Logique & Analyse **171–172** (2000), 283–300

BELIEVING IN STRONGLY COMPACT CARDINALS

WIM MIELANTS

Abstract
The classical argument in favor of the existence of strongly compact cardinals is the principle of uniformity. Here we give an other argument based on a principle of maximal diversity of reflections. This principle is motivated by Maddy's set theoretical naturalism and is inspired by some maximization principles of Leibniz.

Introduction

We have a very uneasy intuition about large large cardinals, or cardinals which are not consistent with the axiom of constructibility $(V = L)$. We have no intuitively convincing evidence to accept or reject their consistency with ZFC-set theory and certainly not the fact that they should be objects which exists independently of the human mind. The main arguments in favor of their existence are the principles of uniformity and reflection, which are in fact metaphysical arguments.

According to Maddy [22] in her set theoretic naturalism, there exist no extra-mathematical tribunal to decide if large cardinal axioms have to be introduced in set theory. Ontological considerations are external to mathematics and have no place in naturalistic methodology. On a pragmatic way we have to look to the advantages and disadvantages of these axioms as means towards particular mathematical goals.

Axioms about large cardinals have applications already in the theory of projective sets of real numbers, but the most impressive result is certainly this of Harvey Friedman in his paper about the necessary use of large cardinals for theorems about finite functions [7]. It is expected that in Boolean relation theory more combinatorial statements will be found which are equivalent with the consistency (or 1-consistency) of any large cardinal axiom yet considered. Friedman is very optimistic and believes that once mathematics is pushed into the new level of structural ambition that is represented in Boolean relation theory, large cardinals will have so many applications that the whole mathematical community will completely embrace this new level of structural ambition [8]. Of course Friedman has only proved the equivalence between the 1-consistency of certain large cardinals with theorems on finite functions of which we only believe that they are true. If by axioms about the consistency, the 1-consistency or the existence of large cardinals we could shorten the proofs of already proved combinatorial statements, this will still not mean so much since by Gödel's paper "On the length of proofs" [11] we know that any new independent axiom can shorten to an arbitrary extent the proofs of suitable assertions which are provable without that axiom. This result of course weakens a little bit our confidence in the value of pragmatic criteria for truth. But it remains true that such results would make the mathematical community more interested in large cardinals.

So, according to Maddy, if mathematics is allowed to expand freely, and set theory is to play the hoped-for foundational role, then set theory should certainly not impose any limitations on its own, the set theory in which mathematics had to be modelled should be as generous as possible, and the set theoretic axioms from which mathematical theorems are to be proved should be as powerful and fruitful as possible. As basic intuitive principle she propose MAXIMIZE according to which that theory is better that provides a greater variety of isomorphism types of interesting mathematical structures. Perhaps DIVERSIFY would be a better name for this principle (this was already proposed by E. Mendelson in a review of some paper of Maddy (see [23])).

Using a combination of a result of Solovay on strongly compact cardinals and instances of the Generalized Continuum Hypothesis (G.C.H.) [27], and a result of Jónsson on the existence of homogeneous universal relational structures, [10] we prove that the axiom about the existence of strongly compact cardinals implies the existence of a proper class of non-isomorphic homogeneous universal models for any first order theory which has a homogeneous universal model. So in the light of Maddy's DIVERSITY-principle this axiom about the existence of strongly compact cardinals should be introduced in set theory.

As Maddy has remarked extra-mathematical philosophy is still capable of playing an important inspiration role. This was certainly the case for this paper. Once Gödel has characterized his philosophical outlook in this way: "My theory is a monadology with a central monad. It is like the monadology of Leibniz in its general structure" [30]. By reading these words of Gödel I wondered if the maximization principles of Leibniz's monadology could still be a source of inspiration for set theory. If we take a certain Von Neumann-Bernays-Gödel set theory with the axiom of global choice (henceforth a NBG-set theory) as background set theory then all proper classes are equinumerous. If then we take as possible worlds all the models of a certain theory on a first order relational language, with this NBG-set theory as background set theory, what will then be the "best of all possible worlds" in

the sense of Leibniz? It has to maximize together harmony and richesness of substructures. We argue that it has to be, if it exist, the unique absolutely homogeneous universal model of that theory on that language and that NBG-set theory in the sense of Philip Ehrlich [4].
The existence depends mainly on the fact that the Joint Embedding axiom and the (weak) amalgamation axiom hold for that theory. These are in fact axioms on maximal composibility of substructures which can be found back in Leibniz's philosophy. The existence of our model theoretic interpretation of the notion of a unique central monad follows from the fact that the strong amalgamation axiom holds for the theory. This is in fact an axiom about maximization of independence. I have found no metaphysical equivalent of this in the monadology. So I think that this new principle should in fact be added to be other principles of Leibniz's philosophy since it decides an important hypothesis, namely the existence of a unique central monad. The maximal diversity of non-isomorphic monads in the best of all possible worlds is obtained if we have a proper class of non-isomorphic homogeneous universal substructures which reflect the whole universe. This will be the case if GCH holds, but if we don't assume GCH (which is rather a restrictive principle) this will still be the case if we have a proper class of instances of GCH. By Solovay's theorem this will be the case if there exist a strongly compact cardinal in the NBG-background set theory.

Leibniz's monadology clearly suggest that the set theory hierarchy had to be as rich as logic allows, so that absolutely homogeneous universal models have as many interesting reflections as possible. If you believe the monodology this is a sufficient condition to believe also in the strongly compact cardinals. But not a necessary one. Indeed Maddy has adopted a MAXIMIZE principle independently of any metaphysics, and by this principle set theory must be as rich as possible to provide maximal diversity of interesting mathematical structures. Homogeneous universal relational models are certainly very interesting. So you may have deep misgivings about Leibniz's monadology, but nevertheless find Maddy's arguments persuasive. On the other hand you can be interested to see that the general metaphysical principles of the monadology can be used to motivate set-theoretic axioms, while being strongly opposed to Maddy's nauralism.

1. *Strongly compact cardinals*

Tarski and Kreisel consider the semantics of infinitary predicate languages $L_{\lambda\mu}$ (λ and μ are infinite cardinals), and later raised the issue of their possible compactness ([14]). For this proceed as for the usual first order logic, first specify the non-logical symbols: the finitary relation, function and constant symbols. Together with an allowed supply of $\max(\lambda, \mu)$ many variables lead

to terms and atomic formula. Then the usual formula generating rules are expanded to allow conjunctions $\wedge_{x<\alpha}$ and disjunctions $\vee_{x<\alpha}$ of α formulas for any $\alpha < \lambda$, and universal quantifications $\forall_{x<\beta}$ and existential quantifications $\exists_{x<\beta}$ of β variables for any $\beta < \mu$. Finally a formula is an expression so generated with less than μ free variables. Structures for interpreting the language are as for first order logic, and the satisfaction relation is extended to incorporate the new infinitary connectives and quantifiers in the expected way. The languages $L_{\omega,\omega}$ are the classical first order languages. A collection of $L_{\lambda,\mu}$-sentences is satisfiable iff it has a model under the expected interpretation of infinitary conjunction, disjunction and quantification. It is ν-satisfiable iff every subcollection of cardinality less than ν is satisfiable. A cardinal $\theta > \omega$ is called *strongly compact* if any collection of $L_{\theta,\theta}$-sentences which is θ-satisfiable, is satisfiable.
So, if A is a set of $L_{\theta,\theta}$-sentences so that each subset $B \subset A$ with $|B| < \theta$ has a model, then A has a model if θ is strongly compact. For $\theta = \omega$ this is the classical compactness theorem of first order logic.
Chang [2] has given a natural generalisation of the notion of a Gödel-constructible set for any cardinal $\theta > \omega$. If $D^\theta(A)$ is the set of θ-definable (or θ-constructible) subsets of a set A, then for each ordinal α we define the term C^θ_α as follows:

$$C^\theta_0 = \emptyset, C^\theta_{\alpha+1} = D^\theta C^\theta_\alpha \text{ and } C^\theta_\eta = \bigcup_{\alpha<\eta} C^\theta_\alpha$$

if η is a limit ordinal. Let C^θ denote the class of θ-constructible sets or $C^\theta(x) \longleftrightarrow \exists\alpha(x \in C^\theta_\alpha)$. C^ω is generally denoted by L (the class of Gödel-constructive sets).
The axiom of constructibility is $V = L$ (where V is the class of all sets). The axiom of θ-constructibility is then the formula $\forall x C^\theta(x)$ or $V = C^\theta$. It has been proved by Scott that the existence of a measurable cardinal implies the negation of the axiom of constructibility [26].
If x is a set we denote the set of all sets which are Gödel-constructible starting from that set by $L(x)$ or $C^\omega(x)$. So $L = L(\emptyset)$. We denote the set of all sets which are θ-constructible starting from a set x by $C^\theta(x)$. So $C^\theta = C^\theta(\emptyset)$. Vopeňka and Hrabácek have proved that the existence of a strongly compact cardinal implies $\sim \exists x : V = L(x)$, or the universe of all sets is not ω-constructible starting from any set ([29]).
Kunen has generalized this result and has proven that if θ is a strongly compact cardinal then $\sim \exists x : V = C^{\theta^+}(x)$, where θ^+ denotes the successor cardinal of θ ([15]). According to Wang [30], Gödel has said once at the end of his life: "Generally I believe that in the last analysis, every axiom of infinity should be derived from the (extremely plausible) principle that V is

indefinable, when definability is to be taken in a more and more generalized and idealized sense".
The result of Kunen proofs that the existence of a strong compact cardinal implies that the universe of all sets V is already indefinable in a very strong sense.
The most important theorem about strongly compact cardinals which will need here is a very nice and deep theorem of Solovay [27]. It is in fact the only theorem about large cardinals which has implications on instances of GCH which we have at the moment.
Solovay's theorem says that if θ is a strongly compact cardinal and if λ is a singular strong limit cardinal greater than θ, then $2^\lambda = \lambda^+$. As a corollary we have that $\{\lambda \| 2^\lambda = \lambda^+\}$ is a proper class. So the existence of a strongly compact cardinal implies the existence of a proper class of instances of GCH and of successor cardinals λ^+ so that $\mu < \lambda^+ \Rightarrow 2^\mu \leq \lambda^+$. If for some $\alpha < \lambda^+$ we have $2^\alpha = \lambda^+$ then for each β with $\alpha \leq \beta < \lambda^+$ we have $2^\beta = \lambda^+$. This fact will be used later on.
If we denote by MC and SCC respectively the axioms that there exist a measurable cardinal or a strongly compact cardinal then Maddy [22] has given some arguments why ZFC+MC is better than ZFC+(V=L) from the point of view of settheoretic naturalism. The first theory maximizes the second one, it is richer and has some more benefits as the second theory without loosing the benefits of that second theory. The same can be done by comparing the theory $ZFC + SCC$ with $ZFC + (\exists x(V = L(x))$.
We can also consider the weaker theories: $ZFC +$ Consis$(ZFC + MC)$ and $ZFC +$ Consis$(ZFC + SCC)$ where the axioms of the consistency of measurable cardinals, respectively strongly cardinals in ZFC are added. This means in fact that measurable cardinals or that strongly compact cardinals are possible in ZFC-set theory. But these theories will no longer imply $\sim (V = L)$, respectively $\sim \exists x(V = L(x))$.

The main (philosophical) argument in favor of the existence of strongly compact cardinals is the principle of uniformity. This principle says that the universe of sets does not change in character substantially as one goes from smaller to larger cardinals, that some or an analogous state of affairs reappear again and again (perhaps in a more complicated version) in an eternal return of successive domains as envisioned already by Zermelo [31]. If this principle is true then G.C.H. (the Generalized Continuum Hypothesis) becomes very implausible since for the finite cardinals $2^n = n+1$ only for $n = 0$ and $n = 1$. So $\aleph_0$ would then be an accident as the only unique cardinal where GCH didn't hold below it but holds for all higher cardinals.
The continuum hypothesis (CH) is still possible since also in the finite case there are some exceptions (0 and 1). The existence of strongly compact cardinals becomes however very plausible by this principle of uniformity.

Indeed, together with the compactness theorem for first order predicate logic there are many other theorems which are equivalent with it and which can be stated so that they express a property of $\aleph_0$. Examples are the Boolean Prime Ideal Theorem, the Stone representation for Boolean algebra's and the Tychonov product theorem for compact Hausdorff spaces. If in a suitable formulation they are again true for some class of larger cardinals, then it is the same class, namely the strongly compact cardinals. So if strongly compact cardinals exist then $\aleph_0$ will no longer be an exceptional accident for them.

2. *A model theoretic interpretation of some maximization principles*

The model theoretic interpretations which we present here are inspired by Leibniz's philosophy. However it is certainly not our aim to give interpretations which are close to the original ideas of Leibniz. This is certainly not the case here. For instance we understand here "possible worlds" in the way it is employed by most logicians today as the models of a certain theory on a certain language. For Leibniz a possible world is a maximal class of concepts which can be realised together.
Our notion of a "possible metaworld" as the class of all models of a theory on a given language and with a given background set theory for the models is a little bit closer to it. We call this a metaworld because the background set theory for the models is described by the metalanguage which defines the whole semantics we use.
This background set theory for the models must have the property that all proper classes are equinumerous, so that we can speak of the "cardinal": ON of all proper classes. So for each background set theory we will take a Von Neumann-Bernays-Gödel set theory with the axiom of global choice (henceforth NBG-set theory). This is always a conservative extension of some ZFC-set theory.
Let T be a theory on some first order relational language L and let S be some NBG-set theory.
We denote the class of all models of T on L with S as background NBG-set theory by $\text{Mod}(L, T, S)$. If $M \in \text{Mod}(L, T, S)$, then we call the cardinality of the universe of that model M the power of M and denote it by $|M|$. If α is a cardinal of S with $\alpha \leq |M|$, then we denote by $\text{Mod}(\alpha, L, S)$,$\text{Mod}(< \alpha, L, TS)$ and $\text{Mod}(\leq \alpha, L, T, S)$ the classes of all models of $\text{Mod}(L, T, S)$ of power respectively $\alpha, < \alpha$ and $\leq \alpha$. We denote the class of all isomorphism classes of models of $\text{Mod}(L, T, S)$ which are substructures of M (or which can be embedded in M) by $\text{Age}M$ and we call this the age of M. If $\alpha \leq |M|$ then we denote by $\text{Age}(\alpha, M)$,$\text{Age}(< \alpha, M)$ and $\text{Age}(\leq \alpha, M)$

respectively the class of all isomorphism classes of models of Age$M$ whose members have power respectively $\alpha, < \alpha$ or α.
We call a model $M \in$Mod(L, T, S) with $|M| \geq \alpha$ α-*universal* if and only if each $M' \in$Mod $(\leq \alpha, L, T, S)$ can be embedded in M. So in this case Age$(\leq \alpha, M)$ is the class of all isomorphism classes of Mod$(\leq \alpha, L, T, S)$.
If $\alpha = |M|$ we call M a *universal model* of T.
If $\alpha = |M| = ON$ we call M an *absolutely universal model of* T (on L and S).
If M is an absolutely universal model of T on L and S then each model of T on L and S is a substructure of M. This is our model theoretic interpretation of the notion of a possible world which maximizes richesness in substructures.
Let M' and M'' be two members of a same element of Age $(< \alpha, M)$. Then there exist an isomorphism $f : M' \to M''$.
If $M \in$Mod(L, T, S) and $\alpha \leq |M| < ON$ then we denote the group of all automorphisms (or symmetries) of M by Aut$M$.
We call M α-*homogeneous* if in this case there exist always a $g \in$Aut$M$ so that $g_{|M'} \equiv f$. Of course M' and $g(M')$ are always isomorphic but here we ask that if M' and M'' are isomorphic that there exist always a symmetry g of the whole structure so that $g(M') = M''$.
So this can be seen as a maximization of symmetry. If $\alpha = |M| < ON$ we call M a *homogeneous model* of T on L and S.
If $|M| = ON$ or if the universe of the model M is a proper class we cannot speak of the group of all symmetries of M, but the notion of a symmetry of M makes sense. We call two submodels M' and M'' of M equivalent if and only if there exist a symmetry g of M so that $g(M') = M''$.
We call M an *absolutely homogeneous model* of T on L and S if any two isomorphic submodels of M (whose universes are sets) are equivalent.
Any two homogeneous universal models of T on L and S of the same power $\alpha \in S$ with $\aleph_0 \leq \alpha \leq ON$ are isomorphic. Indeed a homogeneous universal model M is characterized by the following property (A): If M' and M'' are submodels of M with M' a submodel of M'' then each embedding $f : M' \to M$ can be extended to an embedding $g : M'' \to M$ (see [1]). If property (A) holds then homogeneity of M can be proved by using a classical back-and-forth argument and for the universality of M we need only a forth argument.
If M_1 and M_2 are two homogeneous universal models of T on L and S of the same power, then again by considering a back-and-forth argument (and transfinite induction) we can prove that M_1 and M_2 are isomorphic (by universality they have the same age).
So if α is a cardinal of S there exist up to an isomorphism at most one homogeneous universal model of T on L and S of power α. If $\alpha = ON$ we call this the absolutely homogeneous universal model of T on L and S.

If we interpret Mod(L, T, S) as the class of possible worlds, then if it exists, we interpret the absolutely homogeneous universal model of T on L and S as the "best of all possible worlds" in the sense of Leibniz.
According to Leibniz it is that unique possible world which maximizes together harmony and richesness of substructures.
Again, probably Leibniz had something else in mind with the notion of harmony. We indeed identify here harmony with symmetry. In fact Leibniz had even doubts about the notion of an actual infinity and here we use proper classes. But Leibniz's philosophy is only a source of inspiration for the maximization properties we use here.

The problem what the necessary and sufficient conditions are for a first order theory T on a relational language L to have homogeneous universal models, and to determine those cardinals α for which there exist homogeneous universal models of power α of T was solved by Jónsson and Vaught [10],[28].

Let L be a relational first order language where all the relation symbols have finite rank and where $|L| < ON$. Let T be a $\forall\exists$-first order theory (or a Π_2-theory) on L having an infinite model. Let S be an NBG-background set theory for the models of T, and α a cardinal of S ($\alpha = ON$ is also admitted). Let D =Mod$(< \alpha, L, T, S)$ and let D^* be the class of all isomorphism classes of D.
Let T have the following 5 properties.

(1) The Joint Embedding property: If M_1 and $M_2 \in D$ then there exist an $M_3 \in D$ and embeddings $f_1 : M_1 \to M_3$ and $f_2 : M_2 \to M_3$.
(2) The weak amalgamation property: If $M_1, M_2, M_3 \in D$ and $f_1 : M_1 \to M_2$ and $f_2 : M_1 \to M_3$ are embeddings then there exist and $M_4 \in D$ and embeddings $g_1 : M_2 \to M_4$ and $g_2 : M_3 \to M_4$ so that $g_1 f_1 \equiv g_2 f_2$.
(3) Closeness with respect to directed subfamilies: If λ is an ordinal and $M_\beta \in D$ for every $\beta < \lambda$ and if M_β is a substructure of M_γ whenever $\beta < \gamma < \lambda$ then $\bigcup_{\beta<\lambda} M_\beta \in D$.
(4) $|D^*| < \alpha$.
(5) For any $M \in D$ and $X \subseteq M$, if β is a cardinal with $|X| \leq \beta \leq \alpha$ there exist a $M' \in D$, $|M'| = \beta$ such that $X \subseteq M' \subset M$.

A theory T having this 5 properties is called a Jónsson theory.
Then T has a (up to an isomorphism) a unique homogeneous universal model M of power α with Age $(< \alpha, M) = D^*$ if α satisfies the following 3 properties.

a. $|L \cup \omega| < \alpha$

b. α is regular (or cf $(\alpha) = \alpha$)

c. $\beta < \alpha \rightarrow 2^{\beta} \leq \alpha$.

This theorem has been proved by Jónsson, and Vaught [10], [28], for $\alpha < ON$ and by Ehrlich [4] for $\alpha = ON$.
Of course the conditions a, b, c are trivial for $\alpha = ON$ or α an arbitrary inaccessible cardinal, but there are other interesting classes of cardinals satisfying these 3 properties (see chapter 4).
For the countable case ($\alpha = \aleph_0$), which is excluded by property a, the necessary and sufficient conditions have been established by Fraïssé [6] (or see Cameron [1] p. 32). In this case the countable homogeneous universal model of T is called the Fraïssé limit of D. Here also the weak amalgamation property is the most important condition. So for instance the dense countable linear order without endpoints is the Fraïssé limit of the class of finite linear orders.
The Joint Embedding property and the weak amalgamation property are in fact maximization properties for composibility of substructures, an idea which we can find back in Leibniz's philosophy.
If M is a homogeneous universal model of power α there is a 1-1 correspondence with the elements of Age(β, M) with $\beta < \alpha$ and the equivalence class of submodels of power β of M, where two submodels are called equivalent if and only if there exist a symmetry of M transforming the one in the other. If $|M| < ON$ these equivalence classes are in fact the orbits of Aut$M$ in his natural action on the submodels of M of power β.
Now, since by Leibniz's principle of indescernibles two structures which cannot be distinguished from each other are considered as identical we shall call an equivalence class of models of power β a monad of power β.
If there exist a homogeneous universal submodel of power β of M then all homogeneous universal submodels of M of power β are equivalent. We call the corresponding monad the unique homogeneous universal monad of power β of M.
Of course, this is no longer true if $\beta = \alpha$. In general, there exist many of non-isomorphic homogeneous universal monads of power $\beta = |M|$. However as we shall see in the following section, if the Jónsson theory T satisfies one more condition (the strong amalgamation property) there exist a special unique homogeneous universal monad of power $|M|$ which we will interpret as the unique central monad of the "best of all possible worlds" M.
We denote the class of all is isomorphism classes of Mod(L, T, S) by Mod* (L, T, S) and we call it the metaworld of the theory T on the language L

which is associated to a specific NBG-background set theory S for the models. This set theory is described by the metalanguage which defines the semantics which we use in our model theory.
So if T is a Jónsson theory each metaworld of T contains a unique element which is the absolutely homogeneous universal model of T and which we interpret as the unique best of all possible worlds of that metaworld.

3. *The principle of maximal independence*

Let M be an absolutely homogeneous universal model of a Jónsson theory T on a language L with S as background set theory. Then we know that for each cardinal $\alpha < ON$ there exist at most one equivalence class of homogeneous universal submodels of power α. We consider as an example the absolutely homogeneous universal linear graph on an arbitrary NBG set theory S and we prove that there exist a proper class of equivalence classes of homogeneous universal submodels of power ON. So let T be the theory of linear graphs on the language L with a unique binary relation symbol E. The axioms are that E is irreflexive and symmetric. It is trivial to verify that T is a Jónsson theory. Since E is symmetric we will denote xEy also by $\{x, y\} \in E$. We call the elements vertices and if $\{x, y\} \in E$ we say that the vertices x and y are adjacent and that $\{x, y\}$ is an edge. The absolutely homogeneous universal graph with a given NBG-set theory S as background set theory is characterized by: for any two disjoint sets of vertices A and B there exist always a vertex z so that $\{z, a\} \in E\ \forall a \in A$ and $\{z, b\} \notin E$ $\forall b \in B$. This is in fact the property (A) for linear graphs.
Indeed, if this holds for a linear graph M whose universe is a proper class, then if (C, E') and (D, E'') are linear graphs where C and D are sets and where (C, E') is a subgraph of (D, E''), we have that each embedding $f : (C, E') \to M$ can be extended to an embedding $g : (D, E'') \to M$. This is necessary and sufficient for M to be the absolutely homogeneous universal graph. Let S be the universe of all sets and define E as the class of all sets $\{x, y\}$ with $x \in S, y \in S$ and $x \in y$ or $y \in x$. We prove now that (S, E) is the absolutely homogeneous universal graph (the a.h.u.-graph). Let A and B be two arbitrary disjoint subsets of the proper class S. We prove that there exist always a $z \in S$ so that $\{z, a\} \in E\ \forall a \in A$ and $\{z, b\} \notin E\ \forall b \in B$. The vertices of the graph are all the sets and two vertices are adjacent if one of the two vertices is an element of the other vertex. If max(rank$A$, rank$B$)=μ (some ordinal $< ON$), let u then be an arbitrary set of rank $\mu + 1$ disjoint from B, and consider the set $z = A \cup \{u\}$. If $a \in A$ then $a \in z$ and so $\{a, z\} \in E\ \forall a \in A$. If $b \in B$ we have $b \notin z$ since $A \cap B = \phi$ and $u \cap B = \phi$. But also $z \notin b$ since rank $z >$rank b. So $\{z, b\} \notin E\ \forall b \in B$. Hence (S, E) is the a.h.u-linear graph with S as background set theory.

Let (X_0, E_0) be an arbitrary linear graph. X_0 can be a proper class or a set, also $X_0 = \phi$ is good. Let $X_1 = P_s(X_0)$ be the set of all subsets of X_0. Also if X_0 is a proper class this makes sense. Let $V_0 = X_0, V_1 = X_0 \cup X_1, X_2 = P_s(V_1), V_2 = V_1 \cup X_1$. By induction $X_{\alpha+1} = P_s(V_\alpha), V_{\alpha+1} = V_\alpha \cup X_{\alpha+1}$ and if μ is a limit ordinal $X_\mu = \bigcup_{\alpha<\mu} X_\alpha, V_\mu = X_\mu$.

Let $X = \bigcup_{\alpha<ON} X_\alpha$. Define E' as the class of all sets $\{x, y\}$ with $x \in X, y \in X$, and $x \in y$ or $y \in x$. Then by the same reasoning as before (X, E') is the a.h.u.-graph. But also $(X - X_0, E'')$ where E'' is the restriction of E' on $\begin{pmatrix} X - X_0 \\ 2 \end{pmatrix}$ is the a.h.u.-graph, again by the same reasoning. So $(S, E) \cong (X, E') \cong (X - X_0, E'')$.

Since (X, E') is universal we know that each linear graph (X_0, E_0) can be embedded in (X, E'), but by this construction we have proved that also each linear graph $(X_0 E_0)$ can be embedded in (X, E') in such a way that its complement $(X - X_0, E'')$ is isomorphic with the whole linear graph (X, E'). Moreover any symmetry g of (X_0, E_0) has a natural action on $X_1 = P_s(X_0)$ and on X_α and V_α $(\forall\alpha < ON)$, and so g can be extended to a symmetry g^* of (X, E'). So each linear graph (X_0, E_0) can be embedded in (X, E') in such a way that each symmetry of (X, E_0) extends to a symmetry (X, E'). There exist always an embedding so that no symmetries are lost. There is no reason that this property should be true for other a.h.u.-models. It would be very interesting to find the necessary and sufficient conditions for a Jónsson theory T so that the a.h.u-model of T has the property that each model of T can be embedded in it on such a way that no symmetries are lost.

Of course two subgraphs of (X, E') which are isomorphic with (X, E') can only be transformed in each other by a symmetry of (X, E') if their complements are isomorphic graphs. This condition is not sufficient but it is necessary. Hence, there exist a proper class of equivalence classes of subgraphs of (X, E') which are isomorphic with (X, E'), or there exist a proper class of non equivalent homogeneous universal monads of "cardinality" ON in the a.h.u-graph. Now we prove that in this proper class there exists a unique one which has all the properties which we require for a central monad. By a central monad we understand an equivalence class of subgraphs of the a.h.u-graph (X, E') which are all isomorphic with the whole graph (X, E') and which is in "maximal harmony" with all other monads. This is at least an aspect of the notion of the "Ens Perfectissimus" of Leibniz as a monad which reflects perfectly the whole universe and which is in maximal harmony with the rest of the universe. Of course the "Ens Necessarium" aspect, which is very important in Leibniz's metaphysics is deleted here. This aspect has been studied by Gödel in his ontological proof [12]. For more information about the notions of Ens Perfectissimus and Ens Necessarium of Leibniz see [16] and [17] (37–60). There is a natural way to express this maximal harmony.

Consider the language L' obtained by adding to the binary relation symbol E of L one new unary relation symbol U. Then we obtain from the Jónsson theory T a new theory T' whose models are the linear graphs together with a certain subclass of vertices. On a trivial way T' will be again a Jónsson theory. This is not the case for any Jónsson theory but is is certainly the case for the theory of linear graphs. We will give later on the necessary and sufficient conditions for T so that T' will be again a Jónsson theory. We denote the a.h.u.-model of T' by (X, E', U'). Here (X, E') is the a.h.u. graph, and U' is a subclass of X with the property that if A and B are disjoint subsets of X, there exist always a vertex $z \in U'$ and a vertex $v \in coU'$ (the complement of $\mathcal{U}'$ in X) so that $\{b, z\} \in E$, $\{a, v\} \in E\ \forall a \in A$ and $\{b, z\} \notin E$, $\{b, v\} \notin E\ \forall b \in B$.
This is in fact property (A) for linear graphs with a subclass of vertices. Hence if $(X, E, \mathcal{U}')$ and $(X, E, \mathcal{U}'')$ are models of T', then by this property and using a back and forth argument there exist always a symmetry g of (X, E) so that $g(\mathcal{U})' = \mathcal{U}''$.
So any two subclasses of X which define a model of T' on (X, E) are equivalent. We call the corresponding equivalence class of subclasses of vertices of (X, E), the unique central monad of (X, E).

This example of linear graphs can however not be generalized for an arbitrary Jónsson theory T since in general the theory T' obtained by adding a new unary relation symbol to the language will no longer satisfy the weak amalgamation property in which case T' is not a Jónsson theory.
The property which is necessary for T so that T' will be again a Jónsson theory is called the *strong* amalgamation property [1]. We say that a Jónsson theory T has the strong amalgamation property if and only if $D =$Mod$(< On, L, TS)$ and if $M_1, M_2, M_3 \in D$ and $f_1 : M_1 \to M_2$ and $f_2 : M_1 \to M_3$ are embeddings then there exist an $M_4 \in D$ and embeddings $g_1 : M_2 \to M_4$ and $g_2 : M_3 \to M_4$ so that $g_1 f_1 \equiv g_2 f_2$ and so that no further identifications are necessary, or that $g_1(M_2) \cap g_2(M_3) = g_1 f_1(M_1) = g_2 f_2(M_2)$.
This property holds for linear graphs, linear orders, partial orders, coloured graphs, hypergraphs but not for groups or linear ordered fields.
The fact that no further identifications are necessary is a property of independence. Let M be a a.h.u. model of T on L and S, and let A be a subset of M and $a \in M$, $a \notin A$, then we call a dependent of A if the class of all elements $b \in M, b \notin A$ and $A \cup \{b\}$ equivalent with $A \cup \{a\}$ is a set. If this is a proper class we call a independent of A. The fact that an a.h.u. model has the strong amalgamation property if and only if for each subset A each $a \notin A$ is independent of A has been proved in [1] (p. 37).
If M is the a.h.u. graph then for each subset A of M each $a \notin A$ is independent of A. If M is for instance the a.h.u. linear ordered field, or the Conway field of surreal numbers, then this is in a trivial way not the case,

since $2a = a + a$ is certainly not independent of $A = \{a\}$. The main property of Jónsson theories with the strong amalgamation property which we need here is that if T_1 and T_2 are such theories on languages L_1 and L_2 that $T_1 \cup T_2$ is also such a theory on the language $L_1 \cup L_2$. The theory $T_{\mathcal{U}}$ of unary relations is on a trivial way such a theory, whose unique a.h.u.-model is a proper class with a proper subclass whose complement is also a proper subclass. If T is an arbitrary Jónsson theory with the strong amalgamation property then $T \cup T_U$ is also such a theory on the language $L' = L \cup \{\mathcal{U}\}$. The unique a.h.u. model of $T \cup T_{\mathcal{U}}$ is then the a.h.u. model of T with a central monad. So the existence of a central monad is independent of the choice of NBG-background set theory and depends only on the fact that the theory T satisfies the strong amalgamation property.

In the same way the theory $T_\sim$ of equivalence relations is a Jónsson theory satisfying the strong amalgamation property whose unique a.h.u.-model is a partitioning of a proper class in a proper class of equivalence classes, each of which is a proper class. We call a Jónsson theory with the strong amalgamation property a strong Jónsson theory. If T is a strong Jónsson theory on a language L then $T \cup T_\sim$ is also a strong Jónsson theory on the language $L' = L \cup \{\sim\}$ whose unique a.h.u.-model is the a.h.u.-model of T together with a very specific partitioning which is unique up to a symmetry of the a.h.u. model of T.

We call a partitioning of a relational structure of power α a Spinozistic Partitioning if it contains α equivalence classes and if each equivalence class is isomorphic with the whole relational structure.

A homogeneous universal model of a strong Jónsson theory T has in general many non-isomorphic Spinozistic Partitionings but only a unique one of them is isomorphic with the unique a.h.u.-model of $T \cup T_\sim$. We call this partitioning the central Spinozistic Partitioning since in Spinoza's philosophy the world is partitioned in an infinite number of equivalence classes (attributes) where each equivalence class is isomorphic with the whole universe.

In contrast with the Joint Embedding axiom and the weak amalgamation axiom which are in fact principles of maximal composibility of substructures, the strong amalgamation axiom is a principle of maximal independence. For each subset A and each $x \notin A$ we ask that x is independent of A.

Principles about maximal composibility of substructure are well known in Leibniz's metaphysics, but I have never found a principle about maximal independence. In fact such a principle, which is a metaphysical equivalent of the strong amalgamation property should be added to the other principles of Leibniz, it has the same spirit and it decides a very important hypothesis of the monadology, namely the existence of a unique central monad.

So if T is a strong Jónsson theory each metaworld of T contains a unique "best of all possible worlds" who has a unique central monad and a unique central Spinozistic Partitioning.

4. *Maximal diversity of homogeneous universal reflections*

The existence of a central monad or a central Spinozistic Partitioning in the a.h.u.-model of a Jónsson theory is independent of the choice of the N.B.G.-background set theory and depends only of properties of T. This will no longer be the case if we consider maximal diversity of homogeneous universal substructures of the a.h.u.-model of T.

By the theorems of Fraïssé, Jónsson and Ehrlich [6],[10],[4] there exist homogeneous universal relational structures of power m if and only if $m = \aleph_0$, $m = ON$, m is regular and if $n < m$ then $2^n \leq m$.

If m is a successor cardinal or if $m = n^+$ it is necessary and sufficient that $2^n = n^+ = m$ or that we have an instance of G.C.H. for n.

If m is a strong limit cardinal, then since it is regular, it is a strongly inaccessible cardinal.

If m is a weak limit cardinal, then since it is regular it is a weakly inaccessible cardinal which has the supplementary condition that if $n < m$ we have that $2^n \leq m$.

We call homogeneous universal relational structures whose power is respectively countable, On, a successor cardinal, a strongly inaccessible cardinal an a weakly inaccessible cardinal: countable, absolute, successor, strongly inaccessible and weakly inaccessible homogeneous universal structures.

What are the possibilities for an homogeneous universal model whose power is the continuum? In general $2^{\aleph_0}$ will not satisfy the conditions and so in general there exist no homogeneous universal model whose power is the continuum.

Of course $2^{\aleph_0}$ cannot be a strongly inaccessible cardinal, so if it is a limit cardinal it has to be a weakly inaccessible cardinal m with the property that if $n < m$ we have $2^n \leq m = 2^{\aleph_0}$. Hence for each cardinal α with $\aleph_0 \leq \alpha < 2^{\aleph_0}$ we have that $2^\alpha = 2^{\aleph_0}$. It has been proved that if $2^{\aleph_0}$ is a real valued measurable cardinal we have always the property that $\aleph_0 \leq \alpha < 2^{\aleph_0}$ implies $2^\alpha = 2^{\aleph_0}$. [3]

If $2^{\aleph_0}$ is a successor cardinal or if $2^{\aleph_0} = n^+$ then since $r < n^+ \rightarrow 2^r \leq n^+$ and since $2^n > n$ we have $2^n = n^+$. Hence for each cardinal α with $\aleph_0 \leq \alpha < n^+$ we have $2^\alpha = 2^{\aleph_0}$.

If $n = \aleph_0$ we have the continuum hypothesis. If for instance $n = \aleph_m$ then $2^{\aleph_0} = 2^{\aleph_1} = \cdots = 2^{\aleph_m} = n^+$.

If we have for instance a set theory where $\aleph_0 < \aleph_1 < 2^{\aleph_0} = \aleph_2 < 2^{\aleph_1} = \aleph_3$ then there exist no homogeneous universal models on the continuum.
In a set theory where $\aleph_0 < \aleph_1 < 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 < 2^{\aleph_2} = \aleph_3$ each Jónsson theory has a model on the continuum.
Of course a homogeneous universal model on the continuum in a set theory where $2^{\aleph_0}$ is a real valued measurable cardinal is incredibly much richer as in a set theory where the continuum hypothesis holds.

The existence of strongly and weakly inaccessible cardinals is independent of the axioms of ZFC (or N.B.G.). So the choice of the background set theory is very important here.

Foreman and Woodin [5] have proved that if there exist a supercompact cardinal (this is a special strongly compact cardinal, for a definition see [13]) then one can construct a model of ZFC (or N.B.G.) in which $2^\alpha = \alpha^{++}$ for each infinite cardinal α (singular or regular). In such a set theory there are no instances of G.C.H. So in this set theory of Foreman and Woodin there exist no successor homogeneous universal models. But there exist also no weakly inaccessible cardinals since if m is a limit cardinal and $n < m$ then $2^n = n^{++} < m$ and since m is regular it is a strongly inaccessible cardinal. Hence in such a set theory there exist only countable, strongly inaccessible and absolute homogeneous universal models.

In his paper: "Strongly compact cardinals and the GCH [27], Robert Solovay has proved that the existence of a strongly compact cardinal θ implies that for each singular strong limit cardinal λ with $\lambda > \theta$ we have that $2^\lambda = \lambda^+$. As a corollary we have then that $\{\lambda \| 2^\lambda = \lambda^+\}$ is a proper class.
Hence with a set theory S which contains a strongly compact cardinal θ as background set theory we have that each Jónsson theory T has a proper class of successor homogeneous universal models of power λ^+. So in this case the a.h.u.-model of T has a proper class of non-isomorphic homogeneous universal monads.
So a strongly compact cardinal has strong maximization properties and by Maddy's MAXIMIZE (or DIVERSIFY) principle the axiom about the existence of strongly compact cardinals should be taken as an axiom of set theory. From Leibniz's point of view: if the world contains strongly compact cardinals there exist a maximal number of distinct monads reflecting this world.

Of course, all this could also be seen as an argument in favor of the thesis that the Generalized Continuum Hypothesis (G.C.H.) is a maximizing principle. Indeed if G.C.H. holds then there exist homogeneous universal structures of each infinite cardinality. The problem is that we have no formal definition of the notions of a "maximizing principle" and a "restrictive principle". We can indeed also say that G.C.H. is a restrictive principle, since if

G.C.H. holds then the powers of homogeneous universal structures are the only infinite cardinals.

Joel Friedman proved that G.C.H. is equivalent to the assumption that every "local universe" contains all it smaller-cardinality subsets [9]. A "local universe" is defined as a collection closed under Pairing, Union and Replacement. "Local universes" are Joel Friedman's interpretation of Leibniz's monads. So in his view G.C.H. is analogous to Leibniz's maximization principle that every monad mirrors the universe by perceiving the maximum possible from its point of view. The problem however is that G.C.H. indeed maximizes Replacement, but that this happens at the expense of the Power Set operation.

For an overview of arguments pro or contra the fact that G.C.H. should be a maximizing principle we refer to Maddy [19].

If G.C.H. is a maximizing principle then we have given in this paper no new arguments in favor of the existence of strongly compact cardinals. Of course by the MAXIMIZE-principle the existence of any large cardinal is positive (the more existence the better).

But if G.C.H. is a restrictive principle (as Gödel and Cohen believed) the existence of strongly compact cardinals becomes much more important. Indeed, by the MAXIMIZE-principle we shall then not accept G.C.H., but as a consequence of the axiom about the existence of strongly compact cardinals there will still exist a proper class of non-isomorphic homogeneous universal structures.

So if you believe that the Generalized Continuum hypothesis is restrictive, then there are two distinct new ways to motivate the set-theoretic axiom which posits the existence of strongly compact cardinals.

The first one is to believe in the analogy with general properties of Leibniz's monadology.

The second one is to agree with Maddy's justification of the maximizing maxim in set theory.

Department of Pure Mathematics and Computeralgebra
University of Ghent
Krijgslaan 281 - S22
B-9000 Gent (Belgium)
E-mail: wm@cage.rug.ac.be

REFERENCES

1. Cameron, P., "Oligomorphic Permutation groups", *Lecture Notes in Math.* Vol. 152, Cambridge University Press 1991.

2. Chang, C.C., "Sets constructible using $L_{x,x}$", *Proceed. Symposia. Pure. Math. Vol. XIII. Part I. Axiomatic Set Theory*, Amer. Math. Soc. 1971 (1–8).
3. Drake, F., "Set Theory. An introduction to large cardinals", *Studies in Logic and the foundations of math.* Vol. 76 1974, North-Holland Publ. Comp.
4. Ehrlich, P., "Absolutely saturated models", *Fund. Math.* 133 (1989) (39–46).
5. Foreman, M. & Woodin, W.H., "The generalized continuum hypothesis can fail everywhere", *Annals of Math.* 133 (1991) (1–35).
6. Fraïssé, R., "Sur certaines relations qui généralisent l'ordre des nombres rationnels", *C.R. Acad. Sci. Paris* Vol. 237 (1953) (540–542).
7. Friedman, H., "Finite functions and the necessary use of large cardinals", *Annals of Math.* (2) 148 (1998) 3 (803–893).
8. Friedman, H., F.O.M. e-mail list. Questions grown from Friedman talks (http://www.math.psu.edu/simpson/fom/postings/01200/mgs00006.html) February 3, 2001.
9. Friedman, J., "The Generalized Continuum Hypothesis is equivalent to the Generalized Maximization principle", *The Journal of Symbolic Logic* Vol. 36, Number 1 (1971) (39–54).
10. Jónsson, B., "Homogeneous universal relational systems", *Math. Scand.* 8 (1960) (137–142).
11. Gödel, K., "Über die Länge vom Beweise", *Ergebnisse eines mathematischen Kolloquium* 7 (23–24) (1936) (Collected Works, Vol. I).
12. Gödel, K., *Collected Works, Vol. III, Unpublished Essays and Lectures.* Ed. S. Fefferman, Oxford University Press (1995).
13. Kanamori, A., *The higher infinite, Large Cardinals in Set Theory from Their Beginnings*, Springer Verlag, Berlin 1994.
14. Kreisel, H.J. & Tarski, A., "From accessible to inaccessible cardinals", *Fund. Math.* 53 (1964) (225–308).
15. Kunen, K., Inaccessible properties of cardinals. Ph.D. Thesis. Stanford Univ. 1968.
16. Leibniz, G.W., *Sämtliche Schriften und Briefe*, Prussian Academy of Sciences 1923. VI iii 395–396, 571–579 (Text of 1676).
17. Leibniz, G.W., *Monadology and other Philosophical Essays*, A.M. Schrecker, Indiapolis. Bobbs-Merill 1965.
18. Leibniz, G.W., *Philosophical Papers and Letters.* Ed. Leroy E. Loemker, D. Reidel 1969.
19. Maddy, P., "Believing the Axioms I", *Journal of Symbolic Logic* Vol. 53, Number 2, (1988) (481–511).
20. Maddy, P., "Naturalism and Ontology", *Philosophia Mathematica* 3;3, (48–270) (1995).

21. Maddy, P., "Set theoretic Naturalism", *Journal of Symbolic Logic* 61, (490–514) (1996).
22. Maddy, P., *Naturalism in Mathematics*, Oxford University Press (1997).
23. Mendelson, E., "Math. Sci. Net review 2000 f:03146 of Maddy, P: $V = L$ and MAXIMIZE", *Lecture Notes Logic* 11 (1998).
24. Mielants, W., "Can the best of all possible worlds be a random structure?", *Logique et Analyse* 141–142 (1993) (61–73).
25. Rutherford, D., *Leibniz and the relational order of Nature*, Cambridge University Press 1995.
26. Scott, D.S., "Measurable cardinals and constructible sets", *Polish Acad. Sci. Math.* 9 (1961) (521–524).
27. Solovay, R.M., "Strongly compact cardinals and the G.C.H.", *Proceedings of the Tarski Symposium, Proc. Symposia in Pure Math.* Vol. XXV. Ed. L. Henkin 1974 (365–372).
28. Vaught, R., "Homogeneous universal models of complete theories", *Notices A.M.S.* 5 (1958) 775.
29. Vopenka, P. & Hrbácek, K., "On strongly measurable cardinals", *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astrom. Phys.* 14 (587–591) (1966).
30. Wang, Hao, *A Logical Journey. From Gödel to Philosophy* (1995), The MIT Press, Cambridge, Massachusetts.
31. Zermelo, E., "Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre", *Fund. Math.* 16 (1930) (29–47).