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ATOMIC AND MOLECULAR PARACONSISTENT LOGICS

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Abstract
An unusual classification of paraconsistent logics is presented. The classification is based on the distinction between atomic and molecular paraconsistent logics. An example of a molecular paraconsistent logic with the property that $\vdash \neg A \supset (\neg\neg A \supset B)$ is given. Some difficulties connected with a definition of a paraconsistent logic are stressed.

1. *Some history*

The Russian logician N.A. Vasiliev is generally accepted to be the precursor of paraconsistent logic (together with J. Łukasiewicz). One of his most profound ideas is the idea concerning the subdivision of the laws of logic into two worlds: the world of metalogic and the world of empirical laws. In the first case the laws are invariable while at an empirical level the laws of logic can vary and even *the law of a non-contradiction* and *the law of tertium non datur* do not hold or are non-existent in the second world. This subject is under a thorough investigation in his article "The imaginary (non-Aristotelian) logic" [Vasiliev 1912]. This work was done under a strong influence of the researches in the non-Euclidean geometry, that has been done by Vasiliev's great countryman N.I. Lobatchevsky. The first thorough work on Vasiliev's logical ideas was V.A. Smirnov's article [Smirnov 1962]. The detailed review of this article was published in the "Journal of Symbolic logic" [Comey 1965]. This review attracted the attention of a famous Brazilian logician, A. Arruda, and, as the result, there appeared the work [Arruda 1977] containing various types of reconstruction of Vasiliev's logical ideas. Let us concentrate our attention on the first of three logical systems introduced by Arruda which is called $\mathsf{V1}$. Gilbert's axiomatisation is given for this system (see also [Tuziak 1997]), a number of metatheorems are proved and it is noticed that $\mathsf{V1}$ has the following three-valued characteristic matrix with two designated truth-values 1 and $^1\!/_2$:

x	$\neg x$
1	0
$1/2$	1
0	1

$\supset$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	1	1	1

$\wedge$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	0	0	0

$\vee$	1	$1/2$	0
1	1	1	1
$1/2$	1	1	1
0	1	1	0

$\equiv$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	0	0	1

2. *Definition of paraconsistent logic* (see [Priest 2002])

The major impetus for the invention of paraconsistent logic has always been the thought that in certain circumstances we may be in a situation where our theory is inconsistent, and yet where we are required to draw inferences in a sensible fashion. Let $\vdash$ be any relationship of logical consequence. Call it *explosive* if it satisfies the condition that for all A and B, $\{A, \neg A\} \vdash B$, *ex contradictione quodlibet* (ECQ). Both classical and intuitionistic logics (and also Łukasiewicz's many-valued logics) are explosive. Clearly, if $\vdash$ is explosive it is not a sensible inference relation in an inconsistent context, for applying it gives rise to *triviality* i.e. everything. Thus, a minimal condition for a suitable inference relation in this context is that $\vdash$ must not be explosive. Such inference relationships, and the logics that have them, have come to be called *paraconsistent*.

3. *Sette's logic* $\mathsf{P1}$

A. Sette [Sette 1973] constructed the logical calculus $\mathsf{P1}$ which is obtained from the classical propositional calculus C_2 (see [Kleene 1952]) when the axiom

$$(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$$

is replaced by the axiom

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset A).$$

This logical system has the same characteristic matrix as Arruda's calculus $\mathsf{V1}$.

For the first time such truth-tables appeared in [da Costa 1963], where these have been used for the refutation of some tautologies of $\mathsf{C_2}$ which do not hold in the paraconsistent logic of N.C.A. da Costa, $\mathsf{C_1}$. See also [da Costa & Alves 1981], where $\mathsf{P1}$ was called F. And finally $\mathsf{P1}$ was independently found also by C. Mortensen in 1979, who called it $\mathsf{C_{0.1}}$ (see [Mortensen 1989, p. 299]).

Sette showed that logical connectives $\wedge$, $\vee$ and $\equiv$ are defined by means $\neg$ and $\supset$:

$$\begin{aligned} A \wedge B &= (((A \supset A) \supset A) \supset \neg((B \supset B) \supset B)) \supset \neg(A \supset \neg B), \\ A \vee B &= (A \supset \neg\neg A) \supset (\neg A \supset B), \\ A \equiv B &= (A \supset B) \wedge (B \supset A). \end{aligned}$$

There are several different axiomatizations of $\mathsf{P1}$ with initial connectives $\neg$ and $\supset$. See, for example, [Sette 1973] and [Loparic & da Costa 1986]. The most simple is the following [Sette & Alves 1996]

$$\begin{aligned} &Ax1. \quad A \supset (B \supset A), \\ &Ax2. \quad (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)), \\ &Ax3. \quad (\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset A), \\ &Ax4. \quad (A \supset B) \supset \neg\neg(A \supset B). \end{aligned}$$

Inference rule: *modus ponens*.

4. *Properties of* $\mathsf{P1}$

(1) $\mathsf{P1}$ is a three-valued logical calculus (see above);
(2) $\mathsf{P1}$ is maximal in the following sense: any extension of $\mathsf{P1}$ by a new classical tautology gives classical propositional logic $\mathsf{C_2}$ [Sette 1973] (see also [Mortensen 1989, Theorem 5.4]).
(3) The matrix logic $\mathsf{P1}$ is the combination of two three-valued isomorphs of $\mathsf{C_2}$. Moreover, just these two isomorps are contained in Bochvar's three-valued nonsense logic $\mathsf{B_3}$ [Karpenko 2000].
(4) Although the logic $\mathsf{P1}$ is paraconsistent, in 1997 Prof. E.K. Voijshvillo and Béziau [Béziau 1997] independently discovered that from $\neg A$ and $\neg\neg A$ follows B. It is easy to check that the formula $\neg A \supset (\neg\neg A \supset B)$ is verified by the three-valued tables for $\neg$ and $\supset$ in $\mathsf{P1}$. This formula in the form $A \supset \neg A \supset (\neg\neg A \supset B)$ was already known to Łukasiewicz (see [Jaskowski 1948]). So, let us denote it as Luk. But Voijshvillo's result is much more stronger because only Ax 1 and Ax 3 with *modus ponens* are

used to deduce B from $\neg A$ and $\neg\neg A$. Note that formula Luk is verified in the four-valued logic V (see [Puga & da Costa 1988]), which also is a reconstruction of Vasiliev's ideas.
(5) Logic P1 is paraconsistent only for atomic formulas [Béziau 1997]. This follows immediately from the properties of truth-tables for P1.

5. *Atomic and molecular paraconsistent logics*

(1) L is an atomic paraconsistent logic if there is formula B that does not follow from p and $\neg p$, where p is propositional variable;
(2) L is a molecular paraconsistent logic if there are formulae A and B such that B does not follow from A and $\neg A$, where A, B are are molecular formulae.
There are four possible combinations:
(a) Logic L satisfies the definitions (1) and (2). This is true for most paraconsistent logics, for example N.C.A. da Costa's logics C_n [da Costa 1963], relevant logics [Anderson & Belnap 1975], etc;
(b) Logic L does not satisfy definitions (1) and (2). This is true, for example, for classical logic C_2, intuitionistic logic, Łukasiewicz's many-valued logics [Łukasiewicz & Tarski 1930], etc;
(c) Logic L does not satisfy (1) but satisfies (2). It is easy to show that under some natural assumptions (rule of *modus ponens*, rule of adjunction, deduction theorem) there are no such logics;
(d) Logic L satisfies (1) but does not satisfy (2). This is true for Sette's logic P1.

6. *Molecular paraconsistent logic with property* Luk

Now we build a paraconsistent logic that satisfies definitions (1) and (2) and verifies the formula Luk. For this purpose let us take Łukasiewicz's three-valued logic [Łukasiewicz 1920] in which the negation $\sim$ (involution) is replaced by Sette's negation $\neg$, conjunction $\wedge$ is $min(x, y)$, disjunction $\vee$ is $max(x, y)$. Łukasiewicz's implication $\rightarrow$ is defined in the following way:

$\rightarrow$	0	$1/2$	1
0	1	1	1
$1/2$	$1/2$	1	1
$1*$	0	$1/2$	1

7. *Difficulties connected with the definition of paraconsistency*

Since we have derivations $\neg A, \neg\neg A \vdash B$ and $A, \neg A \vdash \neg B$ in some paraconsistent logics it is necessary to introduce restrictions for the rejection of the principle ECQ. The last derivation holds in Johanson's minimal logic [Johanson 1936]. Therefore one might try to use a stronger constraint, for instance: for no syntactically definable class of sentences (for example, negated sentences) Σ, do we have $A, \neg A \vdash \beta$, for all $\beta \in \Sigma$ [Priest 2002]. Further attempts to tighten up the definition of a paraconsistent logic along this line can be found in [Urbas 1990].

Now let us return to Sette's logic $\mathsf{P1}$. Since this logic is atomic paraconsistent, some logicians do not take into account the derivation $\neg A, \neg\neg A \vdash B$. But in virtue of the discovery of the molecular paraconsistent logics with property Luk the problem of the definition of paraconsistency arises again. Along this line D. Batens suggests to restrict this notion: a logic with the formula Luk is not strictly paraconsistent, i.e. for some A, B is derivable from A and $\neg A$. On the other hand, E.K. Voijshvillo suggests to generalize the notion of paraconsistency: a logic is paraconsistent, if there is no finite set of formulas from which an arbitrary formula B is derivable in this logic.

It seems that the question what a paraconsistent logic is, remains a challenging open problem.

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