Logique & Analyse **173–174–175** (2001), 165–188

ALL PREMISES ARE EQUAL, BUT SOME ARE MORE EQUAL THAN OTHERS

LIZA VERHOEVEN*

Abstract

This paper proposes two adaptive approaches to inconsistent prioritized belief bases. Both approaches rely on a selection mechanism that is not applied to the premises as they stand, but to the consequence sets of the belief levels. One is based on classical compatibility, the other on the modal logic T of Feys. For both approaches the two main strategies of inconsistency adaptive logics are formulated: the reliability strategy and the minimal abnormality strategy. All four systems are compared and found useful.

1. *Introduction*

The context of a reasoning process gives information about the status of the premises. All premises are premises, but some might be more premise than others. It occurs that some premises are more recent or more reliable or plainly more important than others. When they are ranked according to their priority they form a prioritized belief base. Here we use the ordered set $\Sigma = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ to represent a prioritized belief base where the elements grouped in the set Γ_i are of higher priority than the ones grouped in Γ_j if $i < j$.

The aim of this paper is to elaborate propositional logics that can handle an inconsistent prioritized belief base and that have a consequence relation that is as classical as possible, that is, that coincides with Classical Logic

*I am a Research Assistant of the Fund for Scientific Research – Flanders (Belgium). The research for this paper also benefited from a bilateral scientific exchange project funded by the Ministry of the Flemish Community (project BIL01/80) and the Polish State Committee for Scientific Research. I am greatly indebted to Guido Vanackere for his helpful critiques and his stimulating confrontations and also to Diderik Batens for insisting on the usefulness of the modal version.

(henceforth CL) for consistent belief bases.[1] The proof theories of the logics presented here are adaptive, which means that the proofs are dynamic. In dynamic proofs the set of derived formulas can not only increase during the course of the proof. Previously derived formulas may be withdrawn, and previously withdrawn formulas may be rehabilitated at a later stage, because insight in the premises has grown. Apart from the fact that dynamic proofs are much closer to human reasoning processes than static proofs, they are very useful for undecidable problems, such as tracing inconsistencies. An introduction to dynamic proofs is given in Section 3. Notwithstanding that the derivation of formulas in the proofs has a provisional or conditional character, a consequence relation can be defined on the basis of final derivability, which, in turn, is defined in terms of derivation in possible extensions of the proof.

We present two approaches. The first relies on a kind of prioritized compatibility that admits inconsistent Γ_i, though information from inconsistent Γ_i is ignored. Before we apply a compatibility criterion we must obtain a clear view on the given information. In order to do so, we have to look for the essential constitutive parts of each belief level, i.e., we have to derive unanalysable formulas from the belief level which are together equivalent to the information in its original (and often accidental) given shape. Thus we better apply our compatibility criterion to all CL-consequences of a belief level, instead of just to the premises as they stand. [2] Hence the way in which the information is presented in the premises does not influence our results. A formal definition of essential constitutive parts is given in Section 2. This definition allows for a straightforward and intuitively correct interpretation of the relevant information of a belief level, which is exactly what we need when we want to handle inconsistency on the base of priority.

Our compatibility criterion relies on a selection mechanism. Within the field of adaptive logics, two selection strategies are well developed, namely, the reliability strategy and the minimal abnormality strategy. The former is best understood on the proof theoretic level. It suspects all (sub)formulas that are involved in the derivation of some abnormality — in CL all abnormalities surface as inconsistencies — to be unreliable. The latter strategy is best understood on the semantic level. The models are minimally abnormal, i.e. they do not verify more abnormalities than needed in order to verify the premises. In Section 2, we present the procedure of prioritized compatibility for both strategies. As was already mentioned, in Section 3, the mechanism and importance of dynamic proofs are explained. The dynamic proof theory

[1] The first of the systems described here is also given in [7].

[2] The latter is what is usually done, as in [5], [3], [6] and [2].

of the adaptive logic for reliability based prioritized compatibility is formulated in Section 4, followed by an example. The semantics for the minimal abnormality strategy are given in Section 9.

The second approach is based on a translation of the information of the prioritized belief base into the language of modal logic. A basic idea is that we should take into account all consequences of a belief level. This is also a basic idea of our first approach, and so it is the main idea of this paper. This approach is based on an elaboration of some logics for diagnosis that are given (for the predicative case) in [2]. In a style related to Jaskowski's approach to inconsistency, the use of the modal operators blocks the application of the rule *ex contradictione quodlibet*, but in contrast with Jaskowski's approach, the adaptive character of the logic allows for the derivation of all classical consequences whenever no inconsistencies are involved. One of the advantages is that the priorities can be entered in the modal language. Moreover the modal language disposes of the instruments to compose a syntactical recognizable form for the inconsistencies. The only disadvantage of the present version is that the Γ_i are supposed to be consistent themselves. Here, also, the two main strategies for adaptive logics are given. The semantics and the proof theory of the reliability strategy for the modal version are presented in Section 7. In Section 8, the logical equivalence of the reliability strategy for prioritized compatibility and that for the modal version is proved (in the case of consistent Γ_i). That Section concludes with the qualitative and interpretational differences between both. The minimal abnormality strategy for this approach is also introduced in Section 9 by way of its semantics. The examples in Section 10 show that for this strategy the two approaches do differ in result. A conclusion can be read in Section 11.

2. *The Procedure of the Prioritized Compatibility Approach*

The purpose of this Section is to present a procedure for the propositional case that is completely defined by means of CL. We refer to this procedure with the name "the Procedure". In Section 5, we show that the adaptive logic presented in Section 4 is equivalent to the Procedure. This Section is important in that it gives interesting insight in the adaptive logic. Moreover the Procedure allows for an intuitive interpretation of the difference between the reliability strategy and the minimal abnormality strategy — which are well known to people familiar with adaptive logics. Throughout the paper we use the ordered set $\Sigma = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ to represent a prioritized belief base, where the elements grouped in the set Γ_i belong to $\mathcal{W}$ (which here denotes the set of well formed formulas of the standard propositional language), and are of higher priority than the ones grouped in Γ_j if $i < j$.

The Procedure proceeds step by step by selecting from each lower prioritized level the relevant information that is compatible with the information collected from the higher belief levels. In each step i we have two sets: the already obtained information of the levels Γ_1 to Γ_{i-1} — let us call this $P_{i-1}(\Sigma)$ — and the set Γ_i from which information is to be selected. As mentioned in the introduction, we want to work with all the CL-consequences of Γ_i, so we really make a selection of information that does not depend on the presentation of the information. In case of an inconsistent Γ_i, the consequence set of Γ_i is trivial and as we want to work with CL, the only requirement is that such a level should be ignored, what is an acceptable treatment for an inconsistent belief level. Let us first look at our compatibility criterion.

It is obvious that we do not select the consequences that are not compatible, but we do not want to select all consequences that are compatible either. Let us illustrate why. Suppose A and B are separately compatible with $P_{i-1}(\Sigma)$, but not jointly. In that case we surely do not want to retain both. The criterion must be such that only those CL-consequences of Γ_i that are jointly compatible with $P_{i-1}(\Sigma)$ are selected.

Still, if we plainly apply this criterion to $Cn_{\mathsf{CL}}(\Gamma_i)$, we do not obtain the desired results, even not in some very simple situations, for example $P_{i-1}(\Sigma) = Cn_{\mathsf{CL}}(\{p\})$ and $\Gamma_i = \{\neg p, q\}$. Though q is an item to be selected according to our intuitions, it is not jointly compatible with $\neg p \vee \neg q$ while each formula separately is compatible with $P_{i-1}(\Sigma)$. The problem here is that $\neg p \vee \neg q$ is not a relevant consequence of $\neg p$, it is a weakening that brings in irrelevant information. Before applying the compatibility selection, the irrelevant information should be left out.

One way to trace the irrelevant information is to analyse the formulas by using a normal form. When we consider the three connectives $\wedge$, $\vee$ and $\neg$, the following observations can be made. The conjunction is purely connecting: $A \wedge B \vdash_{\mathsf{CL}} A$, $A \wedge B \vdash_{\mathsf{CL}} B$ and $A, B \vdash_{\mathsf{CL}} A \wedge B$. Unlike the conjunction, the disjunction is not decomposable. The negation is the only unary connective. Keeping this in mind, we can see it is useful to apply the conjunction rule only to undecomposable (or completely decomposed) formulas, or even more simple, not to apply the conjunction rule at all. In view of the equivalence in CL of the at first sight undecomposable formula $\neg(A \vee B)$ and the obviously decomposable formula $\neg A \wedge \neg B$, it is also useful to drive the negation inwards, so that we derive formulas in which subformulas of the form $\neg A$ only occur if A is a propositional constant. In this way we obtain the normal form that has the most transparent structure for this context: conjunctions of disjunctions of propositions and their negations, or even more simple, decomposed disjunctions of (negated) propositions.

By deriving formulas in such (conjunctive) normal form, the different conjuncts can be taken to be the atomic parts of the formula. The atomic parts of all formulas of a level can easily be compared as they are written as disjunctions of atoms (primitive formulas and their negations). An atomic part (of a certain level), i.e., an undecomposable formula that is a disjunction of atoms, is considered relevant if and only if the set of its atoms, which we call the atomic set, is not a superset of the set of atoms of any other atomic part of that level. We use the notation $A \sqsubset B$ for A is an atomic part of the conjunctive normal form of B. $At(A)$ denotes the atomic set corresponding to the atomic part A. On the other hand, a set made up of atoms does not uniquely determine a formula. Because in the logics used here the disjunction is commutative, we write $\overline{A}$ for a chosen representative of all disjunctions corresponding to $At(A)$.[3]

Definition 1: $R_i(\Sigma) = \{\overline{A} \mid A \sqsubset B$ *for some* $B \in Cn_{\mathsf{CL}}(\Gamma_i)$ *and for no* $C \in Cn_{\mathsf{CL}}(\Gamma_i)$ *there is a* $D \sqsubset C$ *such that* $At(D) \subset At(A)\}$. [4]

Definition 2: $\tilde{R}_i(\Sigma) = \{A \in R_i(\Sigma) \mid$ *there is no* $\Theta \subseteq R_i(\Sigma)$ *such that* $P_{i-1}(\Sigma) \cup \Theta \nvdash_{\mathsf{CL}} \bot$ *and* $P_{i-1}(\Sigma) \cup \Theta \cup \{A\} \vdash_{\mathsf{CL}} \bot\}$.

Definition 3: $P_i(\Sigma) = Cn_{\mathsf{CL}}(P_{i-1}(\Sigma) \cup \tilde{R}_i(\Sigma))$.

So the general step of the Procedure is defined. The only thing to be specified is the beginning and the ending. Setting $P_0(\Sigma) = Cn_{\mathsf{CL}}(\emptyset)$ can clearly do the job. $\tilde{R}_1(\Sigma)$ then is the consistent core of $R_1(\Sigma)$ and $P_1(\Sigma)$ the classical closure of $\tilde{R}_1(\Sigma)$. It should also be clear that $P_n(\Sigma)$ is the ending of the Procedure.

Let us now look at an example. Let $\Sigma = \langle\{\neg p \vee \neg q\}, \{p, q\}\rangle$. It is obvious that $\tilde{R}_1(\Sigma) = R_1(\Sigma) = \{\neg p \vee \neg q\}$, $R_2(\Sigma) = \{p, q\}$ and $\tilde{R}_2(\Sigma) = \emptyset$. The results are not that strong. Stronger would be that $p \vee q$ is also a consequence. Now why isn't it? $p \vee q \notin R_2(\Sigma)$ (and hence certainly $p \vee q \notin \tilde{R}_2(\Sigma)$) because it is a weakening of both p and q that are in $R_2(\Sigma)$ and because weakenings are treated as irrelevant information. Why is the stronger result acceptable? When we think of maximal consistent sets of $P_1(\Sigma) \cup R_2(\Sigma)$ that contain $P_1(\Sigma)$, we have $P_1(\Sigma) \cup \{p\}$ and $P_1(\Sigma) \cup \{q\}$ and both CL-entail $p \vee q$. Hence we have good reasons to define another compatibility criterion, which selects the formulas that are derivable from all sets that are

[3] From here on i is an index varying from 1 to n, except when specified otherwise.

[4] The definition $R_i(\Sigma) = \{\overline{A} \mid A$ is a conjunction-free element of $Cn_{\mathsf{CL}}(\Gamma_i)$ (in normal form) and there is no such an element B such that $At(B) \subset At(A)\}$ would do as well, but to have uniformity with the proof theory we prefer Definition 1.

obtained by extending $P_{i-1}(\Sigma)$ with as much as possible jointly compatible elements of $R_i(\Sigma)$. To illustrate this difference we first reformulate the above definition in terms of maximal consistent selections, which we collect in the sets $S_i(\Sigma)$.

Definition 4: *A* maximal consistent subset *of a set is a consistent subset such that it is not a proper subset of another consistent subset.*

Definition 5: $S_i(\Sigma) = \{\Theta \mid \Theta$ *is a maximal consistent subset of* $P_{i-1}(\Sigma) \cup R_i(\Sigma)$ *and* $P_{i-1}(\Sigma) \subseteq \Theta\}$.

Definition 6: $\tilde{R}_i(\Sigma) = \{A \mid$ *for all* $\Theta \in S_i(\Sigma)$, $A \in \Theta\}$.

Now it is easy to compare this with the strengthening.[5]

Definition 7: $\tilde{R}^s_i(\Sigma) = \{A \mid$ *for all* $\Theta \in S_i(\Sigma)$, $\Theta \vdash_{\mathsf{CL}} A\}$.

The above two mechanisms are in fact applications of the reliability strategy and the minimal abnormality strategy. These are the two main strategies of inconsistency adaptive logics. They were first presented in [1] and used in almost every following article on adaptive logics by members of the Ghent Group. For the reliability strategy we formulate the proof theory in Section 4 and for the minimal abnormality strategy the semantics in Section 9.

3. *Dynamic Proofs*

Dynamic proofs are designed to be a more faithful representation of human reasoning and to describe and direct reasoning processes for undecidable problems. It is the latter that is the case here, since compatibility is undecidable. Their main characteristic is that derivations may be made conditionally. Therefore a fifth element is added to the lines of a proof. This keeps track of the conditions that are necessary for the derived formula to be a consequence. At each stage of the proof it has to be checked whether the conditions are fulfilled or not. A condition is not fulfilled when it is explicitly contradicted in the so far accepted derivations.[6] In that case, that derivation is no longer

[5] We only defined the strengthened sets $\tilde{R}^s_i(\Sigma)$, of course this changes also the sets $P_i(\Sigma)$ and hence the sets $S_i(\Sigma)$.

[6] By the latter we mean derivations that are unconditionally made or that have fulfilled conditions at that stage.

a valid conclusion. We indicate this by marking the respective line. When at a later stage the condition is fulfilled, the derivation is again considered as valid and the marking is lifted.

The rules for a dynamic proof can be grouped in three sorts: the premise rule(s), which may be lacking or incorporated in the other sorts, the unconditional rule(s) and the conditional rule(s). All can be applied at any time in the proof, but some only conditionally. The marking definitions regulate the control of the conditions and the subsequent (un)marking if needed.

A line in a dynamic proof consists of five elements: (i) the line number, (ii) the formula derived on that line, (iii) the numbers of the lines used to derive the second element, (iv) the rule applied to derive the second element and (v) the fifth element referring to the condition on which the second element is derived.

As the derivation of formulas in the proofs has a provisional or conditional character, the notion of derivability has to be specified. We call the new concept final derivability. The second elements of lines that have an empty fifth element are finally derived on that line. For conditionally made derivations, it does not suffice that the line is unmarked to be finally derivable. We have no guarantee that it stays unmarked in all extensions of the proof. When there is an extension of the proof in which the line is marked, there should be a further extension possible in which it is unmarked again to accept its second element as finally derived.

4. *An Adaptive Logic for Reliability based Prioritized Compatibility*

4.1. *The Proof Theory*

The Procedure describes subsequent classical closures of the union of (i) already obtained information out of previous levels, and (ii) a selection of the relevant information of the respective level. In a proof from a prioritized belief base Σ this will all be done by two rules: a conditional rule RC and an unconditional rule RU, and two marking definitions. The condition of RC has a double function: one has to do with relevance, the other with compatibility. According to the double condition two sorts of marking are needed, see Definition 8 and Definition 11.

First and foremost we have to search for the relevant atomic parts of the classical closure of Γ_i. Therefore, for each CL-consequence of Γ_i that we introduce, we keep track of the atomic parts of the introduced formula in the fifth element.[7]

[7] Note that we do not have a separate premise rule.

RC If $\Gamma_i \vdash_{\mathsf{CL}} A$, one may add a line consisting of (i) the appropriate line number, (ii) A, (iii) a dash, (iv) RC, and (v) $\{\langle \overline{B}, i\rangle \mid B \sqsubset A\}$.

The condition is the set of atomic parts of the introduced formula labelled by the number of their level. As was already mentioned above, it functions as a double condition. The first one is about the relevance of the atomic parts, they should belong to $R_i(\Sigma)$. We do not have the sets $R_i(\Sigma)$, but we can make a tentative construction of the $R_i(\Sigma)$ while the proof proceeds, namely, by collecting the atomic parts that are in the fifth elements. As we can only base our construction on the already made applications of RC, i.e., on the insight in the CL-closures of the belief levels up till now, we can never be sure that we do not collect irrelevant atomic parts. Therefore, at each stage, the new introduced atomic parts have to be compared with the relevant atomic parts from the stage before, to see whether some atomic parts become irrelevant.[8] This is done as follows. We construct the set $R_{i,s}(\Sigma)$ of relevant atomic parts and the set $I_{i,s}(\Sigma)$ of irrelevant atomic parts, where the subscripts i and s respectively refer to the priority level and the stage of the proof, starting with $R_{i,0}(\Sigma) = \emptyset$ and $I_{i,0}(\Sigma) = \emptyset$.

Construction $R_{i,s}(\Sigma)$ and $I_{i,s}(\Sigma)$

(1) At each stage $s + 1$ following stage s, $R_{i,s+1}(\Sigma)$ is defined as follows. If line $s + 1$ was an application of RC for Γ_k, $R_{k,s}(\Sigma)$ has to be extended with the atomic parts (the conjuncts of the conjunctive normal form) of the introduced formula A: $R_{k,s+1}(\Sigma) := R_{k,s}(\Sigma) \cup \{\langle \overline{B}, k\rangle \mid B \sqsubset A\}$ and $R_{l,s+1}(\Sigma) := R_{l,s}(\Sigma)$ for $l \neq k$. Otherwise $R_{i,s+1}(\Sigma) := R_{i,s}(\Sigma)$.

(2) After each extension of $R_{k,s}(\Sigma)$ to $R_{k,s+1}(\Sigma)$, all elements whose atomic set is a superset of the atomic set of another element with the same label, are transferred to $I_{k,s+1}(\Sigma)$.[9]

In such a way we keep track of which introduced information is either relevant or irrelevant at the respective stage of the proof. When it turns out that some introduced formula contains irrelevant atomic parts, it is marked by the following marking definition.

Definition 8: *A line derived on condition* Θ *is I-marked at stage* s *iff for some* $1 \leq i \leq n$, $\Theta \cap I_{i,s}(\Sigma) \neq \emptyset$.

[8] This comparison is also needed to check whether new introduced atomic parts turn out irrelevant in the light of the present insight in the CL-closures of the belief levels.

[9] Hence: if $A \in I_{k,s+1}(\Sigma)$, then $A \in R_{k,s}(\Sigma)$ and $A \notin R_{k,s+1}(\Sigma)$.

A second selection must be made (in which the condition of RC fulfills its second function): information incompatible with the information obtained from previous levels must be deleted. To have the consequences of all the information collected out of several previous levels at one's disposal and also to take the CL-closure at the end, we need an unconditional rule.

RU If $B_1, \ldots, B_m \vdash_{\mathsf{CL}} A$, and $B_1, \ldots, B_m$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_m$ respectively,[10] then one may add a line consisting of (i) the appropriate line number, (ii) A, (iii) the numbers of the lines on which the B_i are derived, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_m$.[11]

We call the level of a derivation the lowest priority level that is used to derive the second element, i.e., the maximum of the labels of the elements in the condition.[12]

To trace the elements of $R_{i,s}(\Sigma)$ that are jointly incompatible with the information of preceding levels, we look for explicit contradictions at stage s of their joint compatibility. The latter can be expressed in several ways. For $\{A_1, \ldots, A_m\}$ not jointly compatible with a set Θ, we use $\Theta \vdash_{\mathsf{CL}} \neg A_1 \vee \ldots \vee \neg A_m$, because it most transparently expresses that it follows from Θ that at least one of the $A_1, \ldots, A_m$ must be false. As it concerns incompatibility with the information of preceding levels, the form $\neg A_1 \vee \ldots \vee \neg A_m$ should be derived at a level smaller than i. We want this derivation to indicate that the elements $A_1, \ldots, A_m$ are not to be selected at the respective stage, therefore we have to make sure that every disjunct of that form is significant. If for example $\Theta \vdash_{\mathsf{CL}} \neg A_1 \vee \ldots \vee \neg A_{m-1}$ also holds, there is no reason not to select A_m.

Definition 9: *$\neg A_1 \vee \ldots \vee \neg A_m$, abbreviated $Don(\{A_1, \ldots, A_m\})$,*[13] *is at stage s a Don-formula of level i iff $\{\langle A_1, i\rangle, \ldots, \langle A_m, i\rangle\} \subseteq R_{i,s}(\Sigma)$ and the formula is at that stage derived on an unmarked line at a level smaller than i. $Don(\{A_1, \ldots, A_m\})$ is at stage s a minimal Don-formula of level i iff for no $\Theta \subset \{A_1, \ldots, A_m\}$, $Don(\Theta)$ is at that stage a Don-formula of level i.*

[10] For $m = 0$, CL-theorems can be introduced to handle inconsistent Γ_i.

[11] In appropriate heuristics RU should only be applied when there is no $1 \leq i \leq n$ such that $\Gamma_i \vdash_{\mathsf{CL}} A$.

[12] The level of a derivation on an empty condition is 0.

[13] The abbreviation *Don* stands for disjunction of negations.

Definition 10: $U_{i,s}(\Sigma) = \{\langle A, i\rangle \in R_{i,s}(\Sigma) \mid$ *there is at stage* s *a minimal* Don*-formula* $Don(\Theta)$ *of level* i *for which* $A \in \Theta\}$.

Definition 11: *A line derived on condition* Θ *is U-marked at stage* s *iff for some* i, $\Theta \cap U_{i,s}(\Sigma) \neq \emptyset$.

Definition 12: *A formula* A *is finally derived at line* j *at stage* s *of a proof iff line* j *is not marked at stage* s *and any extension of the proof in which line* j *is marked, may be further extended in such a way that line* j *is unmarked again.*

Definition 13: $\Sigma \vdash_{\mathsf{PCOM}^{\mathsf{r}}} A$ *iff* A *is finally derived in a proof of the format described above.*

4.2. *An Example*

$\Sigma = \langle\{u, u \supset \neg p\}, \{r \supset \neg s\}, \{(p \wedge q) \vee s\}, \{r, r \supset t\}\rangle$

1	u	-	RC	$\{\langle u, 1\rangle\}$	
2	$u \supset \neg p$	-	RC	$\{\langle \neg u \vee \neg p, 1\rangle\}$	$\surd_I$
3	$\neg p$	-	RC	$\{\langle \neg p, 1\rangle\}$	
4	$r \supset \neg s$	-	RC	$\{\langle \neg r \vee \neg s, 2\rangle\}$	
5	$(p \wedge q) \vee s$	-	RC	$\{\langle p \vee s, 3\rangle, \langle q \vee s, 3\rangle\}$	
6	r	-	RC	$\{\langle r, 4\rangle\}$	$\surd_U$
7	t	-	RC	$\{\langle t, 4\rangle\}$	
8	s	3, 5	RU	$\{\langle \neg p, 1\rangle, \langle p \vee s, 3\rangle, \langle q \vee s, 3\rangle\}$	
9	$\neg r$	4, 8	RU	$\{\langle \neg p, 1\rangle, \langle p \vee s, 3\rangle, \langle q \vee s, 3\rangle, \langle \neg r \vee \neg s, 2\rangle\}$	
10	$u \supset \neg p$	3	RU	$\{\langle \neg p, 1\rangle\}$	

From the introduction of line 3, line 2 gets I-marked. That is because $\langle \neg p, 1\rangle \in R_{1,3}(\Sigma)$ and hence $\langle \neg u \vee \neg p, 1\rangle \in I_{1,3}$. At stage 9 a Don-formula of level 4 is derived, $\langle r, 4\rangle \in U_{4,9}(\Sigma)$ and hence line 6 is U-marked.

As line 10 illustrates, RU guarantees that the logic is closed under CL.

5. *Equivalence of* $\mathsf{PCOM}^{\mathsf{r}}$ *and the Procedure*

To make the proof more transparent, let us first define the set of unreliable relevant information of each stage.

Definition 14: $Don(\{A_1, \ldots, A_m\})$ *is a minimal* Don*-consequence of level* i *iff* $\{A_1, \ldots, A_m\} \subseteq R_i$, $P_{i-1}(\Sigma) \vdash_{\mathsf{CL}} Don(\{A_1, \ldots, A_m\})$ *and for no* $\Theta \subset \{A_1, \ldots, A_m\}$, $P_{i-1}(\Sigma) \vdash_{\mathsf{CL}} Don(\Theta)$.

Definition 15: $U_i(\Sigma) = \cup\{\Delta \mid Don(\Delta)$ *is a minimal* Don*-consequence of level* $i\}$.

Lemma 1: $\tilde{R}_i(\Sigma) = R_i(\Sigma) \setminus U_i(\Sigma)$

Proof. The following equivalences hold:

There is no $\Theta \subseteq R_i(\Sigma)$ such that $P_{i-1}(\Sigma) \cup \Theta \nvdash_{\mathsf{CL}} \bot$ and $P_{i-1}(\Sigma) \cup \Theta \cup \{A\} \vdash_{\mathsf{CL}} \bot$

iff

there is no $\Theta' \subseteq R_i(\Sigma)$, with Θ' a set of finite cardinality, such that $P_{i-1}(\Sigma) \cup \Theta' \nvdash_{\mathsf{CL}} \bot$ and $P_{i-1}(\Sigma) \cup \Theta' \cup \{A\} \vdash_{\mathsf{CL}} \bot$

iff

there is no $\Theta' \subseteq R_i(\Sigma)$, with Θ' a set of finite cardinality, such that $P_{i-1}(\Sigma) \nvdash_{\mathsf{CL}} \bigwedge(\Theta') \supset \bot$[14] and $P_{i-1}(\Sigma) \vdash_{\mathsf{CL}} \bigwedge(\Theta' \cup \{A\}) \supset \bot$[15]

iff

there is no $\Theta' \subseteq R_i(\Sigma)$, with Θ' a set of finite cardinality, such that $P_{i-1}(\Sigma) \nvdash_{\mathsf{CL}} \neg \bigwedge(\Theta')$ and $P_{i-1}(\Sigma) \vdash_{\mathsf{CL}} \neg \bigwedge(\Theta' \cup \{A\})$

iff

there is no $\Theta' \subseteq R_i(\Sigma)$, with Θ' a set of finite cardinality, such that $P_{i-1}(\Sigma) \nvdash_{\mathsf{CL}} \bigvee \neg(\Theta')$ and $P_{i-1}(\Sigma) \vdash_{\mathsf{CL}} \bigvee \neg(\Theta' \cup \{A\})$.

From these we can conclude that for $A \in R_i(\Sigma)$, $A \in \tilde{R}_i(\Sigma)$ is equivalent to $A \notin U_i(\Sigma)$. ■

Lemma 2: A *is finally derived on line* j *in a* $\mathsf{PCOM}^{\mathsf{r}}$*-proof from* Σ *iff* A *is the second element of line* j *on a condition* Δ *such that for all* $\langle B, j\rangle \in \Delta$, $B \in \tilde{R}_i(\Sigma)$.

Proof. For the first direction, suppose A is finally derived on line j of a $\mathsf{PCOM}^{\mathsf{r}}$-proof from Σ on condition Δ. We know that line j is unmarked and every extension in which it is marked can be extended in such a way that it is unmarked again. Because the sets $I_{i,s}(\Sigma)$ can only grow, we know that there is no extension of the proof in which line j gets I-marked. The latter

[14] For $\Theta = \emptyset$ this should be $P_{i-1}(\Sigma) \nvdash_{\mathsf{CL}} \bot$, and as this is always fulfilled, this clause may be ignored in that case.

[15] The notation $\bigwedge(\Theta)$ is of course ambiguous, but in the context of the logics used here it does not matter.

also implies that $\Delta \subseteq R_1(\Sigma) \cup \ldots \cup R_n(\Sigma)$ in view of the construction of the sets $I_{i,s}(\Sigma)$.

We proceed by induction on the level of the derivation. Suppose that the level of line j is i. The induction hypothesis states that if A is finally derived on line j' at a level smaller than or equal to $i-1$, than the fifth element of line j' is contained in $\tilde{R}_1(\Sigma) \cup \ldots \cup \tilde{R}_{i-1}(\Sigma)$. The initial condition is fulfilled because a condition of level 0 is the empty set. The property of being finally derived corresponds in fact to a property of the fifth element, here Δ. From the induction hypothesis we know that $\Delta \cap (R_1(\Sigma) \cup \ldots \cup R_{i-1}(\Sigma)) \subseteq \tilde{R}_1(\Sigma) \cup \ldots \cup \tilde{R}_{i-1}(\Sigma)$ (otherwise line j could not have been finally derived, it could have been U-marked in an appropriate extension for which no further extension would have existed in which the marking would be lifted). We only have to prove that $\Delta \cap R_i(\Sigma) \subseteq \tilde{R}_i(\Sigma)$. As line j is finally derived, it also follows that $\Delta \cap U_i(\Sigma) = \emptyset$, otherwise an extension of the proof can be constructed (by introducing the appropriate elements of $R_i(\Sigma)$ and by deriving an appropriate minimal Don-consequence) in which line j is U-marked and can not be unmarked in any extension of the proof. From this observation we can derive that $\Delta \cap R_i(\Sigma) \subseteq R_i(\Sigma) \setminus U_i(\Sigma) = \tilde{R}_i(\Sigma)$.

The other direction is obvious. ■

Theorem 1: $\Sigma \vdash_{\mathsf{PCOM}^r} A$ *iff* $A \in P_n(\Sigma)$.

Proof. Immediately from Lemma 2 and Definition 3. ■

6. *Note on the Semantics of* PCOM^r

The semantics can be defined in various ways. The Procedure can be translated semantically, but as this does not bring any additional insight we do not do this here. Another way is to make successive selections for each level that select those models that verify all the reliable elements of $R_i(\Sigma)$, that is $R_i(\Sigma) \setminus U_i(\Sigma)$. These sets should of course be defined semantically. As it is straightforward how this can be done, and as the semantics for the equivalent system $\mathsf{T}^r_\diamond$ are given in Section 7, we skip the semantics of PCOM^r.

7. *A Modal Adaptive Logic for Prioritized Belief Sets based on Reliability*

7.1. *The Semantics of* $\mathsf{T}^r_\diamond$ *and* T^r

Although the logic presented below can only handle consistent Γ_i and is in that case equivalent to PCOM^r, it has some apparent advantages. Firstly, the priorities can be entered in the modal language. A formula A of priority level 3 is introduced as $\Diamond\Diamond\Diamond A$, abbreviated $\Diamond^3 A$. Secondly, the modal language disposes of the instruments to compose a syntactical recognizable form for the inconsistencies raising from combining different priority levels, for example $(\Diamond p \wedge \neg p) \vee (\Diamond^2 \neg p \wedge p)$. Thirdly and generally, the alternative formalism provides us with another point of view, which gives us new insight in the problem and the solution presented in this paper.

In [2] a modal logic for diagnosis is described. In an analogous way as for the latter logic, the semantics of the logic $\mathsf{T}^{\mathsf{r}}_{\Diamond}$ are obtained by defining a specific subset of the T-models of Σ. I repeat the semantic construction below only for the propositional case, whereas in [2] the predicative version is given. The only difference is how to define a T-model of Σ.

The general plot is to interpret the prioritized belief base within the modal logic T of Feys (which is von Wright's M).[16] Several propositional versions of T may do.

Let $\mathcal{L}^M$ be the standard propositional modal language with $\mathcal{S}$ and $\mathcal{W}^M$ the sets of sentential letters and wffs (well formed formulas).

A T-model M is a quadruple $\langle W, w_0, R, v\rangle$ in which W is a set of worlds, $w_0 \in W$ the real world, R a binary relation on W and v an assignment function. The accessability relation R is reflexive. The assignment function v is defined by:

C1 $\quad v : \mathcal{S} \times W \mapsto \{0, 1\}$

The valuation function $v_M : \mathcal{W}^M \times W \mapsto \{0, 1\}$, determined by the model M is defined by:

C2.1 $\quad$ where $A \in \mathcal{S}$, $v_M(A, w) = v(A, w)$
C2.2 $\quad v_M(\neg A, w) = 1$ iff $v_M(A, w) = 0$
C2.3 $\quad v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$
C2.4 $\quad v_M(\Diamond A, w) = 1$ iff $v_M(A, w') = 1$ for at least one w' such that Rww'.

A model M verifies $A \in \mathcal{W}^M$ iff $v_M(A, w_0) = 1$. A is valid iff it is verified by all models.

In [2] a T-model of $\langle \Gamma_0, \ldots, \Gamma_n\rangle$ was defined as a T-model of $\{\Diamond^i A \mid A \in \Gamma_i\}$ ($\Diamond^i$ stands for a sequence of i diamonds).[17] Here we need the following definition.

[16] The essential thing is that the accessibility relation R is not transitive. So, K would do just as well. However, T allows for a simpler formulation of the formal machinery below.

[17] By M being a T-model of a set of formulas, we mean that the real world w_0 of M verifies that set of formulas.

Definition 16: *A* T*-model of* $\langle\Gamma_1,\ldots,\Gamma_n\rangle$ *is a* T*-model of* $\{\Diamond^i A \mid \Gamma_i \vdash_{\mathsf{CL}} A\}$.

This is the only modification that is needed. An extra restriction on the belief base has to be imposed here: the Γ_i should be consistent themselves, otherwise there would be no models for such Σ.

Abnormalities of a model and the hereupon based selection of models are defined in exactly the same way. Let $\mathcal{F}^p$ be the set of primitive formulas (sentential letters), and let $\mathcal{F}^a$ be the set of atoms (primitive formulas and their negations). An abnormality is a formula of the form $\Diamond^i A \wedge \neg A$, where $A \in \mathcal{F}^a$. For the semantics, for each T-model M of $\Sigma = \langle\Gamma_1,\ldots\Gamma_n\rangle$, a set of abnormal parts is defined (where $1 \leq i \leq n$):

Definition 17: $Ab^i(M) =_{df} \{A \in \mathcal{F}^a \mid v_M(\Diamond^i A \wedge \neg A, w_0) = 1\}$

The adaptive models of Σ are obtained by making a selection of its T-models, first with respect to the sets $Ab^1(M)$, next with respect to the sets $Ab^2(M)$, etc. In fact, disjunctions of abnormalities are needed. It turns out that the attention may be restricted to disjunctions of formulas of the form $\Diamond^i A \wedge \neg A$, with the same i in each disjunct and $A \in \mathcal{F}^a$. By $Dab^i(\Delta)$, the disjunction $\bigvee\{\Diamond^i A \wedge \neg A \mid A \in \Delta\}$ is denoted. $Dab^i(\Delta)$ is said to be a Dab^i-consequence of Σ iff all T-models of Σ verify $Dab^i(\Delta)$. A Dab^i-consequence $Dab^i(\Delta)$ of Σ is called minimal iff there is no $\Delta' \subset \Delta$ such that $Dab^i(\Delta')$ is a Dab^i-consequence of Σ.

The $\mathsf{T}^{\mathsf{r}}_{\diamond}$-models of Σ are the T-models of Σ in which only unreliable formulas behave abnormally. The sets of unreliable formulas are defined with respect to the minimal Dab^i-consequences ($1 \leq i \leq n$) of Σ:

$U^i(\Sigma) =_{df} \bigcup\{\Delta \mid Dab^i(\Delta)$ is a minimal Dab^i-consequence of $\Sigma\}$

Let $\mathcal{M}_\Sigma$ be the set of all T-models of Σ. The $\mathsf{T}^{\mathsf{r}}_{\diamond}$-models of Σ are obtained by defining n selections of $\mathcal{M}_\Sigma$ as follows:

$$\sigma^0(\mathcal{M}_\Sigma) =_{df} \mathcal{M}_\Sigma,$$

$$\sigma^i(\mathcal{M}_\Sigma) =_{df} \{M \in \sigma^{i-1}(\mathcal{M}_\Sigma) \mid Ab^i(M) \subseteq U^i(\Sigma)\}.$$

The $\mathsf{T}^{\mathsf{r}}_{\diamond}$-models of Σ are the members of $\sigma^n(\mathcal{M}_\Sigma)$. It is possible to characterize them directly as follows:

$M \in \mathcal{M}_\Sigma$ is a $\mathsf{T}^{\mathsf{r}}_{\diamond}$-model of Σ iff $Ab^i(M) \subseteq U^i(\Sigma)$ for $1 \leq i \leq n$.

Definition 18: $\Sigma \models_{\mathsf{T}^{\mathsf{r}}} A$ *iff* $\Sigma \models_{\mathsf{T}^{\mathsf{r}}_{\diamond}} A$ *and* $A \in \mathcal{W}$.

7.2. *The Proof Theory of* $\mathsf{T}^{\mathsf{r}}_{\diamond}$ *and* T^{r}

The only modification to obtain the generic rules that govern the dynamic proofs from $\Sigma = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ is the broadening of the condition for writing down premises from $A \in \Gamma_i$ to $\Gamma_i \vdash_{\mathsf{CL}} A$.

PREM If $\Gamma_i \vdash_{\mathsf{CL}} A$, then one may add a line consisting of
- (i) the appropriate line number,
- (ii) $\Diamond^i A$,
- (iii) $-$,
- (iv) Prem, and
- (v) $\emptyset$.

RU If $B_1, \ldots, B_m \vdash_{\mathsf{T}} A$ and $B_1, \ldots, B_m$ occur in the proof with the conditions $\Delta^1, \ldots \Delta^m$ respectively,[18] then one may add a line consisting of
- (i) the appropriate line number,
- (ii) A,
- (iii) the line numbers of the B_i,
- (iv) RU, and
- (v) $\Delta^1 \cup \ldots \cup \Delta^m$.

RC If $B_1, \ldots, B_m \vdash_{\mathsf{T}} A \vee Dab^k(\Theta)$ and $B_1, \ldots, B_m$ occur in the proof with the conditions $\Delta^1, \ldots \Delta^m$ respectively, then one may add a line consisting of
- (i) the appropriate line number,
- (ii) A,
- (iii) the line numbers of the B_i,
- (iv) RC, and
- (v) $\{\Diamond^k A \mid A \in \Theta\} \cup \Delta^1 \cup \ldots \cup \Delta^m$.

It is obvious in view of the rules that A is derivable in a proof from Σ on the condition Δ for which the maximum of the diamonds of its elements is i iff $A \vee Dab^i(\Delta)$ is T-derivable from Σ.

While the selection of models proceeds in terms of the minimal Dab^i-consequences of Σ, the marking definitions proceed in terms of the minimal Dab^i-formulas that have been derived in the proof (at that stage). $Dab^i(\Delta)$ is a minimal Dab^i-formula at a stage iff, at that stage, it has been derived on

[18] If $\Delta^i = \emptyset$, we consider it to have 0 as the maximum of the diamonds of its elements.

the condition $\emptyset$ and $Dab^i(\Theta)$ has not been derived on the condition $\emptyset$ for any $\Theta \subset \Delta$.

From the set of minimal Dab^i-formulas at stage s, one defines $U^i_s(\Sigma)$ in the same way as $U^i(\Sigma)$ was defined from the minimal Dab^i-consequences of Σ. Next, the marked lines are defined (for each stage).

Definition 19: *Marking for $\mathsf{T}^{\mathsf{r}}_{\Diamond}$: Line i is marked at stage s iff, for some $\Diamond^j A$ in its fifth element, $A \in U^j_s(\Sigma)$.*

Definition 20: $\Sigma \vdash_{\mathsf{T}^{\mathsf{r}}_{\Diamond}} A$ *iff A is finally derived in a proof of the format described above.*

Definition 21: $\Sigma \vdash_{\mathsf{T}^{\mathsf{r}}} A$ *iff* $\Sigma \vdash_{\mathsf{T}^{\mathsf{r}}_{\Diamond}} A$ *and $A \in \mathcal{W}$.*

7.3. *An Example*

$\Sigma = \langle\{u, u \supset \neg p\}, \{r \supset \neg s\}, \{(p \wedge q) \vee s\}, \{r, r \supset t\}\rangle$

1	$\Diamond^1 u$	-	Prem	$\emptyset$	
2	$\Diamond^1 \neg p$	-	Prem	$\emptyset$	
3	$\Diamond^2(r \supset \neg s)$	-	Prem	$\emptyset$	
4	$\Diamond^3((p \wedge q) \vee s)$	-	Prem	$\emptyset$	
5	$\Diamond^4 r$	-	Prem	$\emptyset$	
6	$\Diamond^4 t$	-	Prem	$\emptyset$	
7	u	1	RC	$\{\Diamond^1 u\}$	
8	$\neg p$	2	RC	$\{\Diamond^1 \neg p\}$	
9	$r \supset \neg s$	3	RC	$\{\Diamond^2 \neg r, \Diamond^2 \neg s\}$	
10	$(p \wedge q) \vee s$	4	RC	$\{\Diamond^3 p, \Diamond^3 q, \Diamond^3 s\}$	
11	s	8, 10	RU	$\{\Diamond^1 \neg p, \Diamond^3 p, \Diamond^3 q, \Diamond^3 s\}$	
12	$\neg r$	9, 11	RU	$\{\Diamond^2 \neg r, \Diamond^2 \neg s, \Diamond^1 \neg p, \Diamond^3 p, \Diamond^3 q, \Diamond^3 s\}$	
13	r	5	RC	$\{\Diamond^4 r\}$	$\surd$
14	$Dab^4(\{\neg p, p, s, \neg s, \neg r, r\})$	2, 3, 4, 5	RU	$\emptyset$	
15	t	6	RC	$\{\Diamond^4 t\}$	

No other minimal Dab-formulas are derivable, only $U^4_{15}(\Sigma) \neq \emptyset$, namely, $U^4_{15}(\Sigma) = \{\neg p, p, s, \neg s, \neg r, r\}$. All lines that have a fifth element composed of four diamonds and an element of this set, are marked.

8. *The relation between* $\mathsf{T}^{\mathsf{r}}_{\Diamond}$ *and* $\mathsf{PCOM}^{\mathsf{r}}$

8.1. *The equivalence of* T^{r} *and* $\mathsf{PCOM}^{\mathsf{r}}$

Lemma 3: *For* $A \in \mathcal{W}$ *whose conjunctive normal form does not contain a tautology as a subformula,* $\vdash_{\mathsf{T}} (\Diamond^i A \supset A) \vee Dab^i(\Theta)$ *holds where* $\Theta = \{C \mid C \in At(B), B \sqsubset A\}$ *and there is no* $\Theta' \subset \Theta$ *such that* $\vdash_{\mathsf{T}} (\Diamond^i A \supset A) \vee Dab^i(\Theta')$.

Proof. It is easy to verify that $\Diamond(A_1 \vee A_2) \vdash_{\mathsf{T}} (A_1 \vee A_2) \vee (\Diamond A_1 \wedge \neg A_1) \vee (\Diamond A_2 \wedge \neg A_2)$, while $\Diamond(A_1 \vee A_2) \nvdash_{\mathsf{T}} (A_1 \vee A_2) \vee (\Diamond A_1 \wedge \neg A_1)$ and $\Diamond(A_1 \vee A_2) \nvdash_{\mathsf{T}} (A_1 \vee A_2) \vee (\Diamond A_2 \wedge \neg A_2)$, and that $\Diamond(A_1 \wedge A_2) \vdash_{\mathsf{T}} (A_1 \wedge A_2) \vee (\Diamond A_1 \wedge \neg A_1) \vee (\Diamond A_2 \wedge \neg A_2)$, while $\Diamond(A_1 \wedge A_2) \nvdash_{\mathsf{T}} (A_1 \wedge A_2) \vee (\Diamond A_1 \wedge \neg A_1)$ and $\Diamond(A_1 \wedge A_2) \nvdash_{\mathsf{T}} (A_1 \wedge A_2) \vee (\Diamond A_2 \wedge \neg A_2)$. Putting A in conjunctive normal form and applying the above properties gives the desired result. ■

Lemma 4: *If* $\langle \Gamma_1, \Gamma_2 \rangle \vdash_{\mathsf{T}} Dab^2(\Theta)$, *with* Θ *minimal, then for* $\Delta = \{A \mid A \in R_2(\Sigma), At(A) \subseteq \Theta\}$, $\Gamma_1 \vdash_{\mathsf{CL}} \vee\neg(\Delta)$, *with* Δ *minimal.*[19]

Proof. Suppose $\langle \Gamma_1, \Gamma_2 \rangle \vdash_{\mathsf{T}} Dab^2(\Theta)$, to simplify this we can take the minimal needed elements $e_1, \ldots, e_k$ of $R_1(\Sigma)$ and $e'_1, \ldots, e'_l$ of $R_2(\Sigma)$ such that

$$\Diamond e_1, \ldots, \Diamond e_k, \Diamond^2 e'_1, \ldots, \Diamond^2 e'_l \vdash_{\mathsf{T}} Dab^2(\Theta).$$

From the minimality of Θ it follows that the atoms of Θ occur in pairs of a primitive formula and its negation. The consistency of Γ_1 and of Γ_2 and the minimality of the chosen elements of $R_1(\Sigma)$ and $R_2(\Sigma)$ imply that no elements of the same pair are both disjuncts of elements of $R_1(\Sigma)$, respectively $R_2(\Sigma)$. We also know that

$$\Diamond e_1, \ldots, \Diamond e_k, \Diamond^2 e'_1, \ldots, \Diamond^2 e'_l \vdash_{\mathsf{Triv}} Dab^2(\Theta),$$

and hence that

$$e_1, \ldots, e_k, e'_1, \ldots, e'_l \vdash_{\mathsf{CL}} \bot.$$

The deduction theorem gives us that

$$e_1, \ldots, e_k \vdash_{\mathsf{CL}} (e'_1 \wedge \ldots \wedge e'_l) \supset \bot,$$

or equivalently

$$e_1, \ldots, e_k \vdash_{\mathsf{CL}} \neg e'_1 \vee \ldots \vee \neg e'_l.$$

[19] By Θ and Δ minimal we mean that for no proper subset the respective formulas hold.

From the minimality of $\{e'_1, \ldots, e'_l\}$ and the structure of $R_2(\Sigma)$, we know that $\{e'_1, \ldots, e'_l\} = \{A \mid A \in R_2(\Sigma), At(A) \subseteq \Theta\}$. The minimality of the set Δ follows from the minimality of Θ and of the subset $\{e'_1, \ldots, e'_l\}$ of $R_2(\Sigma)$. ■

Lemma 5: *For $A \in \mathcal{W}$ and Γ_1, Γ_2 consistent, $\langle \Gamma_1, \Gamma_2 \rangle \vdash_{\mathsf{PCOM}^r} A$ iff $\langle \Gamma_1, \Gamma_2 \rangle \vdash_{\mathsf{T}^r} A$.*

Proof. For the first direction, suppose we have a proof in which A is finally derived on condition Δ. We can assume that the following lines occur in the proof, not necessarily successively:

k	B_1	-	RC	$\Delta_1 = \{\langle \overline{C}, 1\rangle \mid C \sqsubset B_1\}$
l	B_2	-	RC	$\Delta_2 = \{\langle \overline{C}, 2\rangle \mid C \sqsubset B_2\}$
m	A	k, l	RU	$\Delta_1 \cup \Delta_2 = \Delta$

From Lemma 2 we know that for all $\langle C, j\rangle \in \Delta$, $C \in \tilde{R}_1(\Sigma) \cup \tilde{R}_2(\Sigma)$. In a $\mathsf{T}^r_\diamond$-proof from $\langle \Gamma_1, \Gamma_2 \rangle$, we can introduce the following lines, relying on Lemma 3:

a	$\Diamond B_1$	-	Prem	$\emptyset$
b	$\Diamond^2 B_2$	-	Prem	$\emptyset$
c	$B_1 \vee Dab^1(\Theta^1)$	a	RU	$\emptyset$
d	$B_2 \vee Dab^2(\Theta^2)$	b	RU	$\emptyset$
e	B_1	c	RC	$\{\Diamond^1 A \mid A \in \Theta^1\}$
f	B_2	d	RC	$\{\Diamond^2 A \mid A \in \Theta^2\}$
g	A	c, d	RU	$\{\Diamond^1 A \mid A \in \Theta^1\} \cup \{\Diamond^2 A \mid A \in \Theta^2\}$

for the respective Θ^1 and Θ^2 (see Lemma 3). Suppose line g is not finally derived. There must be a Θ', such that $\langle \Gamma_1, \Gamma_2 \rangle \vdash_{\mathsf{T}} Dab^2(\Theta')$, Θ' minimal and $\Theta' \cap \Theta^2 \neq \emptyset$. From Lemma 4 it then follows that we have a Δ', such that $\Gamma_1 \vdash_{\mathsf{CL}} Don(\Delta')$, Δ' minimal and $\Delta' \cap \Delta_2 \neq \emptyset$. The latter means that line m can not be finally derived in the PCOM^r-proof, a contradiction.

For the other direction, an analogous construction is possible, the only additional difficulty in that construction is that all relevant information should be introduced by RC and the needed irrelevant information by RU. ■

Theorem 2: *If all Γ_i are consistent, $\Sigma \vdash_{\mathsf{PCOM}^r} A$ iff $\Sigma \vdash_{\mathsf{T}^r} A$.*

Proof. This follows immediately by successively applying Lemma 5. ■

8.2. *The difference between* $\mathsf{T}^{\mathsf{r}}_{\Diamond}$ *and* $\mathsf{PCOM}^{\mathsf{r}}$

Though the consequence sets of both logics restricted to $\mathcal{W}$ are the same for a given Σ containing only consistent Γ_i, there are some fundamental differences, apart from the fact that there is not (yet) a predicative generalization of $\mathsf{PCOM}^{\mathsf{r}}$ available.

The language of $\mathsf{T}^{\mathsf{r}}_{\Diamond}$ is much more expressive. Firstly the priorities can be expressed in the language itself, namely, by a sequence of $\Diamond$. In $\mathsf{PCOM}^{\mathsf{r}}$ on the other hand, only indices that implicitly refer to sets can indicate the priority. Secondly, the abnormalities can be expressed in the modal language itself, namely, by their syntactical form $\Diamond^i A \wedge \neg A$. The situation is different in $\mathsf{PCOM}^{\mathsf{r}}$; again sets are needed with respect to which abnormalities can be defined, i.e., the sets $R_i(\Sigma)$.

On the other hand, the action of $\mathsf{PCOM}^{\mathsf{r}}$ is more complete. It takes into account the context in which the atoms occur, that is the minimal disjunctions in the consequences of a belief level to which they belong. Because for the reliability strategy, the unreliability of a disjunction of atoms in $\mathsf{PCOM}^{\mathsf{r}}$ corresponds to the unreliabilities of the individual atoms in $\mathsf{T}^{\mathsf{r}}_{\Diamond}$ and vice versa (see Lemma 4), and because they have similar implications in the respective systems (see Lemma 5), the effects of this more complete action do not manifest in the results, only in the interpretation. That is not the case for the minimal abnormality strategy, as we will see in Section 10.

9. *The Minimal Abnormality Strategy*

Another frequently used adaptive strategy is the minimal abnormality strategy. For this strategy it is more clarifying to look at the semantics. Because of its greater transparency and also because we have already given the proof theories for the reliability strategies, we restrict ourselves to the semantic definitions.

In this case it could be defined in two different ways, one that considers abnormalities with respect to elements of the set $R_i(\Sigma)$ and one that considers abnormalities with respect to atoms, as was done in the modal version. They do not both give the same results. First we give the semantics of the one that coincides with the strengthened Procedure from Section 2.

9.1. *The Semantics of an Adaptive Logic for Minimal Abnormality based Prioritized Compatibility*

The semantic translation of the strengthened Procedure from Section 2 is straightforward. We can define for each level i a family of $\mathsf{CL_i}$-models that selects the $\tilde{R}_i^s(\Sigma)$.

Definition 22: *A* $\mathsf{CL_i}$*-model of* Σ *is a* CL*-model of* $P_{i-1}(\Sigma)$ *that verifies a maximal consistent subset of* $P_{i-1}(\Sigma) \cup R_i(\Sigma)$.

Definition 23: $\Sigma \models_{\mathsf{PCOM^m}} A$ *iff* $\Sigma \models_{\mathsf{CL_n}} A$.

The semantic consequence relation $\models_{\mathsf{CL_i}}$ can be defined without constructing the sets $Cn_{\mathsf{CL}}(P_{i-1}(\Sigma) \cup \tilde{R}_i(\Sigma))$ explicitly, namely, by setting $P_0(\Sigma) = Cn_{\mathsf{CL}}(\emptyset)$ in analogy with the Procedure (or simply $P_0(\Sigma) = \emptyset$), and recursively constructing $P_i(\Sigma) = \{A \mid \Sigma \models_{\mathsf{CL_i}} A\}$. In this way the definition is purely semantical.

The standard way to define minimal abnormality strategies is somewhat different, but easily seen to be equivalent. It focuses on the abnormalities of a model and selects those models that do not contain more abnormalities than any other model. That has to be done for each level here. The abnormalities of a model at level i are the elements of $R_i(\Sigma)$ that are not verified in that model.

Definition 24: $Ab_i(M) =_{df} \{A \in R_i(\Sigma) \mid M \nvDash A\}$

Definition 25: $P_0(\Sigma) = Cn_{\mathsf{CL}}(\emptyset)$, *and* $P_i(\Sigma) = \{A \mid \Sigma \models_{\mathsf{CL_i}} A\}$.

Definition 26: *A* $\mathsf{CL_i}$*-model of* Σ *is a* CL*-model* M *of* $P_{i-1}(\Sigma)$ *for which there is no* CL*-model* M' *of* $P_{i-1}(\Sigma)$ *such that* $Ab_i(M') \subset Ab_i(M)$.

Definition 27: $\Sigma \models_{\mathsf{PCOM^m}} A$ *iff* $\Sigma \models_{\mathsf{CL_n}} A$.

9.2. *The Semantics of a Modal Adaptive Logic for Prioritized belief Sets based on Minimal Abnormality*

The only modification needed is in fact already given in this paper. Semantically, the logic $\mathsf{T}_{\diamond}^{\mathsf{m}}$ is obtained by stepwise selecting, for each consecutive i, the minimally abnormal (with respect to atoms, see Definition 17 or a copy below) models, starting with the T-models of $\langle \Gamma_1, \ldots, \Gamma_n \rangle$. If $\mathcal{M}_\Sigma$ is again the set of all T-models of Σ, the n selections of $\mathcal{M}_\Sigma$ are defined as follows:

$$\sigma^0(\mathcal{M}_\Sigma) =_{df} \mathcal{M}_\Sigma,$$

$$\sigma^i(\mathcal{M}_\Sigma) =_{df} \{M \in \sigma^{i-1}(\mathcal{M}_\Sigma) \mid \text{for no } M' \in \sigma^{i-1}(\mathcal{M}_\Sigma),\ Ab^i(M') \subset Ab^i(M)\},$$
$$\text{where } Ab^i(M) =_{df} \{A \in \mathcal{F}^a \mid v_M(\Diamond^i A \wedge \neg A, w_0) = 1\}.$$

The $\mathsf{T}^{\mathsf{m}}_{\Diamond}$-models of Σ are the members of $\sigma^n(\mathcal{M}_\Sigma)$.

Definition 28: $\Sigma \models_{\mathsf{T}^{\mathsf{m}}} A$ *iff* $\Sigma \models_{\mathsf{T}^{\mathsf{m}}_{\Diamond}} A$ *and* $A \in \mathcal{W}$.

9.3. *The interpretation*

It is obvious that the minimal abnormality strategies are stronger than the reliability strategies. For the reliability strategy every atomic part, resp. atom that is involved in a cluster of abnormalities (a Don- or a Dab-formula) is rejected to be selected. Conversely, for the minimal abnormality strategy, as few as possible atomic parts, resp. atoms that are involved in a cluster of abnormalities, are rejected.

The difference between $\mathsf{PCOM}^{\mathsf{m}}$ and T^{m} is that the latter is based on a selection of atoms, whereas the former is based on a selection of atomic parts, the shortest disjunctions that are consequences of the respective level. What difference does this make? In $\mathsf{T}^{\mathsf{m}}_{\Diamond}$, all atoms are treated equally; atoms that only occur in disjunction with other atoms in the consequences of a level are handled in exactly the same way as atoms that do occur isolated in the consequences of that level. $\mathsf{PCOM}^{\mathsf{m}}$, on the other hand, focusses on the shortest disjunctions of atoms in the consequences of a level and so takes into account the context in which the atoms occur. An atom — here in fact considered as a trivial disjunction — that occurs in the consequences of a level counts as much as a whole of atoms that occur in a minimal disjunction of that level.

10. *Examples*

10.1. *Example 1*

$$\Sigma = \langle \{p\}, \{\neg p\}, \{p \supset q\} \rangle$$

It is easily seen that the essential consequences for all four logics are p and q.

10.2. *Example 2*

$$\Sigma = \langle \{p\}, \{\neg p, p \supset q\} \rangle$$

It is easily seen that the only essential consequence for all four logics is p. The difference between the above examples illustrates that a difference in priority ranging really makes a difference.

10.3. *Example 3*

$$\Sigma = \langle \{\neg q \vee \neg r\}, \{q, r\} \rangle$$

The reliability strategies only select $\neg q \vee \neg r$. As the atomic parts to be selected or not are atoms, the minimal abnormality strategies have the same results: $\neg q \vee \neg r$ and $q \vee r$.

10.4. *Example 4*

$$\Sigma = \langle \{p\}, \{\neg q \vee \neg r\}, \{p \supset q, r\} \rangle$$

The reliability strategies result in p and $\neg q \vee \neg r$. Rather surprisingly the minimal abnormality strategies also give the same results: p, $\neg q \vee \neg r$ and $q \vee r$.

10.5. *Example 5*

$$\Sigma = \langle \{p\}, \{\neg q \vee \neg r\}, \{p \supset q, p \supset r\} \rangle$$

The reliability strategies and the modal version of minimal abnormality result in p and $\neg q \vee \neg r$. Note that for T^{m}, this contrasts strongly with the results in the previous example, although the premises are almost the same and ask for an analogous interpretation. The procedural version of minimal abnormality gives something more: also $q \vee r$ is a consequence. What is the matter here is that the $\mathsf{T}^{\mathsf{m}}_{\diamond}$-models that verify p, $\neg q \vee \neg r$ and $\diamond^3 \neg p$ block the information flow from $\{p \supset q, p \supset r\}$. This difference between the two abnormality strategies illustrates the shortcoming of the modal version.

11. *Conclusion*

We conclude that both approaches are useful, certainly when opposed to [5], [3], [6] and [2]. The logics presented here are more appropriate in certain contexts, namely, when the implications of the premises are of major concern and the formulation of the premises is insignificant. Both approaches presented in this paper have their advantages. The interpretation of prioritized compatibility is more complete — for the minimal abnormality strategy this leads also to more complete results — whereas the expressive power of the modal language is greater. Also both strategies are meaningful, though minimal abnormality is the strongest.

What is interesting is that the main idea, considering the consequences of the levels Γ_i and acting on them instead of working only with the sets Γ_i themselves, is in fact in a way also the step from PCOM^r to PCOM^m. That can be observed in Definitions 6 and 7 from Section 2.

One thing that still has to be done is the extension of prioritized compatibility to the predicative case. For the modal version it is achieved immediately by adopting the respective definitions from [2].

Another interesting extension would be a system based on a paraconsistent logic such that the inconsistent levels bring in some information instead of being ignored.

Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
E-mail: Liza.Verhoeven@rug.ac.be

REFERENCES

[1] Diderik Batens. Inconsistency-adaptive logics. In Orłowska [4], pages 445–472.
[2] Diderik Batens, Joke Meheus, Dagmar Provijn, and Liza Verhoeven. Some adaptive logics for diagnosis. *Logique et Analyse*, To appear.
[3] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case. In Orłowska [4], pages 473–511.
[4] Ewa Orłowska, editor. *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*. Physica Verlag (Springer), Heidelberg, New York, 1999.

[5] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.
[6] Guido Vanackere. Preferences as inconsistency-resolvers: the inconsistency-adaptive logic PRL. *Logic and Logical Philosophy*, in print.
[7] Liza Verhoeven. Changing one's position in a discussion – some adaptive approaches. *Logic and Logical Philosophy*, To appear.