

ATOMIC AND MOLECULAR PARAconsistent LOGICS

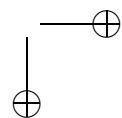
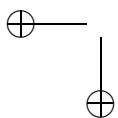
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Abstract

An unusual classification of paraconsistent logics is presented. The classification is based on the distinction between atomic and molecular paraconsistent logics. An example of a molecular paraconsistent logic with the property that $\vdash \neg A \supset (\neg\neg A \supset B)$ is given. Some difficulties connected with a definition of a paraconsistent logic are stressed.

1. *Some history*

The Russian logician N.A. Vasiliev is generally accepted to be the precursor of paraconsistent logic (together with J. Łukasiewicz). One of his most profound ideas is the idea concerning the subdivision of the laws of logic into two worlds: the world of metalogic and the world of empirical laws. In the first case the laws are invariable while at an empirical level the laws of logic can vary and even *the law of a non-contradiction* and *the law of tertium non datur* do not hold or are non-existent in the second world. This subject is under a thorough investigation in his article “The imaginary (non-Aristotelian) logic” [Vasiliev 1912]. This work was done under a strong influence of the researches in the non-Euclidean geometry, that has been done by Vasiliev’s great countryman N.I. Lobatchevsky. The first thorough work on Vasiliev’s logical ideas was V.A. Smirnov’s article [Smirnov 1962]. The detailed review of this article was published in the “Journal of Symbolic logic” [Corney 1965]. This review attracted the attention of a famous Brazilian logician, A. Arruda, and, as the result, there appeared the work [Arruda 1977] containing various types of reconstruction of Vasiliev’s logical ideas. Let us concentrate our attention on the first of three logical systems introduced by Arruda which is called V1. Gilbert’s axiomatisation is given for this system (see also [Tuziak 1997]), a number of metatheorems are proved and it is noticed that V1 has the following three-valued characteristic matrix with two designated truth-values 1 and $\frac{1}{2}$:



x	$\neg x$	\supset	1	$\frac{1}{2}$	0	\wedge	1	$\frac{1}{2}$	0
1	0	1	1	1	0	1	1	1	0
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	1	1	0
0	1	0	1	1	1	0	0	0	0

\vee	1	$\frac{1}{2}$	0	\equiv	1	$\frac{1}{2}$	0
1	1	1	1	1	1	1	0
$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1	1	0
0	1	1	0	0	0	0	1

2. Definition of paraconsistent logic (see [Priest 2002])

The major impetus for the invention of paraconsistent logic has always been the thought that in certain circumstances we may be in a situation where our theory is inconsistent, and yet where we are required to draw inferences in a sensible fashion. Let \vdash be any relationship of logical consequence. Call it *explosive* if it satisfies the condition that for all A and B , $\{A, \neg A\} \vdash B$, *ex contradictione quodlibet* (ECQ). Both classical and intuitionistic logics (and also Łukasiewicz's many-valued logics) are explosive. Clearly, if \vdash is explosive it is not a sensible inference relation in an inconsistent context, for applying it gives rise to *triviality* i.e. everything. Thus, a minimal condition for a suitable inference relation in this context is that \vdash must not be explosive. Such inference relationships, and the logics that have them, have come to be called *paraconsistent*.

3. Sette's logic P1

A. Sette [Sette 1973] constructed the logical calculus P1 which is obtained from the classical propositional calculus C₂ (see [Kleene 1952]) when the axiom

$$(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$$

is replaced by the axiom

$$(\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset A).$$

This logical system has the same characteristic matrix as Arruda's calculus V1.

For the first time such truth-tables appeared in [da Costa 1963], where these have been used for the refutation of some tautologies of C_2 which do not hold in the paraconsistent logic of N.C.A. da Costa, C_1 . See also [da Costa & Alves 1981], where P1 was called F. And finally P1 was independently found also by C. Mortensen in 1979, who called it $C_{0.1}$ (see [Mortensen 1989, p. 299]).

Sette showed that logical connectives \wedge , \vee and \equiv are defined by means \neg and \supset :

$$\begin{aligned} A \wedge B &= (((A \supset A) \supset A) \supset \neg((B \supset B) \supset B)) \supset \neg(A \supset \neg B), \\ A \vee B &= (A \supset \neg\neg A) \supset (\neg A \supset B), \\ A \equiv B &= (A \supset B) \wedge (B \supset A). \end{aligned}$$

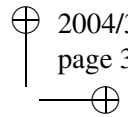
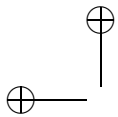
There are several different axiomatizations of P1 with initial connectives \neg and \supset . See, for example, [Sette 1973] and [Loparic & da Costa 1986]. The most simple is the following [Sette & Alves 1996]

$$\begin{aligned} Ax1. & A \supset (B \supset A), \\ Ax2. & (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)), \\ Ax3. & (\neg A \supset \neg B) \supset ((\neg A \supset \neg\neg B) \supset A), \\ Ax4. & (A \supset B) \supset \neg\neg(A \supset B). \end{aligned}$$

Inference rule: *modus ponens*.

4. Properties of P1

- (1) P1 is a three-valued logical calculus (see above);
- (2) P1 is maximal in the following sense: any extension of P1 by a new classical tautology gives classical propositional logic C_2 [Sette 1973] (see also [Mortensen 1989, Theorem 5.4]).
- (3) The matrix logic P1 is the combination of two three-valued isomorphs of C_2 . Moreover, just these two isomorphs are contained in Bochvar's three-valued nonsense logic B_3 [Karpenko 2000].
- (4) Although the logic P1 is paraconsistent, in 1997 Prof. E.K. Voijshtillo and Béziau [Béziau 1997] independently discovered that from $\neg A$ and $\neg\neg A$ follows B . It is easy to check that the formula $\neg A \supset (\neg\neg A \supset B)$ is verified by the three-valued tables for \neg and \supset in P1. This formula in the form $A \supset \neg A \supset (\neg\neg A \supset B)$ was already known to Łukasiewicz (see [Jaskowski 1948]). So, let us denote it as Luk. But Voijshtillo's result is much more stronger because only Ax 1 and Ax 3 with *modus ponens* are



used to deduce B from $\neg A$ and $\neg\neg A$. Note that formula Luk is verified in the four-valued logic V (see [Puga & da Costa 1988]), which also is a reconstruction of Vasiliev's ideas.

(5) Logic $P1$ is paraconsistent only for atomic formulas [Béziau 1997]. This follows immediately from the properties of truth-tables for $P1$.

5. Atomic and molecular paraconsistent logics

- (1) L is an atomic paraconsistent logic if there is formula B that does not follow from p and $\neg p$, where p is propositional variable;
- (2) L is a molecular paraconsistent logic if there are formulae A and B such that B does not follow from A and $\neg A$, where A, B are are molecular formulae.

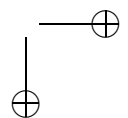
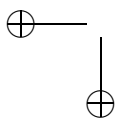
There are four possible combinations:

- (a) Logic L satisfies the definitions (1) and (2). This is true for most paraconsistent logics, for example N.C.A. da Costa's logics C_n [da Costa 1963], relevant logics [Anderson & Belnap 1975], etc;
- (b) Logic L does not satisfy definitions (1) and (2). This is true, for example, for classical logic C_2 , intuitionistic logic, Łukasiewicz's many-valued logics [Łukasiewicz & Tarski 1930], etc;
- (c) Logic L does not satisfy (1) but satisfies (2). It is easy to show that under some natural assumptions (rule of *modus ponens*, rule of adjunction, deduction theorem) there are no such logics;
- (d) Logic L satisfies (1) but does not satisfy (2). This is true for Sette's logic $P1$.

6. Molecular paraconsistent logic with property Luk

Now we build a paraconsistent logic that satisfies definitions (1) and (2) and verifies the formula Luk . For this purpose let us take Łukasiewicz's three-valued logic [Łukasiewicz 1920] in which the negation \sim (involution) is replaced by Sette's negation \neg , conjunction \wedge is $\min(x, y)$, disjunction \vee is $\max(x, y)$. Łukasiewicz's implication \rightarrow is defined in the following way:

\rightarrow	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1^*	0	$\frac{1}{2}$	1



7. Difficulties connected with the definition of paraconsistency

Since we have derivations $\neg A, \neg\neg A \vdash B$ and $A, \neg A \vdash \neg B$ in some paraconsistent logics it is necessary to introduce restrictions for the rejection of the principle ECQ. The last derivation holds in Johanson's minimal logic [Johanson 1936]. Therefore one might try to use a stronger constraint, for instance: for no syntactically definable class of sentences (for example, negated sentences) Σ , do we have $A, \neg A \vdash \beta$, for all $\beta \in \Sigma$ [Priest 2002]. Further attempts to tighten up the definition of a paraconsistent logic along this line can be found in [Urbas 1990].

Now let us return to Sette's logic P1. Since this logic is atomic paraconsistent, some logicians do not take into account the derivation $\neg A, \neg\neg A \vdash B$. But in virtue of the discovery of the molecular paraconsistent logics with property LUK the problem of the definition of paraconsistency arises again. Along this line D. Batens suggests to restrict this notion: a logic with the formula LUK is not strictly paraconsistent, i.e. for some A, B is derivable from A and $\neg A$. On the other hand, E.K. Vojshvillo suggests to generalize the notion of paraconsistency: a logic is paraconsistent, if there is no finite set of formulas from which an arbitrary formula B is derivable in this logic.

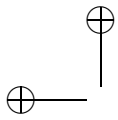
It seems that the question what a paraconsistent logic is, remains a challenging open problem.

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REFERENCES

- [Anderson A.R. & Belnap N.D., jr. 1975] *Entailment: The logic of relevance and necessity*, Vol. 1. Princeton.
- [Arruda A.I. 1977] On the imaginary logic of N.A. Vasiliev, in: *Non-Classical Logics, Model Theory and Computability*, North-Holland, Amsterdam, pp. 3–24.
- [Béziau J.-Y. 1997] What is paraconsistent logic?, *Relatorios de Pesquisa e Desenvolvimento*, Dezembro (Preprint 50/97).
- [Comey D.D. 1965] Review of V.A. Smirnov 1962, *The Journal of Symbolic Logic* 30, pp. 368–370.
- [Costa N.C.A. da 1963] Calculus propositionnels pour les systèmes formels inconsistants, *Comptes Rendus Acad. Sci.* 257, pp. 3790–3792.

- [Costa N.C.A. da & Alves E.N. 1981] Relations between paraconsistent logic and many-valued logic, *Bulletin of the Section of Logic* 10, pp. 185–191.
- [Jaskowski S. 1948] Rachunek zdań dla systemów dedukcyjnych sprzecznych, *Studia Societatis Scientiarum Torunensis* 1, N 5, Sectio A (English translation: Propositional calculus for contradictory deductive systems, *Studia Logica* 24 (1969), pp. 143–157).
- [Johansson I. 1936] Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus, *Compositio Mathematicae* 4, pp. 119–136.
- [Karpenko A.S. 2000] Maximal paraconsistent logic: the combination of two three-valued isomorphs of classical propositional logic, in: *Frontiers of Paraconsistent Logic* (eds. D. Batens *et al.*). King's College Publications. London.
- [Kleene S.C. 1952] *Introduction to Metamathematics*. N.Y.
- [Loparic A. & N.C.A. Costa da 1986] Paraconsistency, paracompleteness and induction, *Logique et Analyse* 113, pp. 73–80.
- [Łukasiewicz J. 1920] O logice trójwartościowej, *Ruch Filozoficzny* 5, s. 170–171. (English translation: On three-valued logic, in: Łukasiewicz J. *Selected works*. Warszawa, 1970, pp. 87–88).
- [Łukasiewicz J. & Tarski A. 1930] Untersuchungen über den Aussagenkalkül, *Comptes rendus de la Société des Sciences et des Lettres de Varsovie* 23, pp. 1–21. (English translation: Investigations into the sentential calculus, in: Łukasiewicz J. *Selected works*. Warszawa, 1970, pp. 131–152).
- [Mortensen C. 1989] Paraconsistency and C1, in: *Paraconsistent logic: Essays on the inconsistent* (eds. G. Priest, R. Routley, J. Norman). Philosophia Verlag. München, pp. 289–305.
- [Priest G. 2002] Paraconsistent logic, in: *Handbook of Philosophical Logic*. Vol. 6 (eds. D. Gabbay, F. Guenther). Kluwer Academic Publishers. Dordrecht. 2nd ed.
- [Puga L.Z. & N.C.A. da Costa 1988] On the imaginary logic of N.A. Vasiliev, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 34, pp. 205–211.
- [Sette A.M. 1973] On propositional calculus P1, *Mathematica Japonica* 16, pp. 173–180.
- [Sette A.M. & Alves E.H. 1996] On the equivalence between some systems of non-classical logic, *Bulletin of the Section of Logic* 25, pp. 68–72.
- [Smirnov V.A. 1962] Logical views of N.A. Vasiliev, in *Outlines of the History of Logic in Russia*, Moscow, pp. 242–257 (in Russian).
- [Tuziak R. 1997] Finitely many-valued paraconsistent systems, *Logic and Logical Philosophy* 5, pp. 121–127.
- [Urbas I. 1990] Paraconsistency, *Studies in Soviet Thought* 39, pp. 343–354.



[Vasiliev N.A. 1912] Imaginary (non-Aristotelian) logic, in: Vasiliev N.A. *Imaginary Logic. Selected works*. Moscow, 1989, pp. 53–93 (in Russian).

