

## MATHEMATICS AND FICTION I: IDENTIFICATION

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### 1. *Introduction*

The object of this paper and its sequel is to study and account for the analogy, drawn both positively and negatively, between mathematics and fiction and to consider fictionalism, the categorization of mathematics as fiction. A great deal of philosophy of mathematics is concerned with foundations in the old sense of a non-mathematical grounding of mathematics, which, in common with Carnap, sometimes Putnam, and many mathematicians, I think mathematics does not need, and with epistemology or ontology, which are important and unavoidable but little discussed here. Independent of foundations and ontology but fundamental to epistemology is the matter of meaning, something of importance to non-philosophers and non-mathematicians. I intend to shed a little —only a little— light on what is going on in mathematics, how one can understand it a bit other than experientially by writing a thesis, as I was told to do by a mentor (sensibly enough). This intention is philosophical and the way that I am attempting to accomplish it is also philosophical, but I write as a mathematician. Applied mathematics typically works by a direct comparison between some mathematics and some area of application. A mathematical approach might be simply to consider the analogy between mathematics and fiction as a bald contingent fact. Philosophy typically compares things by considering them both as instances of some common category; this is explanatory in a way that a mere comparison is not. The kind of explanation I am attempting here requires the more philosophical approach if it is possible, and my view of mathematics makes it so. I shall attempt to explain how it is possible and even natural for mathematics and fiction to have such limited similarity as they have. Some similarity has been noted by a large number of philosophers, Balaguer, Bunge, Field, Papineau, Resnik, Tharp, Tiles, Torretti, Vaihinger, van Fraassen, von Freytag-Löringhoff, and Wang, to list only those mentioned in this paper. No philosopher that I have found has considered this comparison, positively or negatively, sufficiently deeply to make sense of it to my satisfaction.<sup>1</sup> In

<sup>1</sup> John Allen Paulos has written a book-length consideration [1998] of stories and statistics, narrative and numbers, in which he treats them more as polar opposites (differences

the sequel I shall discuss some of these comparisons, mainly those of Hodes, Körner, Resnik, Tharp, and Wagner. Here I shall discuss the rationale and the further step of identification made in particular by Vaihinger, Bunge, and Field. The goal of this paper is to show how it can be that, to put it crudely, Vaihinger and Bunge are right and Field wrong in saying ‘mathematics is fiction’: obviously by meaning different things, but none of them by meaning what the person in the philosophical street would mean by the phrase.

In order to study the comparison with fiction, I shall offer for discussion a novel view of what mathematics is about that I think has merit. In particular, I think that it accounts for some ontological difficulties concerning mathematics over and above those concerning everything else (not to be elaborated here). I must begin with some indication of what I have just called a novel view, which is by no means altogether new or only mine. I am concerned not to cut off from inclusion any of the sorts of mathematics that are practised by honest-to-goodness mathematicians and also not to cast aspersions on any sort of inferential programme, since I am in no position to legislate on such matters. I shall say something about certain sorts of fiction within the more general context of narrative. I am aiming at a partial picture of mathematics that will be recognizable to mathematicians and helpful to others. Using that picture, I hope to show that what Vaihinger and Bunge mean by fiction is acceptable and what Field means is not. Mathematics is not fiction in anything like the word’s normal sense.

## 2. *Mathematics and Relations*

Fully expecting to have my attitude corrected at least in part, I think that both mathematicians and philosophers have been deceived to a greater extent than they would normally allow into regarding mathematics as being about mathematical objects. This has led to 2500 years of argument about the reality or otherwise of these supposed objects, an argument I do not want to add to. The reason that I think the subject of the argument does not matter is that I think mathematics is about relations rather than objects. I think that this subject matter accounts for much that is distinctive about mathematics, for its objectivity, for its comparative success as an intellectual discipline evolving over two and a half millennia and inspiring virtually all others, natural science in particular.

pp. 23 f., 32 f., 87 ff., 116–124) that are codependent than as independent analogues, more negative than positive. While I disagree with little that he says on topic (they *are* related) and admire his expert popular treatment, my task is different and was carried out almost completely before I read his book. We are both trying to round out understanding of mathematics, neither of us by confusing what is distinct.

I disclaim any originality in making this claim; it is often remarked on but then ignored. Scott Buchanan said, ‘Mathematics is the science of relations as such.’ (p. 133 of his [1962]) I illustrate with quotations from Newton to Atiyah. Dan Isaacson [1994], in espousing a similar view, quotes from Poincaré the most succinct statement of it.

Mathematicians do not study objects, but the relations between objects. To them it is a matter of indifference if these objects are replaced by others, provided that the relations do not change. ([1902], p. 20)

Russell states it fairly clearly if with a tendency to go overboard all the way to structuralism.

It is, however, the logic of relations which must serve as a foundation for mathematics, since it is always types of relation which are considered in symbolic reasoning; that is, we are not required to consider such and such particular relation, with the exception of those which are fundamental to logic (like  $\in$  and  $\supset$ ), but rather relations of a certain type—for example, transitive and asymmetrical relations, or one-one relations. ([1956], pp. 3 f.)

Netz [1999] claims to ‘offer a historical vindication of Russell’s claim’ (p. 197) by his consideration of ancient Greek mathematics. Both Hilbert and von Neumann were apparently in agreement on this point, the latter preferring functions—as does Saunders Mac Lane—to the more general term ‘relations’ to describe what is primitive, or can be taken to be primitive. Hao Wang takes much the same line as Poincaré, Russell, and Hilbert in his [1974], p. 345, ‘... it is appropriate to think of mathematics, not as a special branch of knowledge, but as a refinement of general language, supplementing ordinary verbal expressions, which may be too imprecise and cumbersome, with new tools to represent relations’. A more recent example is Sir Michael Atiyah’s address [1995] as President of the Royal Society of London on November 30, 1994. In considering how and why it works, he said:

Mathematics takes the process [of abstraction, which is used in science too] to its ultimate conclusion: the identity of the players is ignored, only their mutual relations are studied. It is this abstraction that makes mathematics such a universal language: it is not tied to any particular interpretation.

Finally, a previous President, despite his predating the invention of pure mathematics, is cited by Frege as taking a relational view of number,

Newton proposes to understand by number not so much a set of units as the relation in the abstract between any given magnitude and another magnitude of the same kind which is taken as unity. (*Grundlagen* §19, Austin translation. Austin gives the reference *Arithmetica Universalis*, Vol. I, cap. ii, 3.)

Newton makes the process of abstraction explicit. It has been suggested that even the widely acknowledged reducibility of mathematical language to set-theoretical language is based on this characteristic of mathematics. On FOM, Martin Davis wrote [1998],

... There is no great mystery (as some have suggested) in the reducibility of mathematics to set theory. Mathematical patterns have to do with relations among arbitrary objects. So what mathematics needs is enough objects and the possibility of expressing arbitrary relations among them. With the definability of ordered pair, set theory gives both.

One might justly ask why, if this view is sound, no one has established a programme of showing that mathematics is the science of relations. The answer to this is that mathematics, as ordinarily done, is precisely such a programme. Mathematical definitions, axioms, and work in general are carried out in terms of relations. That is why mathematicians say that this is characteristic of mathematics. It is simply an observation, and so it has not generated the discussion that less accurate views have that are philosophically more interesting<sup>2</sup> or tractable.

This view need not be uniquely right for the consequences of taking it up to be valuable. If there has been an overemphasis on objects at the expense of relations since the idealists of the nineteenth century, then it might do some good to get folks thinking about relations more and objects less for the sake of an appropriate balance. Set theory looks like a reason to think about objects, but the iterative hierarchy of pure sets is based on the empty set, which seems to me a poor reason for a basis in things of any sort. Geoffrey Hellman [2001] claims with references that a number of set theorists (Zermelo, Gödel, Fraenkel, Bar-Hillel, and Levy) have regarded the empty set as a ‘fiction’. And I cannot take seriously a basis in nothing but notation.

My aim is not definitive but clarificatory; Hao Wang quotes Kant (*KrV*, A727–9, B755–7) ‘... an empirical concept cannot be defined at all, but only made explicit. ... Consequently, mathematics is the only science that

<sup>2</sup>If this view is subsumed by eliminative structuralism, as it might be by an eliminative structuralist in spite of my intention to dig a little deeper, then it would fall under the recent stricture of Akiba [2000] that eliminative structuralism is of ‘little interest’. I consider verisimilitude a higher value than intrinsic interest.

has definitions.’ Being an empirical concept, mathematics itself has no definition. So far as I can see, my stress on relations does not fall into any of the metaphysical categories set out by Stephen Pepper in *World Hypotheses*. What I want, in fact, to do is to avoid taking up any metaphysical stance — other than a minimal one on the importance of relations— since I think that the relational view is compatible with and can be useful to thought in terms of any standard world view.

In view of philosophers’ usual meaning for ‘postulate’, I need a special sense of postulate with which we can postulate mathematical objects for the sake of argument without any further ontological commitment to them.<sup>3</sup> I shall call this the *conversational* sense of postulate. Physicists postulate objects all the time, and many of them are thought to be physical and so to be searchable for. When they are found, they are thought to exist. They may be thought to exist before that (ontological commitment). On the other hand, some of them are not thought to be physical, not found, and not thought to exist in the physical world. The choice betokens the sort of non-committal existence that we need for mathematical objects. On one interpretation, they are the sort of thing Russell called ‘logical fictions’ (in ‘Philosophy of Logical Atomism’, in his [1956], pp. 270 f.), a term that Russell applied much more widely than just to mathematical objects, as have those following him. It was recently attached paradoxically by Christopher Williams to facts ([1992], p. 111). As with the physicists’ objects, some may exist; I do not mean, by using the word ‘postulate’, to deny or affirm the existence of the postulated things. So when I say that we postulate objects to exhibit<sup>4</sup> certain relations we are interested in, I do not mean to attach or deny existence to them, just to ‘introduce discourse about them’ as Michael Resnik puts it ([1997], p. 185). He says that in postulating we do more, if only gradually and eventually; I’ll come to that in part II and call it the *philosophical* sense of postulating.

I need to indicate what I mean by the relations that mathematics studies (free of context or other identification as indicated above). I mean decontextualized relations, that is, the relations abstracted from the things that they relate. At low levels some idealization is required, but at higher levels these things are usually mathematical things, and so no idealization is needed in the abstraction. At the lower levels, definition is typically implicit

<sup>3</sup>The so-called ontological commitment made by merely speaking of things Stephan Körner considers ‘advisable’ to call “quasi-ontological” or “*prima facie* ontological” in order to indicate the possibility of different metaphysical interpretations of ideal existence in mathematics and elsewhere by Platonists, empiricists and others’ (Körner [1966], p. 113).

<sup>4</sup>This exhibiting is a less precise, less narrow, notion than Zalta’s [1983] ‘encoding’ which, in opposition to the usual postulates, is done by objects that must *not* exist.

by axioms, whereas higher up definitions can be explicit.<sup>5</sup> In either case, it is an emaciated sort of definition, since it does not specify its definienda uniquely. (I return to this point at the end of the section.) One of the reasons that my relational view of mathematics is not as distinctive as it might be is that science since Galileo has made a great success by imitating the mathematical method, not concerning itself with what things are but studying their relations (abstracted to a degree and idealized). This process may have achieved its high point with Carnap's view of science that was so structuralist as to reduce it to what is (interest aside) mathematics.<sup>6</sup> Carnap is one of those thought to have reduced mathematics to if-thenism, considered a fictionalism by Roberto Torretti [1981].<sup>7</sup> Gregory Bateson called gravity an explanatory principle ([1969], p. 39) because it was an agreement to seek no further; it was good enough to be able to calculate the force exerted by gravity, whatever gravity was. Physics abstracts relations among things from the things, but it keeps the kind of relations. That is, for example, distances are distances still in spite of their not being between any *particular* places or things. Similarly for other scalar quantities. Likewise vector quantities; one talks of and calculates with vector quantities without having the vectors attached to any particular things in the world, but typically the vector quantities are gravitational force, or displacement, or something else identifiable. When one abstracts the rest of the way and studies the vector quantities without dimensions one may be doing mathematics. I say 'may' because 'mathematics' is an evaluative term like 'literature' not a formal term like 'prose'.<sup>8</sup> If the abstraction is more apparent than actual, then one is still doing physics because no one is interested in what one is doing for its own sake. Physicists

<sup>5</sup> This idea is elaborated with philosophical and mathematical background in a message to FOM from Michael Detlefsen [1998], in particular its second half considering ideas of Kant, Lambert, Pasch, Dedekind, and Hilbert, in particular Hilbert's earlier 'geometrical' formalism (not his later 'arithmetical' formalism—that of the programme.)

<sup>6</sup> This is the view of H. Wang [1974], p. 40 and n. 2 to that page, citing R. Carnap, *The logical structure of the world*, §15, p. 27. For Wang's withering view of Carnap's—and Quine's—account of mathematics, see his [1986] *passim* and succinctly p. 19. Cf. part II, n. 28.

<sup>7</sup> This comparison is unusual but is also considered by David Papineau [1988].

<sup>8</sup> In drawing this distinction in much the same place, Azzouni forgets that the distinction is not merely formal ([1994], p. 107), but he does notice that rational mechanics can be mathematics (p. 108). The formal distinction, as far as it goes, does show in a kind of diagrammatic way how mathematics can be so close to the sciences and yet be distinguishable. Mathematics is like the boundary of the sciences, among them if one regards them as a closed region, outside them if one regards them as open, but approachable within them as closely as one likes.

have come to frown on this style, as van Fraassen notes [1991]. On the other hand, if what one is doing is of interest more widely, then it is mathematics, pure or applied—not a distinction I regard as important philosophically rather than culturally and politically. Physicists' objects, for instance the points at which a gravitational field can be calculated, are surrogate physical places or things, but when a mathematician studies a vector field it is defined at points of a mathematical space, which is just points postulated to have certain relations among themselves with no built-in connection to the physical world. (Quine's posits to explain the physical world are to explain physical relations.) The results of such abstract study—whether interesting enough to be mathematics or not—can then be applied to the physical world by the freedom we always have to apply what we say to what we wish, as when I serve the Sunday joint to my wife and remark that 'Mary has a little lamb.' Mathematical application depends solely on the relevance of the relations among the mathematical objects to relations among the physical or financial or whatever kind of objects the mathematics is applied to. It is the application of relations to relations;<sup>9</sup> nothing is gained by concentrating on the mathematical objects used to hold up those relations because they are left behind in the application process.

Here is an example. When ellipses are applied to planetary orbits, one can compare the Sun and the Earth to geometrical points, which does not seem very reasonable, or one can compare the metric relations between the Sun and Earth and the metric relations between geometrical points—no trouble at all. This is typical of science; what corresponds is relational statements from mathematics and relational statements about the world.<sup>10</sup> Such comparison is not altogether different from comparisons made routinely between stories and reality. Mark Balaguer supplies an excellent example of this in his [1998]. George Orwell's *Animal Farm* can, as Balaguer says, be used to talk and think about Stalin's Soviet Union.

We can say something roughly true about Stalin by uttering the sentence 'Stalin was like the pig Napoleon', even though this sentence is, strictly speaking, false. In other words, it seems that the *historical content* of this sentence is roughly true, despite the fact that there was never any such pig as Napoleon and, hence, that the *Animal Farm content* of this sentence is fictional. ([1998], p. 140)

<sup>9</sup> Azzouni calls this 'mathematical vocabulary [mixing] freely with the vocabulary of the sciences and everyday life' ([1994], pp. 88 f.). At bottom, the so-called mixture—all being concerned with the relational aspects of whatever worlds are being discussed—is homogeneous.

<sup>10</sup> Körner [1966] is a good discussion of the subtlety of this.

The comparison is similar to the mathematical one because, as Balaguer seems not to notice, the similarities that matter are structural, Napoleon's *position* in *Animal Farm* is similar to Stalin's *position* in the Soviet Union in the senses of relations entered into. The object-to-object likenesses that they are both male and mammals are of no more relevance than the object-to-object differences that Napoleon, unlike Stalin, has a tail and Stalin, unlike Napoleon, has a mustache.

Let me be a bit more specific about relations that ground mathematics in my view.<sup>11</sup> We know about different ways of specifying the same things; it gives us set equality. We know about membership and inclusion; they give us set membership and subset inclusion, which took some time to distinguish. We know about ordering and counting; they give us ordinal and cardinal numerals. By reification of relations new contexts are formed from which again relations can be abstracted; neo-logicism studies this phenomenon originally studied by Frege. Numbers come from reifying the cardinals (originally a kind of poem in the apt phrase of Peter Caws) —among the Greeks as late as the time of Plato according to Kneale and Kneale ([1963], chapter six, section two, quoted by Burgess and Rosen [1997], p. 228). Dedekind [1888] made new things of partitions of the rationals.<sup>12</sup> Points, lines, planes, and their relations achieved their abstraction as long ago as Euclid; Netz [1999] mentions 'inequality, proportional inequality, similarity, congru[ence], addition, and subtraction (p. 191)'. But in Euclid, what we should call fractions were preserved as Eudoxian ratios, relations of sizes between magnitudes of the same kind, linear, areal, or volumetric (*Elements*, Defn V.3), despite the arithmetic tradition's already treating fractions as numbers in the common view disputed by David Fowler [1998]. We all know the history of the reification of functions,<sup>13</sup> which were previously treated as relations (formulas then graphs). The relation between a sequence and its limit is one of the most productive relations ever abstracted. If Zeno had just talked of the point at which Achilles was level with the tortoise, he would have invented the limit

<sup>11</sup> Much of the argumentation of structuralists, who take a similar line further, from at least Hao Wang, [1974], pp. 49, 194, could be cited, but this is not my concern here.

<sup>12</sup> He justified this creative act in terms of the previous in a letter to Weber, 'to understand as number (cardinal number) rather not the class (the system of all similar finite systems) itself, but something New (which corresponds to this class) that is created by the mind. We are of divine descent and we no doubt possess creative power not only in things material (railways, telegraphs), but particularly in things mental.'

<sup>13</sup> *E. g.*, Wilson [1999].

Writing of Dirichlet's introduction of arbitrary functions, Mathieu Marion writes, '... with Dirichlet mathematicians moved from an intensional notion of function-as-a-rule to a purely extensional conception' ([1998], p. 7). That is another way of seeing what was happening.

of a sequence. The iterative set hierarchy and category theory both seem to approach dealing with relations raw without much if any concern that anything be available to be related. The current popular notion of structuralism involves the reification of structures, which can better be thought of as just relations and relations of relations because to do so does not merge structures that are isomorphic but different.<sup>14</sup> Mathematics routinely distinguishes between isomorphic structures. My view seems to me to be closer in spirit to an application to mathematics of the general (and literary) structuralism than to the specifically mathematical structuralisms of Michael Resnik and Stewart Shapiro.

Azzouni thinks puzzling the common enough statements such as that, ‘Intuitionistic arithmetic is not, in practice, very different from classical arithmetic . . .’ (quoting Michael Dummett [1977], p. 35, in [1994], n. 53, p. 183). The mathematical relations studied in the two subjects are hard to distinguish; the differences, such as they are, have other sources.

Stressing relations rather than structure avoids the false emphasis on the whole system (not to mention ontology) that structuralism imports into a discussion of mathematics. For most common mathematics, one is not interested in the whole system in which one is working, and is indeed not even interested in what the bounds of it are, if any. For geometry one just needs a large enough piece of the plane, for instance, to do one’s constructions; for arithmetic one needs numbers up to the largest one needs to use. Mac Lane, in the special issue of *Philosophia Mathematica* in volume 4 (1996) on structuralism, points out (pp. 177 and 180) that much mathematics is not about structure. It is all, however, about relations, and the parts that are not *about* structure concern themselves with ‘objects’ within structures, that is, things in relation, to be sure. Michael Lane, in the introduction to his anthology of structuralist writing, generalizes Bourbaki’s structures in a non-mathematical way in the definition: ‘A structure is a set of any elements between which, or between certain sub-sets of which, relations are defined.’ (Lane [1970], p. 24) This, by the way, is the structuralism that inspired post-structuralism. Turning away from mathematics, Lane says that structure for a social scientist ‘is like a plan which he devises to find his way round a

<sup>14</sup> H. B. Griffiths suggests that specializing an algebraic structure, distinguishing one specific example of a class of isomorphic structures, ‘is like casting a play, and the flavour of the special mathematics corresponds to that of a particular production: all such productions have the same abstract structure.’ (private communication, 1999 05 8) If plays were derived from performances, then one would have the same negative feeling toward them—as compared with the performances—that one has toward mere abstracted structures as compared with real mathematics. As Frege says in *Grundlagen* §70, it is logic; it is not interesting enough to be mathematics. The idea of Griffiths illustrates why; it is like reading a play.

building', adding that, '[t]hough this structure is not itself observable, nevertheless it is derived from observation alone'. Many of the related objects of the social scientist have inherent qualities, their own natures specific to the sort of social science being attempted. It does not seem to occur to him that he has specified the definitive difference between mathematics and other intellectual language-games. Outside of mathematics, he writes, 'As structuralists understand and employ the term, a new importance has been given to the logical priority of the whole over its parts.' (p. 14) In mathematics, priority of the whole over its parts is plainly false in some central and important examples, including the positive integers and Euclidean space. Not only is the whole not prior to the parts, but also much of what is of interest in both of these cases was already done before the respective wholes were much thought of. Lane's next sentence is nevertheless accurate for mathematics, 'They insist that the whole and the parts can be properly explained only in terms of the *relations* that exist between the parts.' For more on structuralism, see Caws [1988] and Rickart [1995], where Caws is taken as an example.

Wilfrid Hodges [1985] pointed out that, in spite of Tarski's contribution to understanding language about them, he was hesitant to talk about structures. In his papers for many years, according to Hodges, Tarski avoided writing of the notion of structure, did not distinguish between variables and non-logical constants, and did not use the 'notion of an uninterpreted constant symbol which gets an interpretation by being applied to a particular structure'. Again according to Hodges, the earliest use of truth 'in something' (a specific structure) is by Skolem in 1933, and the notion in its current use was not quickly adopted.<sup>15</sup>

Unequivocally giving an uninterpreted constant symbol an interpretation is not straightforward at all; for consider the following: a generator of a small cyclic group, a generator of a specific group  $C_4$ , a generator of the group of rotations of a square, a generator of the group of the square with vertices  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$ ,  $(0, -1)$ , a generator of the group of a particular physical square like a chess board, say the counterclockwise quarter turn looked at from above. Each of these is an interpretation of the previous; any attempt to claim that an interpretation has been given can be overruled by further interpretation, and none but the last is even unique. Hodges concludes that for Tarski and even for himself such terms are mere indexicals, a notion close to mine and attributed to the ancient Greeks by Netz [1999].

<sup>15</sup>The use of true and false *simpliciter* of mathematics has been becoming more and more suspicious throughout the twentieth century. Corry [1997] is seriously out of date. Tait [2001] says simply 'truth [as opposed to truth in] plays no role in mathematics'. It is important, I think, to avoid both true and false, and it is possible to speak of the many errors in mathematical thinking without the notion of falsity.

### 3. *Psychology*

I am alleging mental processes here. What do those that study such processes think about mathematics? While what psychologists think about things does not necessarily have much philosophical relevance, I should find their taking a totally different view of what goes on in learning mathematics as a serious discouragement. Long ago I checked for consonance with Piaget's views. Lately I tried my idea on a live psychologist; she regarded it as altogether obvious. I turned to Margaret Donaldson's book [1993], where she has a somewhat more elaborate and less rigid developmental story to tell along Piaget's lines. She divides our thinking-about faculties into five modes (with a couple of other modes that are not for object-directed thinking, using Quine's term), four modes for thinking about the present, past, future and what is not specifically located in space-time, and a further one for relations abstracted from space-time altogether. Her chapter on the intellectual transcendent mode, as she calls the last mode, is shorter than some and to me disappointing on that account because it is about logic and mathematics (p. 126), and she hasn't a lot to say about them. What is this kind of thinking about? Her answer is:

... that logic and mathematics are about *relationships*: relationships of compatibility or incompatibility, of symmetry or asymmetry, of inclusion or exclusion, of equality or inequality, and so on. More than this: they entail the systematic study of *patterns* of relationship. And what we call 'creative mathematics' is the attempt to extend this study so that its previous limits are surpassed. That is, a known pattern is extended or new patterns are revealed. (p. 126)

In a philosophical context I hesitate to speak of something's being correct, but if there is even some truth in this view, it has important educational implications. If teachers don't know that this is what they are trying to enable their pupils to do, if they have never noticed themselves doing it, if they have never done it to speak of, how on earth can they expect to succeed in teaching mathematics? I say never 'to speak of', because all have achieved that mode. Without it, one responds to 'what is two plus three?' as does a young child with 'two plus three what?' (pp. 88–95). Decontextualization is not easy; it does not come naturally.

A shift of concern from things-in-relation to relations themselves is made easier if the dominance of the 'things' can be diminished. But that dominance is powerful indeed, and hard for our minds to reduce. (p. 127)

All of the lower modes depend on things. I think that Donaldson has made significant progress in fineness of grain and flexibility beyond Piaget, whom she quotes as saying that mathematical thought

implies the subordination of the real to the realm of the possible and consequently the linking of all possibilities to one another by necessary implications that encompass the real, but at the same time go beyond it. (Piaget [1972], quoted p. 134)

This area may not have been one Mary Warnock was thinking of when she wrote:

... it is the cultivation of imagination which should be the chief aim of education, and in which our present systems of education most conspicuously fail, when they do fail. ([1976], p. 9)

As the remainder of what I have to say will indicate, intellectual and imaginative engagement, which is obviously trainable, is vital to mathematics as to all other thought. Neither Warnock nor I intend a visual limitation; she writes of melody,

If we recognize it as a melody, not just a jumble of sounds, we thereby perceive it as having a certain shape or form. This entails presenting it to ourselves as shaped in the way it is. This is the function of imagination. ... (*ibid.*, p. 50)

One function of imagination anyway. Since I need occasionally to refer to our imagining things and to our imagination (metaphorically<sup>16</sup>) as doing the work but do not intend a faculty psychology or visual limitation anywhere, I might use Azzouni's slogan, 'What is unimaginable is not necessarily inconceivable' ([1994], p. 201) and talk of conceiving instead. But I do not know what would do the work; a mental uterus is a strained metaphor.

Before we leave the psychologists I want to draw attention to their concern with two matters that I shall come back to. One is pretend play in which children by their second year pretend, for instance, that 'physically present objects are made to stand for—or serve as— others that they in some measure (but perhaps quite remotely) resemble' (Donaldson [1993], p. 65). The foundation is being laid here for the imaginative activity called upon in all sense-making including mathematics (also of course in delusions), which I am concerned to represent as an imaginative activity, not just a deductive activity that could be done by a computer. By imaginative, I do not mean visual imagination in particular. (The whole matter of the similarity of mathematics to play is one that I cannot go into but acknowledge. The literature on play other than in competitive games has convinced me that this aspect of mathematics, drawn attention to particularly by Brian Rotman [1993] and David Wells [1987], is of great importance.) The other psychologists' concern is

<sup>16</sup> According to Daston [2000], 'ambivalence toward the faculty of imagination' (p. 4) has been evident for 250 years, helped by Gulliver's visit to Laputa. But Lakoff and Johnson [1999] point out in fleshing out this parenthesis that Descartes is primarily responsible.

with a specific sense-making activity in which we engage from a very early age. As Donaldson puts it,

we write for ourselves an authorised version of our lives. It can truly amount in the end to a Holy Scripture, to be lived by, to be revered. It is then hardly surprising if it is resistant to change, even when it is doing us harm. (p. 26)

We are not concerned here with the harmful possibilities like repression of the past or limitations on the future but with how basic is the much exercised capacity to absorb, recall, and tell stories.<sup>17</sup> They are what we abstract from the infinite flux of sensual experience; as Mary Warnock puts it,

to put a framework round the moment-by-moment flux of events so that they may be contemplated as displaying a pattern which makes sense.

... Thus, narrative is a natural mode of thought.

... The notion of ourselves as people, self-directed, motivated, responsible for what we do and say, able to assimilate and order even what happens to us by accident or apart from our own will, all this derives from our ability to interpret events according to a 'plot'.

This does not mean rewriting history, but simply writing it, or telling it to ourselves without writing it down. Stories, then, are central to our ability to manage and understand the world. ([1994], pp. 92 f.)

Something that I want to suggest is that our narrative capacity and our logical capacity are thoroughly entangled in our general capacity for communicable knowledge.<sup>18</sup> An argument is different from a story, but *significant* temporal consequence, causal consequence, and logical consequence are too closely

<sup>17</sup> Paulos [1998] emphasizes the importance of imagination fed by stories (pp. 55 ff.) and of stories for self-building and culture (pp. 103, 106); 'We are the stories we tell.' (p. 172) Perhaps the most important insight of his book (if correct) is that embeddedness in stories is essential to meaningfulness at a personal rather than merely abstract level. In the philosophical literature this stance has been taken up by Alisdair MacIntyre [1977] quoting Bruno Bettelheim [1976].

<sup>18</sup> Paul Ricoeur [1984] points out, 'The largest part of our information about events in the world is, in fact, owing to knowledge through hearsay,' so that the arts of narration are imitations of ordinary discourse. This emphasis on narrative stems, for me, from my appreciation of the importance of what is often called the social side of knowledge, something that plays little part in this essay. Generally it is hard to distinguish what has social sources from what we merely have in common —the question of nurture and nature. But normal narrative, as Barbara Herrnstein Smith points out, is a social transaction, 'someone telling someone else that something happened' (Smith [1981], p. 228). Charles Taylor argues extensively in his [1989] that to 'grasp our lives in a *narrative* is a basic condition of our humanity', citing Bruner and Ricoeur, as well as Heidegger's *Being and Time*, Div. II, chaps. 3, 4. Think of the dehumanizing effect of total amnesia.

related for them to be altogether independently acquired notions. Developmental psychologists are working on sorting them out in their context just as philosophers like David Hume had to sort them out in theirs. In and after a talk she gave in Oxford 1999 6 10, Alison Gopnik suggested that the blurred *because* relation that encompasses cause, reason, and logical connection goes all the way down to infancy. A single blurred relation because what took philosophy thousands of years to sort out is not naturally sorted out by children (Gopnik called it cause because what she talked about is what *we* call cause but did not suggest that the children she discussed distinguished among cause, reason, and logical connection). And all the way down because narrative is not more fundamental than this relation—as could be thought—for without this relation there is a list of events not narrative. It is precisely the imposition or noticing of relevant connection that distinguishes narrative from mindless listing, whether in life experience or history (in fiction, of course, the telling itself *creates* connection). If we say ‘Sam woke up and then he got out of bed’, we connect the two acts; mere temporal succession is not connection. We need to make the Hume-detected leap even to tell a story. The absolutely standard use of time as an independent variable in early scientific mathematics conflates all three of the because relations.<sup>19</sup> I am not discouraged from this view by the whole chapter of his book [1986] that Jerome Bruner devotes to distinguishing between narrative and argument, two things that it has never occurred to me to confuse and which I have never seen confused.<sup>20</sup> But I think that the one capacity may be built on the other. Bruner confirms the fundamental rôle played by narrative in our lives, pointing out, as does Donaldson, that psychoanalysis can be little more than improving the patient’s autobiography.<sup>21</sup>

#### 4. *Cognitive Science*

Let us now turn our attention to another study that touches on mathematics, cognitive science, in particular the research programme connected with

<sup>19</sup> H. Weyl wrote that one of the historical roots of the concept of function ‘was suggested by the “natural dependencies” which prevail in the material world—the dependencies which consist, on the one hand, in the fact that conditions and states of real things are variable over *time*, the paradigmatic independent variable, on the other hand, in the *causal* connections between action and consequence.’ ([1987], pp. 45 f.) Archimedean spirals and Newton’s fluxions were defined in terms of time.

<sup>20</sup> The distinction has been harped on since Plato’s *Phaedrus* according to Mary Warnock [1994], pp. 91 f.

<sup>21</sup> This is elaborated by Roy Shafer [1981].

George Lakoff and Mark Johnson ([1980] and [1999]). It tries to say something about the mechanisms of the transition from the material modes of thinking to the abstract or how the mechanisms for the abstract are based on the mechanisms for the concrete. Disciple Mark Turner, in his book [1996], takes the line of thinking as far as it can go, which is a convenient form in which to see it. He says that ‘narrative imagining —story— is the fundamental instrument of thought’ (p. 4). This device is used, by transfer of domain, in what he calls parable.

Parable begins with narrative imagining —the understanding of a complex of objects, events, and actors as organized by our knowledge of *story*. It then combines story with projection: one story is projected onto another. (p. 5)

The bases of this narrative imagining<sup>22</sup> are called ‘image schemas’, tiny events that occur over and over from our earliest days in our sensori-motor experience, both of perception and of our active participation in our environments. Turner is insistent that our categories are not static bins into which we have filed experience but are structured by image schemas.

Partitioning the world into objects involves partitioning the world into small spatial stories because our recognition of objects depends on the characteristic stories in which they appear. (p. 17)

The idiosyncratic particulars that Russell wrote of have to be supplemented by how they were related spatio-temporally when they were experienced. Recognizing animacy in others depends on seeing small spatial stories with an actor other than oneself; we project onto others the small stories that we have acted in. Naturally the world we interact with constrains the projections that work. I haven’t space to explain the result that Turner thinks we achieve with our projections, blended spaces<sup>23</sup> combining features of the story projected and the target space into which the story is projected, one of the commonest of which is signalled by the locution, ‘If I were you ...’ (p. 76). Turner makes categorization depend on blending, on ‘seeing something *as* something’ (p. 112, *cf.* L. Wittgenstein, *Philosophical Investigations*, pp. 194 ff.). He goes so far as to say that propositions of a physical nature are abstractions from stories, which are the more fundamental data structure.

<sup>22</sup> From a philosophical perspective, Mary Tiles has called for ‘rehabilitation of the (productive) imagination within the sphere of the cognitive’ ([1988], p. 191)

<sup>23</sup> An unrelated source of a similar idea is Emily Grosholz [1991], in which she writes of Descartes’ viewing curves ‘as hybrids, which are simultaneously spatially shaped configurations, algebraic equations in two unknowns, and an infinite array of number pairs’. The notion of hybrid is one she has developed [2000].

I turn now to narrative, which has a long history of being compared with mathematics. The comparison is mainly in the form of comparisons with fiction, but much that has been written is more appropriately thought of as comparison with stories, which need not be fictional.<sup>24</sup> The division between stories about real persons and about made-up characters makes surprisingly little difference in narrative as the similar ontological divide makes surprisingly little in mathematics. In both cases, whether one thinks of what is described as real makes some difference, but whether it is actually real makes little difference to the understanding of what is said.<sup>25</sup> The comparison is mainly one way; scholars in the humanities have not often compared history and literature with mathematics.<sup>26</sup> The main reason for current writers on mathematics to mention narrative, usually fiction, is their wish to avoid ontological commitment or to discuss the work of others that lack ontological commitment to the objects of mathematics. But a number of writers that have connected fiction and mathematics have said more than just that they do not believe mathematical objects to exist; that position, after all, is well known as anti-realism and need not involve any comparison with stories. I am going to give some indication of comparisons of mathematics with fiction or other narrative, which has now been going on for nearly two hundred years. But first a word about Aristotle.

### 5. *The Distant Past*

Aristotle had both positive and negative influences on the relational view of mathematics. Van Fraassen points out approvingly that

<sup>24</sup>I seem to be taking a literary turn here. I ought to make clear that I am nowhere here writing about literature, which is an evaluative concept. I refer to Peter Lamarque and Stein Haugom Olsen [1994], where the evaluative nature of literature is explained at page 255 and for the rest of the book. Fiction and history may and may not be literature. There is a whole industry devoted to the literary, which is beside the points I want to make. One way of expressing the distinction would be that my discussion is, from a literary point of view, determinedly superficial or skeletal, depending on the metaphor chosen.

<sup>25</sup>One needs to know about the Devil to understand witch trials, but one requires no ontological commitment (I probably owe this example to someone).

<sup>26</sup>An exception is Northrop Frye in his *Anatomy of Criticism*, quoted in Ricoeur [1977], p. 226. The non-existence of mathematical objects, which Frye takes as given, is his reason for the reverse comparison (in Ricoeur's words, 'suspension of real reference is the condition of access to the virtual mode of reference', the 'proposal, in imaginative, fictive mode, of a world' (p. 229)).

In almost parallel passages in the *Poetics* and the *Physics*, Aristotle tells both the dramatist and the physicist to depict events as part of a causal story ‘proceeding in accordance with necessity or probability’. ([1991], p. 9)

As well as seeing the analogy that this paper is concerned with, Aristotle was not unfriendly to stories and regarded fiction as more philosophical than history on account of dealing with universals. Jonathan Lear has made an attempt [1982] to understand Aristotle’s philosophy of geometry. Aristotle’s idea is that geometrical objects, about which geometers reason, are, as it were, *in* physical objects and are *regarded* as separated, or, as we should say, abstracted. In Lear’s translation,

... the best way of studying geometry is to separate the geometrical properties of objects and to posit objects that satisfy these properties alone.

... Though this is a fiction, it is a helpful fiction rather than a harmful one: for, at bottom, geometers are talking about existing things and the properties they really have (*Metaphysics* M §3 1078a, quoted p. 175)

While the positing part makes a lot of sense, it is much less clear that the triangles Aristotle speaks of are in any but a metaphorical sense *in* the so-called triangular physical objects. Physical objects are regarded *qua* mathematical objects.<sup>27</sup> One is reminded of Maddy’s small sets of eggs. It is only in a few bottom-level cases that mathematical objects are direct abstractions from physical objects. Most are abstractions from mathematical objects that have been abstracted at a previous stage: the iterative conception of abstraction. Moreover, it is typically relations that are abstracted rather than objects in any case.

According to Lear, Aristotle’s foundation is the usefulness of the theory so produced. Aristotle has shown that there is a path from the real world to the geometrical world. It appears on Lear’s account that my attitude to mathematical subject matter and therefore epistemology is broadly Aristotelian, even including not being especially fussed about ontology.

On the other hand, Aristotle is thought by Wang ([1974], p. 135) to have contributed to the general neglect of relations by philosophers, regarded by Russell as hampering philosophy since Spinoza.

Speaking generally, adjectives and common nouns express qualities or properties of single things, whereas prepositions and verbs tend to express relations between two or more things. Thus the neglect of prepositions and verbs led to the belief that every proposition can

<sup>27</sup> As Netz says [1999], this ‘*qua* operation [is] the make-believe at the heart of Greek mathematics’ (p. 198).

be regarded as attributing a property to a single thing, rather than as expressing a relation between two or more things. Hence it was supposed that, ultimately, there can be no such entities as relations between things. ([1912], p. 54)

The two-sidedness of Aristotle may not be just apparent; while putting down relations explicitly he emphasized some importantly with his doctrine of the four causes, four relations, of course.

### 6. *The More Recent Past*

The comparison with fiction until quite recently was not so much making the ontological point that is being made by contemporaries but was using 'fictional' as a way of saying 'abstract'. But saying that what is abstract is humanly constructed. There is a pre-history of Duns,<sup>28</sup> Occam, Vico, Berkeley, Hobbes, Adam Smith, and Kant, but I see no need to go so far back. I mention Jeremy Bentham [1932] largely because he exemplifies the tendency to regard what is not material as fictitious, giving examples like quantity, quality, degree, and relation, as opposed to what he called fabulous entities including generalities like unicorns and particulars like Pegasus. In mentioning motion, his point is not that things do not move but that it is grammatically convenient to have a noun to correspond to the verb to move, and nevertheless that grammatical convenience does not create things despite the long history of mere talk's being taken to have ontological consequences (Anselm). Bentham's notion of a relation is on the one hand extremely broad, being produced by a mind's regarding any pair of things, 'at the same time, passing from the one to the other' (p. xl), and on the other hand limited only to pairs, 'always *between*, never *among*' (p. xl). Abstraction is, for Bentham, an example of a relation; 'relation is the most abstract of all abstractions' (p. lv, n. 3). This current in the use of the notion of fiction is still with us. The contemporary materialist philosopher Mario Bunge regards all formal as opposed to natural science as fictitious, elaborating his view of the distinction between mathematical fictions and, for instance, artistic fictions at some length in his *Treatise on Basic Philosophy*, of which [1985] is the relevant book. On the other side, the epistemological status of mathematics is quite distinct from aesthetic, moral, and religious tenets.<sup>29</sup>

<sup>28</sup> Duns had objects existing in the intellect without real existence but only intentional existence (*Concerning Human Knowledge*, quoted in Mitscherling [1999]).

<sup>29</sup> 'The truth about what there is concretely is only the most minimal (though also in one sense the most fundamental) truth. Much more important for us as human beings is the vast body of moral, aesthetic, and religious truths that, on my view, pertain to other domains of existence. I have emphasized the realm of concrete existence only because of a widespread

Like today's cognitive scientists, the Kantian Hans Vaihinger was concerned with understanding understanding, and like Lakoff and Johnson he regarded most of it as done by seeing one thing as another. His working out of this idea led to the book *The Philosophy of 'As If'* (published in German in 1911), with its title's misleading suggestion (to me) of deception. One of the things he writes about in that book is what we now call mathematical and other modelling. Mathematics is not narrative in form, but one may see something about mathematics by looking at narrative as an analogue. If one is clear that mathematics is not narrative, then this seeing is a fiction in Vaihinger's sense; it is not a *simple* truth. He noticed that there were not enough simple truths to go around. 'All cognition is the apperception of one thing through another.' (p. 29) Many of what he calls fictions I should call insights. He says Aristotle in *Metaphysics* M §3, from which I quoted above, 'is trying to justify the procedure of mathematics against the reproach that its subject-matter is a non-existent entity and not something independent' (p. 142). Vaihinger of course calls ancient Greek postulates fictions (in content). He regards the extensive employment of fictions to be an important aid to the development of science —some of his scientific examples are motion, centres of gravity, and absolute space, but the first place where it was done with 'great results' (p. 148) to be mathematics, which is characterized 'by the freedom with which it forms these fictional constructs' (p. 148). The a priori and deductive procedures of science and of mathematics he regards as different not 'in essence or quality, but quantitatively and in degree' only (p. 227, n. 2). The mathematical examples that he gives indicate the sort of thing that he means. His first example is the algebraic technique of using letters 'as if' they were numbers, which he calls a 'substitutive fiction' (p. 148).

'Thought itself, in general, when operating with words instead of perceptions, makes use of such symbols.' (p. 148) He regards this as a very general technique. He calls the use of co-ordinates for points in the plane 'really the classical example' (p. 148). Linear equations (he calls them 'artificial lines'), differentials and fluxions, the infinitely large, and negative, fractional, imaginary, and irrational numbers are further examples, to which he adds as the most up-to-date the imagining of 'spaces of more than three dimensions' (p. 149) on which the method of determinants depends. He makes the point

and persistent tendency to assume that the truths of morality, religion, etc. must be about peculiar sorts of concretely existing entities. It is this assumption that has been particularly responsible for the ever-growing scepticism, throughout the modern era, about morality and religion. Claims that are profoundly true and important as *interpretations* of human existence have been based on highly implausible claims about special sorts of concrete existents (e.g., gods, values, immortal souls).' Gary Gutting [1978], note to p. 106.

recently emphasized by Y. Rav [1999] that ‘great mathematicians have always been distinguished by the invention of devices’ (p. 149). He claims that ‘every really new discovery in mathematics rests upon such a device’ (p. 149) and remarks that practice, especially with respect to the infinite has [at the end of the nineteenth century] run well ahead of methodology, by which I think he means justification. His summing up on space, which he discusses at some length, makes valuable points on need, abstraction, imagination, and parallelism of method to the other sciences.

For mathematics, however, the concept is necessary, useful and fruitful, because the mathematicians only investigate the characteristics and laws of extended objects, *qua* extended, and not their materiality or other physical properties. The concept of pure space arises from retaining the relation of objects after the things themselves have already been thought away.<sup>30</sup> . . .

Abstraction detaches something which we experience only in something else (whether as property or as relation) from this other entity—from something to which it is so firmly and inextricably bound that when what has been detached is accurately analysed we are forced to admit to ourselves that nothing remains in our hands. . . .

Imagination, by reason of its specific and peculiar gifts, comes to the aid and rescues abstraction which, as described above, has dissolved the given world into nothing and stands looking round helplessly at the result of its activity. Imagination reintroduces into the isolated relation the idea of the related elements, but in a form in which they are only shadows of what we find in reality. It thus provides a support for the product of abstraction and prevents it from falling into the abyss of nothingness.

What we must do, therefore, is to make clear to ourselves that the space of the mathematicians is nothing but a scientific and artificial preparation, which differs from the schematic auxiliary constructs, etc., of other sciences, only in the nature of the objects that are to be investigated and not in method of investigation. This unity of method must be strongly emphasized. Only a methodological approach can purge us of our old prejudices about the objects of mathematics. (pp. 232–233)

Now that we have Vaihinger’s way of seeing one thing as another, which is an elaboration of what we normally do when not doing either mathematics or philosophy, we can mention the three different ways in which, in these three different modes of thought, we *apply*. To see mathematics as fiction one need only see some analogy between two different and distinguished kinds

<sup>30</sup> He *did* claim that Aristotle was on this track.

of text, perhaps only between specific examples of the two. Little generality is needed —though is often inferred inductively. The common-sense application of one idea to another is inherently vague without being useless. As I remarked at the outset, mathematics and philosophy have different more precise ways of dealing with application. Mathematics typically takes mathematical objects and their relations and *maps* them, as is often said, onto some application domain's objects and relations. I need not dwell on the common task of mathematical modelling.<sup>31</sup> The model and target may have things in common, but no one focuses on those generalities. Philosophy typically applies a notion to another by subsuming one under the other. So, a philosophical discussion of the analogy between mathematics and fiction will probably subsume one under the other or both under some more general notion. The latter, which is what I am trying to do, is preferable to what we shall see happens, which is to subsume the one under the other, taking fiction to be the more general category itself instead of seeking a more general category among texts of which mathematics and fiction can be instances, preserving their distinctiveness from each other. This distinctiveness, as I shall be remarking later, is important. There are immense logical complications to fiction, largely because of its complex interactions with the everyday world, that no one concerned with mathematics is likely to feel attracted to.<sup>32</sup> (These characteristics of the contrasting styles of mathematics and philosophy emerged in discussion with Bob Hale.)

Very roughly contemporaneous with Vaihinger (who wrote and published at very different times) is Frege, who also used the idea of fiction, Gareth Evans suggested, as 'a convenient mat under which he could sweep the problem posed for his theory' ([1982], p. 28) by the sense of empty singular terms. This is a use of the notion of fiction certainly different from Vaihinger's and also not as a place to put mathematics, since Frege did not think that mathematical terms were empty. It was his semantic theory that needed somewhere to put embarrassments. Evans calls a 'cover-up' Frege's unjustifiable treatment of any use of an empty singular term as fictional or even poetical use of language. 'This is no momentary aberration; at almost every place where Frege discusses empty singular terms, the idea of myth or fiction, sometimes even poetry is close at hand.' (p. 28) Evans cites eight

<sup>31</sup> Mathematical modelling is discussed from a structuralist point of view in Rickart [1995], chapter 7, where, however, he makes the odd comment that Spinoza's *Ethics*, which is modelled *on* mathematics in the different sense of emulation, is an example of having 'the form without the content of mathematics' (p. 119). Since the important way in which a work like this that merely imitates mathematics differs from mathematics is precisely in *having* content, in this case ethical content, the criticism seems oddly backwards.

<sup>32</sup> Woods [1969] and [1974], Lewis [1983], and postscripts (same volume) pp. 276–280.

places in three different collections of Frege's writings. Has a sentence containing an empty singular term a sense?

Yes: a sentence containing an empty singular term may have a sense, in that it does not necessarily have to be likened to a sentence containing a nonsense-word. But no: it does not really have a sense of the kind possessed by ordinary atomic sentences, because it does not function properly, it is only *as if* it functions properly. (p. 30)

Evans uses Vaihinger's technical term for fiction, 'as if', to explain a situation for which Frege used the mat 'fiction'.

(Russell's talk of non-abstract substantives as fictions, on the other hand, is mere rhetoric, I think. Following upon his attempts to 'construct' mathematical entities, along the lines of 'the definition of real number' (Russell [1954], p. 290), he found that he could 'construct' physical things too. Space, time, thing, matter, [physical] points, instants, space-time, and many mathematical and psychological terms he showed to his satisfaction could be replaced by classes of entities he regarded as more basic, in some cases dispensing with abstraction. In fact, he calls the principle involved 'the principle which dispenses with abstraction'.<sup>33</sup> This is pretty clearly a case of logical analysis, as of objects into the classes of sense-data or potential sense-data corresponding to them, which he phrased as their being logically constructed. These were things the reality of which he did not dispute but whose analysis he wished to discuss because he thought the analysis into ultimate constituents important. The maxim involved, a derivative of Occam's Razor<sup>34</sup> for which he gives credit to Whitehead, being 'Whenever possible, substitute constructions out of known entities for inferences to unknown entities.' ([1985], p. 161) But he did choose to call these things both 'fictions' and even 'myths'; he could use the construction and fiction terms together, 'The persistent particles of mathematical physics I regard as logical constructions, symbolic fictions...'.<sup>35</sup> In discussing Russell's no-class theory in the same volume, K. Gödel (*ibid.*, p. 141) uses the 'fiction' term still, saying that while Russell might at best have shown empty and unit classes to be fictions, he did not show all classes to be fictions. This rhetoric persists, being exemplified by David Bostock [1974] and [1979], where he uses Russell's term 'logical fiction' ([1974],

<sup>33</sup> Russell [1985], p. 161, quoting his [1914], p. 42.

<sup>34</sup> For reference to Thorburn [1918], I am indebted to Yehuda Rav. The usual ontological Occam's Razor does not apply to mathematics because mathematics does not entail ontological commitment. The methodological principle, which is far older than Occam, is just common sense. It is not absolute, and Azzouni, for example, relaxes it ([1994], pp. 101–103).

<sup>35</sup> [1910], p. 128, quoted by Max Black in *The philosophy of Bertrand Russell*, p. 246.

p. 9) to describe what numbers become on his view at the beginning of the work but changes to ‘construction’ later and eventually casts doubt on the propriety of using ‘logical’ ([1979], p. 281).)

In Germany, Vaihinger was not forgotten. Bruno Baron von Freytag-Löringhoff, just after the second World War, wrote a short book in the spirit of Vaihinger but in more up-to-date language based on lectures<sup>36</sup> translated into English as [1951] in which he makes reference to pre-war works by himself and others in German, emanating from Halle where Vaihinger was professor. He carefully distinguishes the philosophical existence question from the mathematical, saying of the latter that it only appears to be ontological ([1951], pp. 23 f.). Taking as given that ‘the objects of pure mathematics are not real in the concrete sense’ (p. 25), he observes

that we think in the same way of both the concretely real and the abstract or non-real; and that the type of Being which occurs in the latter is structurally analogous to Reality in the concrete sense. (p. 25)

He goes on in a Vaihinger-like way:

Whether we speak of real or non-real (in the sense of abstract) Being, we regard both as being entirely independent of whether they are thought by us or not. This, we are bound to do; for otherwise we should be thinking, not of the object of thought, but of ourselves and our thought processes. And this is not the case. The position becomes particularly clear as soon as we encounter errors. What has been erroneously accepted as a fact, although it is actually non-real, is here thought of as being really existent. And we must deal with all non-Reality (or abstract Reality) in this way. In order to be able to think of it at all, we are obliged to ascribe to it an independence of being thought —just as we do in the case of concrete Reality. We must ascribe to it, independent Subsistence-in-itself (*Ansichbestand*). And while in the case of concrete Reality, this is real, in the case of non-Reality (*Unwirklichem*) or abstract Reality, it is fictitious. (p. 26)

He adds that ‘it is just *because* its type of Being is purely fictitious that it can remain identically the same at all places and at all times’ (p. 29).

Gareth Evans mentions the belief-independence of informational states as a ‘fundamental characteristic’ ([1982], p. 359, n. 30) of informational systems. And what does von Freytag-Löringhoff mean by ‘being’ for mathematical objects?

<sup>36</sup> *Gedanken zur Philosophie der Mathematik* (Meisenheim, 1948).

Obviously, an object is a mathematical object and is known as such, insofar as it belongs to, or is a member of, a mathematical system of objects and logical relations. This belonging-to, or membership (*Zugehörigkeit*) is what constitutes the meaning of the expression 'there exists' in mathematical existential propositions; and insofar as the system to which it is referred is a different one, this expression means something different in every system. (p. 30)

He even considers the obvious comparison with ordinary fictions, pointing out that in a novel logic rules too (this is before post-modernism), but because of the superior definiteness of mathematical ideas they can be carried in this way farther than 'the majority of ideas' (p. 30).

There is nothing very definite about a character in a novel of fiction, for example. As regards questions which extend beyond the range of the intrinsic logic of a novel, we have to turn to the writer for an answer. Pure mathematical ideas, on the other hand, must be completely definite: their definitions must supply all information. (p. 31)

And this leads back ultimately to the implicit definitions of axioms. These lectures cover a lot of ground in a short space, including some material on applicability. The final excerpt I shall quote mentions that any mathematical existence statement implicitly calls upon its mathematical status to be understood.

In our interpretation of the mathematical expression 'There exists,' we omitted to mention its reference to the *whole* of mathematics—a reference which occurs whenever we say that such and such objects, considerations, arguments etc., exist in mathematics. Here, it is a question of belonging, not to any particular mathematical system, but to mathematics as a whole, to a totality of which the logical structure is no longer so simple. We might characterize mathematics as the total aggregate of logically possible (i.e. non-contradictory, or self-consistent) systems which are based on implicit definitions. (p. 33)

This seems to be an early suggestion of a special mathematical modality but linked to the fictional modality.

In his sense of the word, which has little to do with stories, Vaihinger is right to say that mathematics about fictions; he acknowledges that he is following Bentham, and Bunge acknowledges his debt to Vaihinger. But this sense of fiction is not one that is widely recognized, even among philosophers.

### 7. Narrative (from the Sanskrit root *gnā*–know)

Before getting down to the story, I need to comment on distinctions. French and German do not distinguish with vocabulary between history and story. And the French use the preterite tense for both sorts of narrative. In English, what happened in the past is called history, but writings about what happened are called histories. One history, many histories. Presumably, to the extent that the histories have got it right, they express the view from nowhere or now and here of what happened. It is less simple with fiction, where the terms legend and myth are used. A number of writers do not make any such distinctions, wasting the term myth on stories or even non-stories that are taken to be untrue. The usefulness here of myth and legend is that what is identified as a *single* myth (made up entirely) or legend (based on someone real, *e. g.*, Faust) can contain many stories that would be in conflict if they were taken to be historically accurate.<sup>37</sup> This conflict is not something wrong, because the sense of truth required in such stories is truth-in-the-story, as we shall call it, rather than some more absolute truth. Note that ‘in’ in this phrase is definitely metaphorical; its elaboration ‘in the world of the story’ is just more elaborately metaphorical. It is understandable to say that when he was with Leda, Zeus was *actually* somewhere else and not in the form of a swan, but it is absurd because Zeus is not actually anywhere.

Between narrative and mathematics, the important similarity that is my subject is how they exemplify similar successful ways to give appropriate discussion to relations. Relations, whether in a family tree or commutative diagram, can be displayed, but that sort of representation, while it can serve as an adjunct to discussion, is unable to indicate much more than what (or who) is related to what (or to whom), and sometimes how. For a *discussion* of relations in the way that people care about, what one needs is to engage the intelligence and imagination (not primarily or necessarily visual) of the reader with entities related by the relations to be discussed. The appropriate discussion that allows a person to engage with persons and care enough about them to follow their relations is narrative. There is no kind of talk more engaging than narrative. This indubitable fact can be illustrated as disparately as by the brisk sales of *Sophie’s World*, a novelized history of philosophy, and the quantity of narrative in the Jewish and Christian scriptures. Typically, narrative follows persons through time<sup>38</sup> and is more than

<sup>37</sup> Hao Wang [1974], p. 80. ‘It is a striking fact that in diverse systems of different strength, we can prove counterparts of all ordinary theorems about real numbers. This suggests that no proof in any system formalizes faithfully the true mathematical result.’ His ‘true’ result is like a myth not a story, and the diverse systems like stories within that myth.

<sup>38</sup> As Ricoeur puts it ([1984], p. 3), ‘portrays the features of temporal experience’. Lamarque and Olsen ([1994], p. 225) quote Prince ([1982], p. 4), ‘narrative is the representation of

just the facts recounted, as Adam Morton has attempted to show in some research I shall report on later in this section. Enough is said about the characters involved to allow the reader to imagine them (I emphasize not just visually) and the world in which they live(d). As described, such characters are seriously incomplete; the reader's imagination is called upon to fill them out. Their situation (relations<sup>39</sup>) is expounded and developed over time to whatever final situation (again relations if still alive) has been chosen for the end of the narrative. Of history, Hayden White writes,

The events must be not only registered within the chronological framework of their original occurrence but narrated as well, that is to say, revealed as possessing a structure, an order of meaning, which they do *not* possess as mere sequence. ([1981], p. 5)<sup>40</sup>

For persons and perhaps history generally, connection is in terms of reasons and intentions; in purely physical matters, connection is in terms of causes. Plots and subplots, however complex, have to be presented in a linear way with devices like flashbacks to fill in out-of-order details.<sup>41</sup> Not everything is given equal weight; the more dramatic episodes are given emphasis, tension builds, conflict is resolved. One of the classic final situations introduces into narrative a device thought more often to be mathematical, uncompleted infinity: 'they lived happily ever after'. 'They lived happily ever after' is a

at least two real or fictive events or situations in a time sequence, neither of which presupposes or entails the other.'

<sup>39</sup>These are ordinary relations. Fiction only works because the relations attributed to the imaginary characters are of the same kind as those in the world. If the relations were unreal too, no one could understand the story.

'Fictive states of affairs [like fictional objects discussed at pp. 42 f.] ... are intensional objects whose nature and very existence [*i.e.*, non-existence] are dependent logically on the descriptions in some originating fictive utterance. This is a simple consequence of the non-extensionality of fictive content and the redirection of attention from reference to sense entailed by the fictive stance.' (Lamarque and Olsen [1994], p. 88)

This view of stories is particularly associated with the nineteenth-century French literary critic Hippolyte Taine, I am told by my colleague J. R. Allen.

<sup>40</sup>It should be noted that, while I am citing White as a philosopher of history, I am doing so only for views that are not idiosyncratic. Even his sternest critic (Noël Carroll, 'Interpretation, history and narrative', *The Monist* 73 (1990), 134–166) would not deny what I am quoting White as saying.

<sup>41</sup>Robert Scholes says 'The object of a story is the sequence of events to which it refers; the sign [using Peircean terms] of a story is the text in which it is told (print, film, etc.); and the interpretant is the diegesis or constructed sequence of events generated by a reading of the text. ... each of these three aspects of "story" has its own temporal structure.' ([1981], p. 206) More common terms for his object and interpretant are *fabula* and *sjuzet* respectively.

common feature of many stories, and for the same reason that the natural numbers are an infinite set. It is easier than thinking about where they stop. ‘They lived happily until Sam developed liver cancer’ is not the way to end a fairy tale. Note, however, that my little sketch of narrative up to that classic but unhistorical ending, has applied equally both to history<sup>42</sup> and to works of fiction.

Turning from the common devices to those peculiar to fiction, we see that, in the simpler sort of fiction (*e.g.*, the Greek myths, fairy tales, many short stories, much in the genres of romance, murders, adventure stories, science fiction, and fantasy), what is important is the relations in which the characters find themselves<sup>43</sup> rather than the characters.<sup>44</sup> The beginning of a work of fiction, the commencement of that modality, is signalled by the theatre stage, or by a standard or conventional opening like ‘once upon a time’ or ‘Aixo era y no era’.<sup>45</sup> Except for the framing, indicating the author’s attitude and prompting the reader’s, fiction can be just the same as non-fiction. Much of what is said of fiction can truly be said of narrative in general. In particular, characters can be real in a sense, though they are usually not. The sense I suggest is that they allude to the real persons and places as props for the game of make-believe. The connection of history or historical fiction to the real world is done *as a whole* rather than by word-for-word reference.<sup>46</sup> Such writing is intended to get at truth of a kind not altogether the same as that of an inventory, a kind of truth that some might hesitate to call truth at all

<sup>42</sup> According to Ricoeur ([1984], p. 161), ‘... fiction and history belong to the same class as regards their narrative structure’. And it is the narrative structure with which we are concerned, what is preserved when a prose or epic story is adapted respectfully to stage or screen.

<sup>43</sup> This is the classical view of drama beginning with Aristotle’s *Poetics*, Chapter 6. Luigi Pirandello brilliantly confirms its wisdom by his creation of an exception, *Six Characters in Search of an Author*, which demonstrates (in the dramatic way —showing— rather than in the deductive way —proving) that starting with the characters you will never get a play. Shakespeare’s history plays are definitely not counterexamples.

<sup>44</sup> ‘Something happened to fiction around 1800’ (Gabriel Josopovici, orally, 1999 5 4) is one way of putting the change from the kind of story I am concerned with here to the kind of story that has developed into the contemporary psychological novel where much of the interest is in getting inside the characters’ heads.

<sup>45</sup> ‘It was and it was not’, traditional exordium of Majorcan storytellers according to Paul Ricoeur [1977], p. 224. John Woods, in [1974], invented the operator ‘O’ (for Latin *olim*) for fictional sentences.

<sup>46</sup> This idea was suggested to me by but not found in S. Hoffman, *Mathematics as Make-Believe* (Chapter 3 of draft Ph. D. dissertation, University of Alberta, 1998).

—as many hesitate to call mathematics truth. Ordinary fiction is even less dependent than historical on what might be called pragmatics; real-world reference is an after-the-fact option. You may or may not want to call someone a Scrooge.<sup>47</sup> Before the work of fiction, say *Othello*, is written, there are no corresponding fictional characters, just the idea of an unusual dramatic triangle. The circumstances of the play's beginning and the unfolding of the plot define the character of Othello (physical and mental) to the extent that it *is* defined. Much is filled in by the playgoer or film or TV viewer, even more by a reader.<sup>48</sup> In Greek tragedy masks remove much of the personality (a concept not yet invented) of the characters, but Shakespeare has soliloquies that give insights into the character that are not displayed in overt speech and action. Likewise in novels the narrator often tells the reader things that cannot be deduced from the action of the characters. This is necessary for what is often called 'realism', because we all know that we have interior lives and that it would be artificial in a bad sense to pretend that the characters had not. It would make them less than human.<sup>49</sup> Much of the interest of twentieth-century novels in particular (despite the contrary opinion and practice of Virginia Woolf) lies in seeing what happens in heads. But it seems to me that what is going on in certain genres (less and less so with the passage of time and the shift from circumstance-driven to character-driven plots) is that situations are being worked out with the aid of the characters rather than the situations' being incidental to the characters. This working out is to a large degree arbitrary in fiction unlike history and mathematics. The plot is contingent on the author's decisions, and so whether the story is worth taking in depends upon the author's skill in creating a story that is satisfying in ways that it is outside my competence even to describe. Fortunately these skills are not important to the present inquiry.

Before turning to mathematics, let me report on Morton's work [1996] on narrative as described in his piece considering 'the likeness and unlikeness of mathematics to the rest of language'. He has conducted an experiment illustrating that the narrative form gives its content something more than just the sum of non-narrative parts. If one tells 'Little Red Riding Hood' backwards or by stating 'what is true of the wolf, what is true of the grandmother, what is true of the basket, *etc.*' (p. 214), it requires prompting for undergraduates

<sup>47</sup> 'Characteristics associated with fictional characters can become paradigms in non-fictional contexts.' (Lamarque and Olsen [1994], p. 89)

<sup>48</sup> 'Much of the pleasure of reading fiction derives from the imaginative "filling in" of character and incident.' (Lamarque and Olsen [1994], p. 89)

<sup>49</sup> This they may be. They may even be mathematical, as in *Flatland*.

even to recognize the story as one they know. He also explains the understanding of a story in terms of the successive interpretation of each sentence on the basis of background information as a little deductive algorithm:

Hear  $s$   
 Assign  $s$  an interpreted logical form, call it  $\sigma$ .  
 Set  $\tau = \sigma$   
 FOR  $v =$  immediate consequence of  $\tau$   
 IF  $v$  is of the form ' $P$  does  $A$  at  $t$ ' and is relevant to the story STOP  
 ELSE set  $\tau = v$  (p. 215)

He points out that his interpretive procedure tells the reader both when to stop deducing consequences and that they have understood. What he does not say is that the deduction for the sake of relevance may be from the background information or a previous sentence rather than from the new sentence or it may require finding or inventing a piece of background presupposition that had not previously seemed relevant or realizing that the new sentence is a sufficiently new point simply to be filed for future use. In short, a very great deal may be required other than but broadly similar to what Morton suggests.<sup>50</sup> I agree, broadly speaking, that what is sought is relevance in the sense of making sense of the total package so far and that deduction is the tool. But the deduction is not necessarily logical; one does not seek merely logical consequences.<sup>51</sup> The consequences may be causal in either direction (to a cause or effect of what one reads about) or, in personal contexts, reason rather than cause. The upshot can easily be to reject all literal interpretations of the new sentence and to seek a figurative interpretation if the need to do so was not initially obvious. His whole paper is a fascinating opportunity to see similarity and difference between the mathematical and narrative modes in examples. From Morton's examples, I draw the conclusion opposite to his.

How are some stories analogous to mathematics? There is the clear modality shift; one knows that it is mathematics that is being done as soon as one has achieved the capacity to do mathematics. One does not ask the question 'two plus three *what?*' because the context is clearly mathematical. The signals are not as literary as 'once upon a time', but they are perfectly clear. For a discussion of mathematical relations in the way that people care about, one needs to engage the intellect and imagination (not primarily or necessarily

<sup>50</sup> Cf. Umberto Eco [1979].

<sup>51</sup> Even within logic itself, what is needed in stories may be different from what is used in mathematics. This is an theme of Paulos [1998], who elaborates (pp. 87 ff., 101 ff., 107, 109) the distinction between the latter extensional logic and the former intensional logic, mentioning Saul Kripke and Mark Turner as well as Jon Barwise, whose situational logic has been semi-popularized by Keith Devlin [1991].

visual) of the reader with entities related by the relations to be discussed. One kind of discussion of mathematical objects is algorithmic, virtually narrative but specifying what is to happen rather than reporting it. Adam Morton ignores his own pseudo-code algorithm (quoted above) in concluding for unlikeness over likeness. The similarity of algorithm to story is so close that I intend to say nothing more about it. Another appropriate discussion that allows a person to engage with mathematical objects and care enough about them to follow their relations is deductive. Mathematical deduction follows a logical progression through time, the reader's time. As Alan Montefiore points out,<sup>52</sup> it is the abstraction, reason, that is timeless; timeless deductions depend upon the thoroughly temporal reasoning of human beings. Enough is specified about the objects involved to allow the reader to imagine them (I again emphasize not necessarily visually) and the space in which they lie and to reason about them. They are types rather than individuals with characters of their own.<sup>53</sup> Their situation (relations) is expounded and developed over time to whatever final situation (again relations) has been chosen as the conclusion of the proof. The deductive structure, however complex, has to be presented in a linear way with devices like lemmas corresponding somewhat to flashbacks if the lemmas have been already proved before the main proof that calls upon them is proved. This makes the text more like a story although the time passing is the reader's. Presentation as a directed graph would be more perspicuous if we were capable of taking in such pictures synchronically; but our input mode is, for most purposes, diachronic, as with narrative. (A proof done entirely by lemmas is more like the directed graph on account of having no overarching proof.) We prefer in both narrative and proof to leap about the undrawn graph. Hypertext (freeing readers from the tyrannical ordering—pun intended—of an author) is currently experimented with.<sup>54</sup> Not everything is given equal weight; the more interesting arguments are given emphasis; tension builds. *Reductio ad absurdum* resolves contradictions. Infinity is often brought into a mathematical situation in order to simplify the expression and thought about the situations described; it is much easier to think of an infinite frieze than to deal with the

<sup>52</sup> 'The Idea of "Crisis" in Philosophy', read at the conference, The Crisis in Analytic Philosophy, University of Southampton, April 12–13, 1999.

<sup>53</sup> This is like fiction; '... *works of fiction*, qua fiction, are (primarily) about kinds rather than particulars ...'. Lamarque and Olsen ([1994], p. 122) write this in support of their approval of Aristotle's contrast between fiction and history in *Poetics*, Chapter 9.

<sup>54</sup> E. Artin, in his review [1953] of the *Algebra* of Bourbaki wrote, 'We all believe that mathematics is an art.' But exposition must always fail. 'Mathematics is logical to be sure; each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that its perception should be instantaneous.'

boundary conditions necessitated by a finite one or to deal with the positive integers than the finite number of them representable in a computer memory. There is obviously mathematics of a non-imagination-engaging sort; here I am concerned with thinking mathematically not *executing* an algorithm. While I can say with confidence that before *Othello* is written there are no *Othello* characters, I have to be less confident about mathematical objects, but my personal inclination is to say the same for the same reasons. The content of a theorem's hypothesis and the unfolding of the proof define the objects for the most part. There are no soliloquies or inner nature to reveal with them. Mathematical conclusions are circumstance-driven as by fate in Greek tragedy. But *how* they are worked out (proved) needs to be invented more like fiction than history, where it is *what happened* that needs to be figured out. The proof more than the conclusion is contingent on the author's decisions, and so whether a proof is a good one depends upon the author's skill in creating a proof that is satisfying in ways that again as with stories it is outside my competence even to describe. Validity is of course the most important thing, but one can choose among valid proofs ones that are better and worse. Second to validity is whether a proof is explanatory<sup>55</sup> rather than just convincing. Explanatory proofs are more like narrative.<sup>56</sup>

No literary critic would be satisfied with my sketch of even the simplest of fictions, but inadequate as it is, it is more elaborate than any I have found in the philosophical literature comparing or identifying mathematics and fiction. It does at least approximate a sense of fiction quite different from that of Vaihinger's.

#### 8. *Mathematics not fiction*

The analogy drawn above is importantly both positive and negative; mathematics is neither fiction nor narrative despite the closeness of algorithms to narrative. My aim is to compare two *different* things and suggest why the comparison is so often made.<sup>57</sup> One of the philosophers to consider one or both comparisons was Leslie Tharp, who began his posthumously published paper, 'Myth and Mathematics: A Conceptualistic Philosophy of Mathematics I', wishing to make 'a comparison with fiction' to illuminate his conceptualistic position on mathematics.

<sup>55</sup> This feature is being studied by Paolo Mancosu ([2000] and [1999]).

<sup>56</sup> MacIntyre [1977] claims that explanation is grounded in narrative.

<sup>57</sup> One might note that the comparison is possible or makes sense because, as Netz [1999] points out of ancient Greek mathematics, it is a literary genre (p. 306).

The comparison is not intended in any pejorative sense whatsoever. Rather, we wish to focus attention on the technical fact that myth and other fiction frequently operate with meaningful everyday concepts, but without objects. In fiction one has all along been using ordinary logical forms and inferences in contexts where no objects are referred to. ([1989], p. 167)

He means, of course, that no real ordinary concrete objects have been referred to, observing the convention that you cannot call reference what you do to things that are not real. But before he even left the first page of his paper, he already slipped into the philosophical mode I mentioned above, exemplified by the phrase ‘mathematics may profitably be considered to be a kind of fiction’ (p. 167).<sup>58</sup> Since, as Vaihinger and others since have observed, comparing things is a primary way in which we come to understanding, this unwillingness is one of the important ways in which philosophy’s credit has fallen among its publics. According to the typical contemporary philosopher, everything is something it is not, which merely gives opportunity to opponent philosophers, who are of course able to deny such an identity, since all such identities are false. As Vaihinger pointed out and Lakoff and Johnson [1999] re-emphasize, it is from many such ‘false’ statements that we learn ‘as if’. The only person generally thought to claim the name ‘fictionalist’ is Hartry Field [1980], who seems to have discovered for himself the question, ‘why regard the axioms as *truths*, rather than as fictions that for a variety of reasons mathematicians have become interested in?’ (p. viii) and the answer, ‘no entities have to be postulated to account for mathematical truth’ (p. viii). Field’s postulation is presumably the stronger ontological kind including what Michael Resnik (who does it) calls ‘to affirm their existence’<sup>59</sup> ([1997], p. 185). Field sounds a bit like Tharp, but Field’s view does not seem to be that mathematics is merely like fiction. As

<sup>58</sup> Writing of the analogy between fictionality and truth, Kendall L. Walton says, ‘There is a persistent temptation to go one step further and to think of fictionality as a species of truth (Imagining might then be regarded as a kind of believing, one appropriate to this species of truth.) The temptation is both reflected in and nourished by the fact that what is fictional is colloquially described as “*true* in a fictional world.” “Fictional worlds” are easily thought of as remote corners of the universe where unicorns really do roam ...’ ([1990], p. 41). Likewise, representation is only *like* reference, not a species of reference (*idem.*, p. 122). No referent is required.

<sup>59</sup> I take this affirmation to be both a propositional attitude (belief) and a degree of emotional commitment. In calling the conversational version of postulation free of belief, I do not mean that intellectual and imaginative engagement is free of emotional commitment. The conclusion to Mary Warnock’s [1976] speaks of the power of imagination, one of the functions of which is to ‘see’ significance; its ‘impetus comes from the emotions as much as from the reason, from the heart as much as from the head’ (p. 196). It would be difficult indeed to spend hours, not to mention years, studying something in a completely uncommitted way. Lakoff and Johnson cite work of A. Damasio (*Descartes’ Error*. New York: Grosset/Putnam,

a committed nominalist, he thinks mathematics is fiction or, it seems to me, lying (formally the same of course since hearers can commit category mistakes). (There is no question that mathematics is put out as a *kind* of truth. Since he says that it *is* not true, he makes liars of mathematicians.) This is illustrated in his second book.

... the fictionalist can say that the sense in which ' $2 + 2 = 4$ ' is true is pretty much the same as the sense in which 'Oliver Twist lived in London' is true: the latter is true only in the sense that it is true *according to a certain well-known story*, and the former is true only in that it is true *according to standard mathematics*. Similarly, the fictionalist *believes* that  $2 + 2 = 4$  only in the sense that he or she believes that standard mathematics *says that* (or, *has as a consequence that*)  $2 + 2 = 4$ ; just as most of us believe that Oliver Twist lived in London only in the sense that we believe that the novel says that or has as a consequence that Oliver Twist lived in London. ([1989], pp. 2f.)

Field is mistaken about the subject matter of mathematics.<sup>60</sup> In his reply to criticisms of Penelope Maddy [1990] he says,

Our different set theories 'have a different subject matter' only in that they are different stories. They differ in subject matter in the way that *Catch-22* and *Portnoy's Complaint* differ in subject matter; these differ in subject matter despite the fact that neither has a real subject matter at all.

The point of comparing mathematics to fiction is simply to make this negative point that neither is properly evaluated in terms of how well it describes a real subject matter. ([1990], p. 207)

The subject matter of mathematics is dealt with in a way sufficiently subtle that it has not been noticed by many that have been deceived by grammar into thinking that its subject matter is different from what it is. The subject matter of novels, as all making a comparison with fiction should know, is neither vacuous nor unreal, only superficially so. Field shortchanges both mathematics and fiction in identifying them.

1994) saying still more than this. But the necessary commitment in mathematics is to the interest not to the *existence* of the material.

<sup>60</sup> Mary Tiles writes. 'This ... is to ignore the way in which mathematical expressions have their content internally generated by their place in a system. When employed in a physical theory they impose and make available precise forms of relation, precise structures for the articulation of theoretical concepts which, as physical concepts, are derived from initial source analogies, paradigm intended applications, experimental and measurement practices etc. (here following Campbell to some extent), but thought through, and thereby structured and schematized by pure mathematical forms.' ([1988], p. 202)

Failing to see that mathematics works out in an objective manner the logical consequences of relations that it postulates objects to have, Field fails to see that there is any objectivity in mathematics. So he wants nothing to do with it and proposes replacing it with mathematics-like theory that is not about abstract objects. While the replacement is admirable work, it has remarkably little to do with fiction and has had the effect of sidetracking discussion of fiction to a considerable degree for twenty years. The rhetoric of fiction has been maintained, as for example in Penelope Maddy's criticism mentioned above, but few have bothered to think about fiction itself. Some that have will be considered in part II. I emphasize that mathematics is relative to choice of subject-matter and inference-programme, but within each such pair is entirely objective as almost everyone agrees, even those that find it puzzling. The main point of this section needs to be repeated, Field's motivating rhetoric is wrong; mathematics is not fiction in the ordinary sense of the word. There are three ways in which this is shown. Bunge has pointed out ([1985] and [1997]) quite enough differences between mathematics (which he regards as fiction in Vaihinger's sense) and what he has to call artistic fiction. With a lot of sympathy for the analogy between mathematics and certain fiction, I have given descriptions of both that indicate correspondences, but there are constant distinctions; I shall pursue this theme in part II. Finally, in order to identify mathematics and fiction, Field has to misdescribe both, saying that they are both about nothing when mathematics is about such relations as succession, triangularity, and compactness and his own example, *Catch-22*, is about the human condition, which is why it's funny, and about humanity's inhumanity (formerly, 'man's inhumanity to man'), running up through *catch-22* (a device almost worthy of mathematics) to war itself. Triangle *ABC* and the aircrew are made up for the purpose; so what? To call the real issues 'nothing' is not just wrong; it's outrageous.

### 9. Conclusion

I have been concerned in this paper to begin a serious comparison between mathematics and fiction, to be concluded in part II, and to perform a ground-clearing exercise to deal with the identifications of mathematics as *about* fictions by Vaihinger and *as* fiction by Field, respectively correct in Vaihinger's Pickwickian sense and incorrect in the standard sense.

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