A}_2SR \,\&\, \mathsf{I}_1SP$, $\neg\mathsf{I}_1QR \,\&\, \mathsf{A}_1PR \,\&\, \mathsf{A}_1SQ \,\&\, \mathsf{I}_1SP$, $\neg\mathsf{I}_1QR \,\&\, \mathsf{A}_2PQ \,\&\, \mathsf{A}_2SR \,\&\, \mathsf{I}_1SP$, $\mathsf{A}_2PQ \,\&\, \mathsf{A}_1SQ \,\&\, \mathsf{I}_1SP$, $\neg\mathsf{I}_2QR \,\&\, \mathsf{A}_2PQ \,\&\, \mathsf{A}_1SR \,\&\, \mathsf{I}_1SP$, $\neg\mathsf{I}_2QR \,\&\, \mathsf{A}_2PR \,\&\, \mathsf{A}_1SQ \,\&\, \mathsf{I}_1SP$, $\mathsf{A}_2PR \,\&\, \mathsf{A}_1SR \,\&\, \mathsf{I}_1SP$. But it is impossible, since the negations of these formulas are provable in IL2 (T7, T12, T13, T8, T15, T16 and T8 correspondingly), and Δ, being consistent, contains them.

When $\mathsf{d}_\Delta(P)*$ and $\mathsf{d}_\Delta(S)$ contain negative literal $\sim p_i$ the proof is similar. Let $\sim p_i \in \mathsf{d}_\Delta(S)$ and $\sim p_i \in \mathsf{d}_\Delta(P)*$ i.e. $p_i \in \mathsf{d}_\Delta(P)$. Then, firstly either C_i is $\neg\mathsf{I}_1 QR$ and [$\mathsf{A}_1 PQ \in \Delta$ or $\mathsf{A}_2 PR \in \Delta$], or C_i is $\neg\mathsf{I}_2 QR$ and [$\mathsf{A}_1 PQ \in \Delta$ or $\mathsf{A}_1 PR \in \Delta$], secondly, either C_i is $\neg\mathsf{I}_1 QR$ and [$\mathsf{A}_2 SQ \in \Delta$ or $\mathsf{A}_1 SR \in \Delta$], or C_i is $\neg\mathsf{I}_2 QR$ and [$\mathsf{A}_2 SQ \in \Delta$ or $\mathsf{A}_2 SR \in \Delta$]. It implies that Δ contains at least one of the conjunctions: $\mathsf{A}_1 PQ$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_1 SP$, $\neg\mathsf{I}_1 QR$ & $\mathsf{A}_1 PQ$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_1 SP$, $\neg\mathsf{I}_1 QR$ & $\mathsf{A}_2 PR$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_1 SP$, $\mathsf{A}_2 PR$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_1 SP$, $\neg\mathsf{I}_2 QR$ & $\mathsf{A}_1 PQ$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_1 SP$, $\neg\mathsf{I}_2 QR$ & $\mathsf{A}_1 PR$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_1 SP$, $\mathsf{A}_1 PR$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_1 SP$, but their negations belong to Δ, since they are provable in IL2 (T7, T11, T14, T8, T17, T18 and T7 correspondingly).

To prove the converse assume $|\ \mathsf{I}_1 SP\ | = 1$ under d_Δ, i.e. $\mathsf{d}_\Delta(P) * \cap \mathsf{d}_\Delta(S) = \emptyset$, but at the same time $\mathsf{I}_1 SP \notin \Delta$. The latter, by (m3), means that $\neg\mathsf{I}_1 SP \in \Delta$. Let the number of $\neg\mathsf{I}_1 SP$ in the ordered list be i. Axioms $\mathsf{A}_1 SS$ and $\mathsf{A}_1 PP$ (A9) also belong to Δ. Hence, $p_i \in \mathsf{d}_\Delta(S)$ and $\sim p_i \in \mathsf{d}_\Delta(P)$, that is, $p_i \in \mathsf{d}_\Delta(P)*$. Thus, an intersection of $\mathsf{d}_\Delta(P)*$ and $\mathsf{d}_\Delta(S)$ is non-empty (it contains p_i), which contradicts the initial assumption.

(4) A is $\mathsf{I}_2 SP$.

First prove that $\mathsf{I}_2 SP \in \Delta \Rightarrow |\ \mathsf{I}_2 SP\ | = 1$ under d_Δ.

Assume $\mathsf{I}_2 SP \in \Delta$, but $|\ \mathsf{I}_2 SP\ | = 0$ under d_Δ, i.e. $\mathsf{d}(P) \cap \mathsf{d}(S) \neq \emptyset$. Therefore, there is a literal (positive or negative), which belongs to both $\mathsf{d}(P)$ and $\mathsf{d}(S)$.

For a positive literal p_i where $p_i \in \mathsf{d}_\Delta(P)$ and $p_i \in \mathsf{d}_\Delta(S)$, it holds, by the definition of d_Δ, that, firstly, either C_i is $\neg\mathsf{I}_1 QR$ and [$\mathsf{A}_1 PQ \in \Delta$ or $\mathsf{A}_2 PR \in \Delta$], or C_i is $\neg\mathsf{I}_2 QR$ and [$\mathsf{A}_1 PQ \in \Delta$ or $\mathsf{A}_1 PR \in \Delta$], secondly, either C_i is $\neg\mathsf{I}_1 QR$ and [$\mathsf{A}_2 SR \in \Delta$ or $\mathsf{A}_1 SQ \in \Delta$], or C_i is $\neg\mathsf{I}_2 QR$ and [$\mathsf{A}_1 SQ \in \Delta$ or $\mathsf{A}_1 SR \in \Delta$]. Then, by (m4) and from the assumption $\mathsf{I}_2 SP \in \Delta$, it follows that Δ contains at least one of the conjunctions: $\neg\mathsf{I}_1 QR$ & $\mathsf{A}_1 PQ$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_2 SP$, $\mathsf{A}_1 PQ$ & $\mathsf{A}_1 SQ$ & $\mathsf{I}_2 SP$, $\mathsf{A}_2 PR$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_2 SP$, $\neg\mathsf{I}_1 QR$ & $\mathsf{A}_2 PR$ & $\mathsf{A}_1 SQ$ & $\mathsf{I}_2 SP$, $\neg\mathsf{I}_2 QR$ & $\mathsf{A}_1 PQ$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_2 SP$, $\neg\mathsf{I}_2 QR$ & $\mathsf{A}_1 PR$ & $\mathsf{A}_1 SQ$ & $\mathsf{I}_2 SP$, $\mathsf{A}_1 PR$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_2 SP$, that is impossible, for the negations of these formulas are the theorems of IL2 (T23, T9, T10, T26, T19, T20 and T9 correspondingly), and, therefore they are contained in Δ.

For a negative literal be in $\mathsf{d}(P)$ and $\mathsf{d}(S)$, it holds, firstly, either C_i is $\neg\mathsf{I}_1 QR$ and [$\mathsf{A}_1 PR \in \Delta$ or $\mathsf{A}_2 PQ \in \Delta$], or C_i is $\neg\mathsf{I}_2 QR$ and [$\mathsf{A}_2 PQ \in \Delta$ or $\mathsf{A}_2 PR \in \Delta$], and secondly, either C_i is $\neg\mathsf{I}_1 QR$ and [$\mathsf{A}_2 SQ \in \Delta$ or $\mathsf{A}_1 SR \in \Delta$], or C_i is $\neg\mathsf{I}_2 QR$ and [$\mathsf{A}_2 SQ \in \Delta$ or $\mathsf{A}_2 SR \in \Delta$]. Then Δ contains at least one of the conjunctions: $\neg\mathsf{I}_1 QR$ & $\mathsf{A}_1 PR$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_2 SP$, $\mathsf{A}_1 PR$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_2 SP$, $\mathsf{A}_2 PQ$ & $\mathsf{A}_2 SQ$ & $\mathsf{I}_2 SP$, $\neg\mathsf{I}_1 QR$ & $\mathsf{A}_2 PQ$ & $\mathsf{A}_1 SR$ & $\mathsf{I}_2 SP$, $\neg\mathsf{I}_2 QR$ & $\mathsf{A}_2 PQ$ & $\mathsf{A}_2 SR$ & $\mathsf{I}_2 SP$, $\neg\mathsf{I}_2 QR$ & $\mathsf{A}_2 PR$ & $\mathsf{A}_2 SQ$

& I_2SP, A_2PR & A_2SR & I_2SP. And again, the negations of these formulas belong to Δ (T24, T9, T10, T25, T21, T22, T10).

The proof from the right to the left is based on the assumptions $\mid \mathsf{I}_2SP \mid= 1$ under the canonical assignment, i.e. $\mathsf{d}_\Delta(P) \cap \mathsf{d}_\Delta(S) = \emptyset$, and $\mathsf{I}_2SP \notin \Delta$. The latter, by (m3), means $\neg\mathsf{I}_2SP \in \Delta$ and $\neg\mathsf{I}_2SP$ belongs to the ordered list (being C_i). Since axioms A_1SS and A_1PP (A9) also belong to Δ, $p_i \in \mathsf{d}_\Delta(P)$ and $p_i \in \mathsf{d}_\Delta(S)$. Therefore, $\mathsf{d}_\Delta(P) \cap \mathsf{d}_\Delta(S) \neq \emptyset$, that contradicts the assumption.

(5). A is A_3SP.

The proof in this case can be reduced to the combination of those for (3) and (4), because Δ is maximal and contains formulas of the type $\mathsf{A}_3SP \equiv \neg\mathsf{I}_1SP$ & $\neg\mathsf{I}_2SP$ (A16).

(6) A is I_3SP.

In this case the proof reduces to (1) and (2). It follows from the fact that Δ is maximal and contains formulas $\mathsf{I}_3SP \equiv \neg\mathsf{A}_1SP$ & $\neg\mathsf{A}_2SP$ (A17).

The proof of inductive step is trivial: it is based on the classical semantics for propositional connectives and on the characteristics of IL2-maximal set mentioned above. ■

Now we have an opportunity to demonstrate completeness of IL2.

Completeness Theorem. *Every valid formula is provable in* IL2.

Proof. Consider an arbitrary valid formula A. Assume it is not provable in IL2. Then the formula $\neg\neg A$ is not a theorem of IL2. Hence, by the definition of IL2-consistent set, the set $\{\neg A\}$ is IL2-consistent. By the Extension Lemma, there is a IL2-maximal extension Δ of this set. And by the Main Lemma, $\mid \neg A \mid= 1$ under d_Δ. Hence, $\mid A \mid= 0$ under d_Δ, that contradicts the initial assumption of its validity. ■

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