Logique & Analyse **167–168** (1999), 221–241

INCONSISTENCY-ADAPTIVE ARITHMETIC

TIMOTHY VERMEIR*

Abstract
This paper concerns a specific inconsistent arithmetic, formulated by Jean Paul Van Bendegem and praised for its so-called outstanding properties by Graham Priest. I shall show that this inconsistent arithmetic, in the form presented by Van Bendegem and Priest, is subject to three major problems. Next, I shall show that these problems may be removed by replacing the underlying logic.

1. *Introduction*

Two articles on a specific system for inconsistent arithmetic are of special interest in this paper: *Strict Finitism as a Viable Alternative in the Foundations of Mathematics* by Jean Paul Van Bendegem (appeared in *Logique et Analyse* [11], see also [10] for an earlier version), and Graham Priest's article in *Mind* entitled *Is Arithmetic Consistent?* [8].

The system for inconsistent arithmetic presented in these papers is obtained by collapsing the standard models of arithmetic. The idea is to define equivalence classes on the domain $\mathcal{D}$, the set of natural numbers, as follows

$$\varepsilon(\mathcal{D}) \;=\; \{[0],[1],[2],\ldots[K],[L,L',L'',\ldots]\} \tag{1}$$

("K" denotes the number that is *classically* the predecessor of L. The number K shall be formally defined in subsection 4.2.) The domain of the collapsed model is the finite $\varepsilon(\mathcal{D})$.

*I am greatly indebted to Diderik Batens and Kristof De Clercq for their helpful comments on earlier versions of this paper. Research for this paper was supported by the Fund for Scientific Research – Flanders.

The resulting model is given the Logic of Paradox (LP)[1] as underlying logic—see [6] and [7]. According to this logic, formulas receive non-empty subsets of $\{0, 1\}$ as truth values. The value $\{1\}$ indicates that a formula is true (only), the value $\{0\}$ that it is false (only), and the value $\{0, 1\}$ that it is both true and false.

In a collapsed model, an equality $t_1 = t_2$ is true if there is a number in the equivalence class $\varepsilon(t_1)$ that is equal to a number in the equivalence class $\varepsilon(t_2)$. The equality is false whenever there are two numbers, one belonging to the first equivalence class, and one to the second, which are not equal in classical arithmetic. It follows that $L = L$ is both true and false, as is the case for $\sim L = L$. In the sequel, I shall employ $\mathrm{IA_C}$ to refer to this particular system for inconsistent arithmetic.

In [8], Graham Priest proves the following lemma (CA abbreviates Classical Arithmetic)

Lemma 1: (The Collapsing Lemma [8]) *For any wff φ, if φ is true/false in* CA*, then φ is true/false in* $\mathrm{IA_C}$.

The intuitive interpretation of $\mathrm{IA_C}$ is that all numbers classically smaller than L behave consistently, that is classically. L and all of its successors on the other hand do not: every property P that holds for one of them, inevitably holds for them all.[2] Thus, intuitively, $0 = 1$ is false, and $1 = 1$ is true according to $\mathrm{IA_C}$, just as the Collapsing Lemma states.

However, lemma 1 should be understood in the light of the logic LP. A correct understanding of this lemma requires that one realizes that, where truth and falsity are taken to exclude each other in the classical case, they definitely do not exclude each other in the context of LP. Hence, the lemma does not prohibit that a formula is true *only* in CA, while both true and false in $\mathrm{IA_C}$.

Both authors have claimed a number of things about $\mathrm{IA_C}$. Among these are that the system is finitely axiomatizable and that it comprises all of classical arithmetic. The central point I want to make in this paper, is that many of these claims are indeed correct, but *only* if one replaces LP by a (very) different logic.

[1] Characteristic for this logic is that there is no *real* implication: the implication is defined by the classical $A \to B =_{\mathrm{def}} \sim A \vee B$, which, in a paraconsistent environment is not detachable.

[2] As such, these numbers are indistinguishable, and we can thus speak of a *finite domain*. The number L is taken to be (by Graham Priest, see [8] and [9]) so large that it does not have any physical meaning, a number "larger than the number of combinations of fundamental particles in the cosmos" ([8], page 338).

In the next section, I shall raise three major objections to the system $\mathrm{IA_C}$. These objections typically derive from properties of the underlying logic LP. In section 3, two logics are briefly described, namely the logic CLuNs (3.1), and ACLuNs2 (3.2). These two logics will form the basis of the different systems that will be constructed in section 4. The following section then describes a final attempt, and the solution to the problems at hand.

2. *Some problems with* $\mathrm{IA_C}$

Gödel's theorem and maximal non-triviality

Both Jean Paul Van Bendegem and Graham Priest have emphasized in their articles on inconsistent arithmetic, that $\mathrm{IA_C}$ is finite, and is, as a result, not affected by Gödel's theorem: both the Gödel sentence and its negation are true in $\mathrm{IA_C}$.

But, there is more to the story of the Gödel-theorem, as is pointed out by Diderik Batens in his [1].[3] The idea behind Gödel's First Theorem is, in Batens' words, that "a domain is only completely described by some theory if the latter is maximally non-trivial (...) Any axiomatization of CA is incomplete because one may add an axiom to it, and still get a non-trivial theory." ([1], section 10).

It can easily be seen that this aspect of the theorem is not avoided by $\mathrm{IA_C}$. If we would add the axiom $t = t'$, for any number t classically between 0 and L, we obtain a number theory that is non-trivial. Hence, $\mathrm{IA_C}$ is not maximally non-trivial, and not complete in the sense described above.

Elaborate axiomatization

As we have just seen, the logic LP, underlying $\mathrm{IA_C}$ as defined by Van Bendegem and Priest, does not include a detachable implication. This is the cause of a next, quite severe problem from which $\mathrm{IA_C}$ suffers.

Since there is no detachable implication, there is no means of expressing rules like "if n and m are not equal, then $f(n)$ and $f(m)$ are not equal", where f is one or other function on the natural numbers.

For instance, we want to assure that all classical inequalities (up to L) remain true in inconsistent arithmetic. All these are captured in the following

[3] In this article, it is also shown that there are problems with this particular interpretation of Gödel's theorem, and the proof Priest hints towards. What to think about the Gödelnumbering for instance? Surely there are wffs with a Gödelnumber larger than L. (All unpublished papers cited here, and many more, are available from the URL `http://logica.rug.ac.be/centrum/writings/`.)

triangle, with in total $(L \cdot (L+1))/2$ entries:

$$
\begin{array}{ccccc}
\sim 0 = 1 & & & & \\
\sim 1 = 2 & \sim 0 = 2 & & & \\
\sim 2 = 3 & \sim 1 = 3 & \sim 0 = 3 & & \\
\vdots & \vdots & \vdots & \ddots & \\
\sim 0^{(L-1)} = L & \sim 0^{(L-2)} = L & \ldots & \sim 1 = L & \sim 0 = L
\end{array}
$$

The first column can be captured by the axiom $(\forall x) \sim x = x'$, the second by $(\forall x) \sim x = x''$, and so on. This series continues up to the statement that every number is not equal to its L^{th} successor: $(\forall x) \sim x = x^{(L)}$. In the absence of a detachable implication, each of these L formulas needs to be an axiom in order to capture these classical theorems.

An interesting consequence of this problem is that, although IA_{C}, which is a finite theory, is finitely axiomatizable, it is not *sensibly* finitely axiomatizable. If, as Van Bendegem would like it, L just is the largest number, then there are not enough numbers to enumerate the axioms needed to formulate the axiomatization of IA_{C}.[4] Enlarging L clearly worsens the problem instead of solving it.

The abnormality of models

In LP, a wff A is regarded as true in a model M, if $1 \in \mathsf{v}_{\mathsf{M}}(A)$. As such, when Priest states that, for instance, $2 + 3 = 5$ is true in all IA_{C}-models, what he actually says is that for all models[5] $\mathsf{M} \in \mathcal{M}_{\mathrm{IA}_{\mathrm{C}}}$: $1 \in \mathsf{v}_{\mathsf{M}}(2+3 = 5)$. This does not exclude there being models in which this equation is also false, i.e. there are IA_{C}-models in which $\mathsf{v}_{\mathsf{M}}(2 + 3 = 5) = \{0, 1\}$. $2 + 3 = 5$ is a *paradoxical* sentence in these models.

This constitutes the third objection against IA_{C}: although the classical truths hold in all models, there are models in which there are more true contradictions then necessary, models which are too abnormal.

[4] A possible reply is to state that we could combine these L axioms into a very long conjunction. But needles to say this is not a real way out of the problem, for this particular axiom would consist of more than L tokens.

[5] Sets of models will be denoted by $\mathcal{M}$, with a subscript specifying the formal system defining the models. This notation will be used throughout the article.

3. *Two paraconsistent logics*

3.1. *The logic* CLuNs

CLuNs is a logic derived from the much poorer paraconsistent logic CLuN, first described in [2].[6] The added 's' denotes the addition of the Schütte-properties, which relate to negation. For a detailed account on CLuNs, I refer to [5]. For our purpose, a brief introduction will do.

The basic idea behind CLuN and CLuNs, is that we drop from CL the consistency-presupposition, leaving only the demand for completeness: of two formulas A and $\sim A$, at least one must be true, but possibly both are. Unlike in Priests LP, a formula can not be true and false at the same time, but may be true together with its negation: $\mathsf{v}_{\mathsf{M}}(A) = \mathsf{v}_{\mathsf{M}}(\sim A) = 1$.

In CLuN, all properties of negation are lost, except for those derivable by positive logic from $\models A \vee \sim A$. The Schütte-properties are a way to reintroduce some central properties of negation.

The language of CLuNs is the language $\mathcal{L}$ of CL, and can be defined by means of the tuple $\langle \mathcal{S}, \mathcal{C}, \mathcal{V}, \mathcal{P}^1, \mathcal{P}^2, \ldots \rangle$ where $\mathcal{S}$ is the set of sentential letters, $\mathcal{C}$ is the set of (letters for) individual constants, $\mathcal{V}$ the set of variables, and $\mathcal{P}^r$ the set of predicates of arity r. $\mathcal{L}$ also contains *bottom*, $\perp$, implicitly defined by $\perp \supset A$.

The axioms and rules of CLuN (see also [2]) are:

MP	From A and $A \supset B$ to derive B
R$\forall$	To derive $\vdash A \supset (\forall\alpha)B(\alpha)$ from $\vdash A \supset B(\beta)$, provided β does not occur in either A or $B(\alpha)$.
R$\exists$	To derive $\vdash (\exists\alpha)A(\alpha) \supset B$ from $\vdash A(\beta) \supset B$, provided β does not occur in either $A(\alpha)$ or B.
A$\supset$1	$A \supset (B \supset A)$
A$\supset$2	$((A \supset B) \supset A) \supset A$
A$\supset$3	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A$\perp$	$\perp \supset A$
A&1	$(A \& B) \supset A$
A&2	$(A \& B) \supset B$
A&3	$A \supset (B \supset (A \& B))$
A$\vee$1	$A \supset (A \vee B)$
A$\vee$2	$B \supset (A \vee B)$
A$\vee$3	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A$\equiv$1	$(A \equiv B) \supset (A \supset B)$

[6] The name "CLuN" denotes the main idea behind it: it is a weakening of Classical Logic (CL), allowing for gl$u$ts with respect to $n$egation.

A≡2 $(A \equiv B) \supset (B \supset A)$
A≡3 $(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$
A∼1 $(A \supset {\sim}A) \supset {\sim}A$
A∀ $(\forall\alpha)A(\alpha) \supset A(\beta)$
A∃ $A(\beta) \supset (\exists\alpha)A(\alpha)$
A=1 $\alpha = \alpha$
A=2 $\alpha = \beta \supset (A \supset B)$ where B is obtained by replacing in A an occurrence of α that occurs outside the scope of a negation by β

An axiomatization of CLuNs is obtained by replacing A=2 by

A=2^s $\alpha = \beta \supset (A \supset B)$ where B is obtained by replacing in A an occurrence of α by β

and adding

A∼∼ ${\sim}{\sim}A \equiv A$
A∼⊃ ${\sim}(A \supset B) \equiv (A\&{\sim}B)$
A∼& ${\sim}(A\&B) \equiv ({\sim}A \vee {\sim}B)$
A∼∨ ${\sim}(A \vee B) \equiv ({\sim}A\&{\sim}B)$
A∼≡ ${\sim}(A \equiv B) \equiv ((A \vee B)\&({\sim}A \vee {\sim}B))$
A∼∀ ${\sim}(\forall\alpha)A \equiv (\exists\alpha){\sim}A$
A∼∃ ${\sim}(\exists\alpha)A \equiv (\forall\alpha){\sim}A$

For the semantics of CLuNs, it is handy to extend the language $\mathcal{L}$ with $\mathcal{O}$, the set of *pseudo-constants*, with the same cardinality as the domain of the largest model one wants to consider. The language $\mathcal{L}+$ is then defined as the tuple $\langle \mathcal{S}, \mathcal{C} \cup \mathcal{O}, \mathcal{V}, \mathcal{P}^1, \mathcal{P}^2, \ldots\rangle$. Define the set $\mathcal{W}+$ as the set of wffs over $\mathcal{L}+$. Furthermore, let ${}^{\sim}\mathcal{S}$ be the set $\{{\sim}A | A \in \mathcal{S}\}$, and ${}^{\sim}\mathcal{P}^r = \{{\sim}\pi^r | \pi^r \in \mathcal{P}^r\}$; these are the sets of negated sentential letters and negated predicates respectively. Note that where in CLuN all ∼-negations "drop from the sky" (i.e. are not truth-functional), in CLuNs this is only the case for negations of *atomic* formulas.

A CLuNs-model M is a couple $\langle \mathcal{D}, \mathsf{v}\rangle$, where $\mathcal{D}$ is the domain, and v is the interpretation function defined by:

S1.1 $\mathsf{v} : \mathcal{S} \rightarrow \{0, 1\}$
S1.2 $\mathsf{v} : \mathcal{C} \cup \mathcal{O} \rightarrow \mathcal{D}$
such that $\mathcal{D} = \{\mathsf{v}(\alpha) | \alpha \in \mathcal{C} \cup \mathcal{O}\}$
S1.3 $\mathsf{v} : \mathcal{P}^r \rightarrow P(\mathcal{D}^r)$
(the power set of the r-th Cartesian product of $\mathcal{D}$)
S1.4 $\mathsf{v} : {}^{\sim}\mathcal{S} \rightarrow \{0, 1\}$

S1.5 for all $r > 0$, $\mathsf{v} : {}^{\sim}\mathcal{P}^r \rightarrow P(\mathcal{D}^r)$

Hence, the valuation function defined by a model $\mathsf{M} = \langle \mathcal{D}, \mathsf{v} \rangle$ is given by the following:

S2.1 $\mathsf{v_M} : \mathcal{W}+ \rightarrow \{0, 1\}$
S2.2 where $A \in \mathcal{S}$, $\mathsf{v_M}(A) = \mathsf{v}(A)$; $\mathsf{v_M}(\perp) = 0$
S2.3 $\mathsf{v_M}(\pi^n(\alpha_1, \ldots, \alpha_n)) = 1$ iff $\langle \mathsf{v}(\alpha_1), \ldots \mathsf{v}(\alpha_n) \rangle) \in \mathsf{v}(\pi^r)$
S2.4 $\mathsf{v_M}(A \supset B) = 1$ iff $\mathsf{v_M}(A) = 0$ or $\mathsf{v_M}(B) = 1$
S2.5 $\mathsf{v_M}(A \& B) = 1$ iff $\mathsf{v_M}(A) = 1$ and $\mathsf{v_M}(B) = 1$
S2.6 $\mathsf{v_M}(A \vee B) = 1$ iff $\mathsf{v_M}(A) = 1$ or $\mathsf{v_M}(B) = 1$
S2.7 $\mathsf{v_M}(A \equiv B) = 1$ iff $\mathsf{v_M}(A) = \mathsf{v_M}(B)$
S2.8 $\mathsf{v_M}((\forall \alpha) A(\alpha)) = 1$ iff $\mathsf{v_M}(A(\beta)) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$
S2.9 $\mathsf{v_M}((\exists \alpha) A(\alpha)) = 1$ iff $\mathsf{v_M}(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$
S2.10 $\mathsf{v_M}(\alpha = \beta) = 1$ iff $\mathsf{v}(\alpha) = \mathsf{v}(\beta)$
S2.11 where ${\sim}A \in {}^{\sim}\mathcal{S}$, $\mathsf{v_M}({\sim}A) = 1$ iff $\mathsf{v_M}(A) = 0$ or $\mathsf{v}({\sim}A) = 1$
S2.12 where $r > 0$, $\mathsf{v_M}({\sim}\pi^r(\alpha_1, \ldots, \alpha_r)) = 1$ iff
$\mathsf{v_M}(\pi^r(\alpha_1, \ldots, \alpha_r)) = 0$, or $\langle v(\alpha_1), \ldots, \mathsf{v}(\alpha_r) \rangle \in \mathsf{v}({\sim}\pi^r)$
S2.13 $\mathsf{v_M}({\sim}{\sim}A) = \mathsf{v_M}(A)$
S2.14 $\mathsf{v_M}({\sim}(A \supset B)) = \mathsf{v_M}(A \& {\sim}B)$
S2.15 $\mathsf{v_M}({\sim}(A \& B)) = \mathsf{v_M}({\sim}A \vee {\sim}B)$
S2.16 $\mathsf{v_M}({\sim}(A \vee B)) = \mathsf{v_M}({\sim}A \& {\sim}B)$
S2.17 $\mathsf{v_M}({\sim}(A \equiv B)) = \mathsf{v_M}((A \vee B) \& ({\sim}A \vee {\sim}B))$
S2.18 $\mathsf{v_M}((\forall x) A(x)) = \mathsf{v_M}((\exists x)({\sim}A(x))$
S2.19 $\mathsf{v_M}((\exists x) A(x)) = \mathsf{v_M}((\forall x)({\sim}A(x))$

In [5], the following theorems are proven:

Theorem 1: CLuNs *is sound with respect to the semantics.*

Theorem 2: CLuNs *is (strongly) complete with respect to its semantics.*

Theorem 3: CLuNs *is maximally paraconsistent.*

It is important to emphasize that CLuNs has a detachable implication, $\supset$, unlike Priests paraconsistent logic LP.

3.2. *The logic* ACLuNs2

Generally speaking, inconsistency-adaptive logics are paraconsistent, non-monotonic logics which interpret the set of premises as consistently as possible.[7] They localize the inconsistencies that follow from the premises and adapt to these. This adaption must be understood as follows: where an inconsistency is involved, the rules of inference are restricted in order to avoid triviality. Where such a threat is not present, all rules can be applied to their full strength.

Inconsistency-adaptive logics are defined from two logics, known as the lower limit logic (a logic capable of handling inconsistencies) and the upper limit logic (which defines logical normality). The combination of these two logics is governed by an *adaptive strategy*. Until now, all inconsistency-adaptive logics studied have CL as upper limit logic.[8] The logic ACLuNs2, which we shall describe here, has CLuNs as lower limit logic, and uses the *minimal abnormality* strategy.[9]

An important aspect of inconsistency-adaptive proofs is that they are *dynamic*. This means that all derivations validated only by the upper limit logic are based on the understanding of the premises provided by a certain stage of the proof. At a later stage, when our insight has grown (for instance by means of further analysis of the premises), these derivations must be re-evaluated using the newly gained knowledge concerning the abnormalities present in the premises. Consider the simple example of a proof in which p and $\sim p \vee q$ occur at a certain stage. From these CL (the upper limit logic) validates the derivation of q. However, if we would learn at a later stage that $\sim p$ also holds (from some other premises for instance), we shall need to revise the derivation of q, since this new information tells us that both p and $\sim p \vee q$ are true without q necessarily being true. In situations like these, where we rely on the consistent behaviour of one or more formulas to derive another formula, we speak of a formula (in this case q) being *conditionally derivable* from the premises (the condition in this case being the consistent behaviour of p). As the example illustrates, conditionally derived formulas are not necessarily definite.

In ACLuNs2-proofs, the abnormalities are captured by the DEK-consequences of the premises. Where $\exists A$ abbreviates A proceeded by an existential quantifier over each individual variable free in A, a DEK-formula is a

[7] See [2], [4] and [3], among others, for a more detailed account on adaptive logics.

[8] Another brand of adaptive logics, the *ampliative adaptive logics*, are an exception in this respect.

[9] Another well known strategy is *reliability*, see [2].

formula of the form

$$\exists(A_1\&{\sim}A_1) \vee \ldots \vee \exists(A_n\&{\sim}A_n)$$

where all A_i are primitive formulas.[10] We shall abbreviate this, for the sake of simplicity, as $DEK\{A_1, \ldots, A_n\}$; where I write $DEK(\Delta)$, Δ is obviously a finite set. Each such A_i is called a *factor* of the DEK-formula. A DEK-formula $DEK(\Delta)$ is called a *minimal* DEK-consequence of a set of premises Γ iff $\Gamma \vdash_{\mathrm{CLuNs}} DEK(\Delta)$ and there is no $\Theta \subset \Delta$ such that $\Gamma \vdash_{\mathrm{CLuNs}} DEK(\Theta)$. It can easily be seen that at least one factor of a minimal DEK-consequence must behave inconsistently in order to validate the entire disjunction.

As a result of the De Morgan properties present in CLuNs, ACLuNs2 has the very interesting property that abnormalities are restricted to primitive formulas only. This means that all abnormalities originate from atomic formulas.

An ACLuNs2-proof consists of a number of lines, each of which contains five elements: a line number, the formula derived, the lines from which the formula is derived, the rule used, and the condition on which this line is derived. This condition is a set of primitive formulas the consistent behaviour of which we rely on in order to derive the formula that is the second element of the line. In the above example, the fifth element of the line on which q is derived would be the set $\{p\}$. Naturally, conditions are *carried along*: when we use a line i, with Δ as fifth element to derive a formula on line j, Δ shall be a subset of the fifth element of line j, independently of the derivation rule used.

Formally, the minimal abnormality strategy is specified by the *integrity criterion*. Let $^\circ\Phi_s$ be the set of all ϕ, where ϕ contains one factor out of each minimal DEK-consequence (at stage s). Define Φ_s from $^\circ\Phi_s$ as the set of those elements of $^\circ\Phi_s$ that are not supersets of other elements.

Definition 1: *Line j, with A as second element, fulfills the integrity criterion (at stage s) iff (i) the intersection of some member of Φ_s and of the fifth element of line j is empty, and (ii) for each $\phi \in \Phi_s$ there is a line k such that the intersection of ϕ and the fifth element of line k is empty and A is the second element of line k.*

The derivation rules for ACLuNs2 are the following.

[10] This limitation to primitive formulas is not common to all inconsistency-adaptive logics.

RU If $\vdash_{\text{CLuNs}} (A_1\& \ldots \&A_m) \supset B$ and $A_1, \ldots, A_m$ occur in the proof at a stage s, then add a new line to the proof, consisting of an appropriate line number, B, the numbers of the lines on which the A_i were derived, "RU", and, as fifth element, the union of the fifth elements of all lines mentioned in the third element.

RC If $\vdash_{\text{CLuNs}} DEK\{C_1, \ldots, C_n\} \vee ((A_1\& \ldots \&A_m) \supset B)$ and $A_1, \ldots, A_m$ occur in the proof at a stage s (where the C_i are primitive formulas), then add a new line to the proof, consisting of an appropriate line number, B, the numbers of the lines on which the A_i were derived, "RC", and, as fifth element, the union of $\{C_1, \ldots, C_n\}$ with the fifth elements of all lines mentioned in the third element.

At any stage s of the proof, a line i is marked (e.g. by means of a "†") iff line i does not fulfill the integrity criterion (at stage s). A marked line is not considered as part of the proof (at that stage), and hence can not be relied upon for further derivations.

Even though the proof theory of ACLuNs2 is dynamic, we are able to define what it means for a formula to be finally ACLuNs2-derived from a set of premises. A formula A is *finally derived* in an ACLuNs2-proof from Γ iff A occurs as the second element of an unmarked line i of the proof, and if line i is marked in an extension of the proof, then it is unmarked in a further extension.

Let us examine a simple example.

1.	$p\&r$	—	PREM	—
2.	$p \supset (s\&q)$	—	PREM	—
3.	$r \supset {\sim}(t \vee q)$	—	PREM	—
4.	${\sim}s \vee {\sim}q$	—	PREM	—
5.	${\sim}s \vee t$	—	PREM	—
6.	p	1	RU	$\emptyset$
7.	$s\&q$	2,6	RU	$\emptyset$
8.	s	7	RU	$\emptyset$
†9.	t	5,8	RC	$\{s\}$
10.	$(s\&{\sim}s) \vee (q\&{\sim}q)$	4,7	RU	$\emptyset$

At this stage of the proof, $\Phi_{10} = \{\{q\}, \{s\}\}$, and hence line 9 must be marked. However, the proof can be further extended:

11.	r	1	RU	$\emptyset$
12.	${\sim}(t \vee q)$	3,11	RU	$\emptyset$
13.	${\sim}t\&{\sim}q$	12	RU	$\emptyset$
14.	${\sim}q$	13	RU	$\emptyset$

15.	$\sim t$	13	RU	$\emptyset$
16.	q	7	RU	$\emptyset$
17.	$q\&\sim q$	14,16	RU	$\emptyset$

We now have derived a new DEK-formula, $DEK\{q\}$ which is minimal at stage 17. As such, $\Phi_{17} = \{\{q\}\}$, and the marking of line 9 must be removed, and it becomes

9.	t	5,8	RC	$\{s\}$

We can still extend the proof with the following line.

18.	$(t\&\sim t) \vee (s\&\sim s)$	5,8,15	RU	$\emptyset$

This DEK-consequence is minimal at this stage, and forces us to mark line 9 again.

†9.	t	5,8	RC	$\{s\}$

This proof nicely illustrates the dynamic character of ACLuNs2-proofs: when more knowledge about the premises became available, by reasoning from them, markings were added and removed again, until a "stable" situation was reached after line 18.

For the model theoretic side of ACLuNs2 I shall be very brief. Define the set of abnormalities of a CLuNs-model M as the set $Ab(\mathsf{M})$ of all *primitive* formulas A for which $\mathsf{v}_{\mathsf{M}}(\exists(A\&\sim A)) = 1$. The ACLuNs2-models of Γ are then defined as the set of those CLuNs-models of Γ for which $Ab(\mathsf{M})$ is minimal. More formally, a CLuNs-model M is an ACLuNs2-model of Γ iff there is no CLuNs-model M' of Γ such that $Ab(\mathsf{M}') \subset Ab(\mathsf{M})$. For all further details concerning the semantics of adaptive logics, I refer the interested reader to the bibliography.

4. *Axiomatizing inconsistent arithmetic*

4.1. *Introduction: not even an attempt*

A good place to start for axiomatizing inconsistent arithmetic is the set of Peano-axioms, which define (when the underlying logic is CL) Peano-arithmetic (hereafter PA). As is clear from previous sections, the underlying logic of any system for inconsistent arithmetic can not be CL, for triviality would be inevitable. Therefore, the logic CLuNs as defined above will be the underlying logic.[11] The Peano-axioms are:

AX1 $(\forall x){\sim}(0 = x')$
AX2 $(\forall x, y)(x' = y' \supset x = y)$
AX3 $(\forall x)(x + 0 = x)$
AX4 $(\forall x, y)(x + y' = (x + y)')$
AX5 $(\forall x)(x \cdot 0 = 0)$
AX6 $(\forall x, y)(x \cdot y' = (x \cdot y) + x)$
AX7 $(A(0)\&(\forall x)(A(x) \supset A(x'))) \supset (\forall x)A(x)$

Although the underlying logic, CLuNs, is paraconsistent, preventing the application of *ex falso quodlibet*, triviality is the result of adding

AX8 $L = L'$

to the above. The cause of this triviality lies in the combination of AX2 with AX8: from $L = L'$ together with $(\forall x, y)(x' = y' \supset x = y)$ one can derive, in $L + 1$ steps, the disastrous $0 = 1$, trivializing everything along the way. At that point, we are at total loss.

In what follows, I will define two versions of inconsistent arithmetic, namely IA_1 (in subsection 4.2) and IA_2 (in subsection 4.3). These will solve the first problem that inconsistent arithmetic as defined by Van Bendegem and Priest, IA_C, suffered: incompleteness.

4.2. *First attempt:* IA_1

What we saw earlier on as a benefit that CLuNs has over LP, namely the presence of a detachable implication, now seems to be the cause of trouble: it is the detachable character of the implication in AX2 that leads to triviality when combined with AX8. A possible adjustment to the axioms can thus be to delete the weaken AX2 to:

AX2' $(\forall x, y)({\sim}x' = y' \vee x = y)$

[11] The reason we take CLuNs instead of CLuN to be the appropriate logic for our purpose, is that the Schütte-properties allow us to derive more formulas. As such, they make the gap between classical logic and paraconsistent logics smaller (the Schütte-properties inhere in CL), making it possible to derive more CA-theorems. The drawback is that these properties, being related to negation, are the cause of a more extensive spreading of inconsistencies. This will turn out not to be a problem for inconsistent arithmetic.

We can define a pseudo-implication as follows

$$A \sqsupset B \ =_{\text{def}} \ \sim A \vee B \tag{2}$$

such that the second axiom becomes

Ax2' $(\forall x, y)(x' = y' \sqsupset x = y)$

Note that $\sqsupset$ is the implication of LP, which Priest writes as "$\rightarrow$".

Since we are at it, we can continue in this fashion of redefining implications. Define

$$(A \rightarrow B) =_{\text{def}} (A \ \supset \ B)\&(\sim B \ \supset \ \sim A) \tag{3}$$

This implication is, in a paraconsistent environment, stronger than the classical implication $\supset$, since it reintroduces modus tollens. We can replace the last Peano-axiom, defining mathematical induction by

$$(A(0)\&(\forall x)(A(x) \supset A(x'))) \rightarrow (\forall x)A(x) \tag{4}$$

This axiom clearly does not lead to any trouble.

However, the gain of replacing the original axiom for mathematical induction by this new version, is minimal. That is, we win modus tollens, allowing to derive from $(\exists x){\sim}A(x)$ that ${\sim}A(0) \ \vee \ {\sim}(\forall x)(A(x) \supset A(x'))$. Even if this would be an important step forward in a classical setting, we have to keep in mind that the derivation of a disjunction is in a paraconsistent environment not very interesting. Since disjunctive syllogism is not permitted in CLuNs, the gain is minimal.

We are now able to rewrite the original Peano-axioms in the following manner:

Ax1 $(\forall x){\sim}(0 = x')$
Ax2' $(\forall x, y)(x' = y' \sqsupset x = y)$
Ax3 $(\forall x)(x + 0 = x)$
Ax4 $(\forall x, y)(x + y' = (x + y)')$
Ax5 $(\forall x)(x \cdot 0 = 0)$
Ax6 $(\forall x, y)(x \cdot y' = (x \cdot y) + x)$
Ax7' $(A(0)\&(\forall x)(A(x) \supset A(x'))) \rightarrow (\forall x)A(x)$
Ax8 $L = L'$

The resulting theory, being these axioms with CLuNs as underlying logic, I shall call IA_1. We denote the consequence relation of IA_1 by $\vdash_{\text{IA}_1}$. If, for a formula ϕ and for all IA_1-models M, $\mathsf{v}_{\mathsf{M}}(\phi) = 1$, then we write $\models_{\mathrm{IA}_1} \phi$. Furthermore, let $\mathcal{M}_{\mathrm{IA}_1}$ denote the set of IA_1-models, i.e. the CLuNs-models verifying AX1 to AX8 (with the second and seventh axiom replaced as stated above).

By the properties of CLuNs, we have:

Theorem 4: *For any set of wffs* Γ *and any wff* ϕ, $\Gamma \vdash_{\mathrm{IA}_1} \phi$ *iff* $\Gamma \models_{\mathrm{IA}_1} \phi$.

IA_1 has already solved one of the problems IA_{C} suffered from:

Theorem 5: *Every model* $\mathsf{M} \in \mathcal{M}_{\mathrm{IA}_1}$ *is maximally non-trivial.*

Proof. Let M be a model of IA_1. If $\mathsf{v}_{\mathsf{M}}(A) = 0$, then $\mathsf{v}_{\mathsf{M}}(A \supset B) = 1$ for all wffs B. Hence, the set $\{\phi \mid \mathsf{v}_{\mathsf{M}}(\phi) = 1\} \cup \{A\}$ is trivial. □

Incidentally, one can not replace *both* implications by the stronger one, which would result in the axiom

$$(A(0) \& (\forall x)(A(x) \to A(x'))) \to (\forall x)A(x) \tag{5}$$

This version of mathematical induction is to weak (due to the strong antecedent) to prove many universally quantified formulas that should come out true.[12]

Even though IA_1 is maximally non-trivial, there remain some unsolved problems. One is the third problem mentioned in section 2. Also, we still suffer from the second of the problems IA_{C} had: the intuitively true inequalities (of the form $\sim n = m$ where neither n nor m is zero) are not provable in IA_1.

4.3. *Second attempt:* IA_2

A possible solution to this particular problem of unprovable inequalities, is to split the (original) second Peano-axiom. If we define, within our object language, a predicate S by which we can express that a number behaves

[12] An example of this is the property

$$M(x) =_{\mathrm{def}} x < x' \tag{6}$$

where $<$ has the classical definition, that is $x < y$ iff $\exists z \neq 0 : x + z = y$. Note that $L < L$ and $\sim(L < L)$ both hold. Let the number K be defined as $K' = L \;\&\; \neg K = L$, where the strong negation $\neg$ is defined by $\neg A =_{\mathrm{def}} A \supset \bot$. Unlike AX 8, (5) does not enable one to prove $(\forall x)M(x)$. Indeed, one can not have $\sim M(L) \supset \sim M(K)$ as $\sim M(L)$ is true whereas $\sim M(K)$ is false.

consistently, then we could split axiom 2 into two distinct axioms, both with the original (detachable) implication: one for the consistent numbers, and one for all the numbers which behave inconsistently (these are the numbers that are both equal and not equal to L). For this purpose we can use the strong negation, $\neg$, as defined in footnote 12. Hence we can say that

$$(\forall x)S(x) \equiv \neg(x = L) \tag{7}$$

Using this, we can define the rest of the numbers, i.e. those who are equal to L, and thus behave inconsistently:

$$(\forall x)B(x) \equiv \neg S(x) \tag{8}$$

Note that for every natural number n, either $B(n)$, or $S(n)$ holds, but definitely not both (due to the strong negation).

With this in mind, we can rewrite the second Peano-axiom so that we can, once we know that certain numbers behave consistently, proceed as in classical arithmetic. And if we know for sure that two numbers both behave inconsistently, we can also use the original second axiom. So now we have

AX2.1 $(\forall x, y : S(x)\&S(y))(x' = y' \supset x = y)$
AX2.2 $(\forall x, y : B(x)\&B(y))(x' = y' \supset x = y)$

This splitting of the second axiom allows us to do as we always do in PA, except when for one of the two numbers involved predicate S holds, and for the other B holds. But in all other cases, this is a great step forward, especially for the "small" numbers. If we know, for instance, that $S(m)$ and $S(n)$, and $m' = n'$, we can derive that $m = n$. A step we could not make in IA_1.

What is more, we can strengthen these two axioms to include the use of modus tollens. This does not lead to the derivation of unwanted formulas. Hence, we can formulate

AX2.1 $(\forall x, y : S(x)\&S(y))(x' = y' \rightarrow x = y)$
AX2.2 $(\forall x, y : B(x)\&B(y))(x' = y' \rightarrow x = y)$

As for mathematical induction, we can preserve the stronger version (AX7') here. The resulting arithmetic will be called IA_2. For IA_2, we make use of notations completely analogous to those of IA_1.

Theorem 6: *For any set of wffs* Γ *and any wff* ϕ, $\Gamma \vdash_{\text{IA}_2} \phi$ *iff* $\Gamma \models_{\text{IA}_2} \phi$.

As for IA_1 we can prove the following theorem (the proof is omitted, for entirely analogous to the proof of theorem 5):

Theorem 7: *Every* $\mathsf{M} \in \mathcal{M}_{\mathrm{IA}_2}$ *is maximally non-trivial.*

This theorem allows us to conclude that IA_2 is complete in the sense of [1].

With IA_2 we managed to reintroduce the detachable implication $\supset$ in the second axiom. As a consequence, IA_2 does not have the problem that IA_1 has: it is possible to derive from $m' = n'$ that $m = n$, *provided* we know that both m and n are either small ($S(m)$ and $S(n)$) or big ($B(m)$ and $B(n)$). This makes the formalism terribly heavy and certainly is weird with respect to the usual mathematical practice.[13]

5. *The solution to the problem*

The solution to our problems will be twofold: for one, a final adjustment must be made to the axioms, which will result in the system for inconsistent arithmetic IA_3. A second alteration will involve the underlying logic (see subsection 5.2).

5.1. *Third attempt:* IA_3

As is clear from the above, the problem that IA_C, IA_1 and IA_2 have in common is the large, if not to say huge, amount of axioms they need in order to preserve all classical theorems. When discussing this problem in the introduction, it was stated that the cause, for IA_C, lies in the absence of a detachable implication in the paraconsistent logic LP. The same holds for the IA_1. For IA_2 there is the need to prove either the property S or the property B for the numbers under consideration, and this *prior* to the application of modus ponens and/or modus tollens.

The idea that will be used here, is captured beautifully in the well-known saying "if you can't beat them, join them": if modus tollens is not allowed in

[13] The obvious objection against this "problem" is to state that L was chosen to be large enough to not have any physical meaning, and thus not to be present in arithmetical practice. Let me reply by saying that many interesting, practical aspects of mathematics, like for instance imaginary numbers, have started out as purely theoretical concepts, with absolutely no "physical meaning". One day L *could* have some significance in our understanding of the world.

a paraconsistent logic, like CLuNs, then we can just as well take up the contraposed version of Ax2 in the list of axioms. As such, we define Ax2^{CP}:

Ax2^{CP} $\quad (\forall x, y)(\sim x = y \supset \sim x' = y')$

Note that we use the $\supset$-implication, since allowing for modus tollens would be disastrous (from $K' = L'$ we could then derive that $K = L$).

This axiom, together with Ax1, will enable us to derive any desired inequality. This solves the second of the problems from section 2: the axiomatization of inconsistent arithmetic given by IA_3 is finite, and does not need more than L axioms.

Ax1	$(\forall x)\sim(0 = x')$
Ax2^{CP}	$(\forall x, y)(\sim x = y \supset \sim x' = y')$
Ax3	$(\forall x)(x + 0 = x)$
Ax4	$(\forall x, y)(x + y' = (x + y)')$
Ax5	$(\forall x)(x \cdot 0 = 0)$
Ax6	$(\forall x, y)(x \cdot y' = (x \cdot y) + x)$
Ax7'	$(A(0)\&(\forall x)(A(x) \supset A(x'))) \to (\forall x)A(x)$
Ax8	$L = L'$

As for IA_1 and IA_2, we can state the following theorems:

Theorem 8: *For any set of wffs* Γ *and any wff* ϕ, $\Gamma \vdash_{\text{IA}_3} \phi$ *iff* $\Gamma \models_{\text{IA}_3} \phi$.

Theorem 9: *Every model* $\mathsf{M} \in \mathcal{M}_{\text{IA}_3}$ *is maximally non-trivial.*

Although IA_3 does not suffer the first two of the three problems from section 2, it is not yet the ideal axiomatization of inconsistent arithmetic. We lack an axiom stating that if two small numbers n' and m' are equal, then n and m are. This is as a matter of fact only half a problem: the axioms stated above suffice to derive all truths in inconsistent arithmetic. But, the direct line between $n' = m'$ and $n = m$ is lost: we have to prove independently of the premise $n' = m'$ that n and m are equal. But this is not a problem, since they *are*. The problem arises when n and m are variables in an equation: in that case there is no way to derive the desired equality in IA_3.

Furthermore, the third problem of section 2, concerning minimal abnormality, still remains.

These two problems will be solved by what we shall call "IAA", the system for inconsistent arithmetic to be defined in the next section.

5.2. *Final attempt and solution:* IAA

To overcome the shortcomings of IA_3, we need to abandon the monotonic paraconsistent logic CLuNs and turn to an inconsistency-adaptive logic, namely ACLuNs2. As was shown in subsection 3.2, the logic ACLuNs2 selects the minimal abnormal models from the set of CLuNs-models of some set Γ. We will call the number-theory defined by the Peano-style axioms of the section above, namely

Ax1	$(\forall x)\sim(0 = x')$
Ax2^{CP}	$(\forall x, y)(\sim x = y \supset \sim x' = y')$
Ax3	$(\forall x)(x + 0 = x)$
Ax4	$(\forall x, y)(x + y' = (x + y)')$
Ax5	$(\forall x)(x \cdot 0 = 0)$
Ax6	$(\forall x, y)(x \cdot y' = (x \cdot y) + x)$
Ax7'	$(A(0) \& (\forall x)(A(x) \supset A(x'))) \rightarrow (\forall x)A(x)$
Ax8	$L = L'$

together with ACLuNs2 as underlying logic, IAA.

The effect of this on inconsistent arithmetic is twofold. For one, there are no models $\mathsf{M} \in \mathcal{M}_{\mathrm{IAA}}$ for which $\mathsf{v}_\mathsf{M}(n = m) = 1$ if n and/or m are classically smaller than L, and classically different of one another. It is not very difficult to see why this is the case. When a formula of the form $\sim A$ is stated in the list of axioms, there are, according to the clauses of the CLuNs-semantics, two possibilities: either A is false ($\mathsf{v}_\mathsf{M}(A) = 0$), making $\sim A$ consistently true in the model; or A is an abnormality, in which case $\mathsf{v}_\mathsf{M}(A) = \mathsf{v}_\mathsf{M}(\sim A) = 1$. In the set of all CLuNs-models both possibilities occur. On the other hand, ACLuNs2 selects minimal abnormal models, which assures that the first possibility is chosen, *unless* the axioms require the second one (as is the case for L and all numbers classically greater than L).

For IAA, this means that if the first axiom states that $\sim 0 = 5$, then in every model M, $\mathsf{v}_\mathsf{M}(0 = 5) = 0$. Since there is no axiom stating that $0 = 5$ must also be true, the abnormality is not demanded by the axioms, and thus $\mathcal{M}_{\mathrm{IAA}}$ does not include models in which the formula *does* behave inconsistently.

The eighth axiom $L = L'$ causes an abnormality, since $\sim L = L'$ follows from Ax1 and Ax2^{CP}. The result is that for all $\mathsf{M} \in \mathcal{M}_{\mathrm{IAA}}$, $\mathsf{v}_\mathsf{M}(L = L') = \mathsf{v}_\mathsf{M}(\sim L = L') = 1$. By the axioms, all equations concerning only numbers larger than L will behave inconsistently.

A second effect of the selection of minimal abnormal models will be that some derivation rules will be reintroduced. As we have seen in the section on ACLuNs2, modus tollens, disjunctive syllogism and some other rules are *conditionally* applicable. This is a great step forward when working with unknown values, as is quite common in arithmetical practice. When we, for instance, know that $n = m$, although we do not have any idea of which numbers n and m actually are, we can derive in IAA that there predecessors $p(n)$ and $p(m)$ are equal, *provided* $n = m$ is a consistent formula, i.e. is not true together with $\sim n = m$.

Since the abnormalities in inconsistent arithmetic are restricted to those formulas containing only numbers equal to or larger than L, the condition

"the formula $n = m$ must behave consistently"

can be reformulated as

"$\neg n = L$ and $\neg m = L$"

As such, the consistency requirement is reduced to the requirement of not being equal to L.

Let us now investigate whether the system we have constructed is able to overcome the three objections we raised against the original number theory $\mathrm{IA_C}$. First of all, it is quite obvious that IAA is complete in the sense described in subsection 2, in view of the following theorem, the proof of which is that of theorem 5.

Theorem 10: *Every model* $\mathsf{M} \in \mathcal{M}_{\mathrm{IAA}}$ *is maximally non-trivial.*

In addition to this theorem, stating maximal non-triviality for all IAA-models, we can prove the set of IAA-theorems to be maximally non-trivial.

Theorem 11: *The set of* IAA*-theorems is maximally non-trivial. That is, for every formula A, either* $\vdash_{\mathrm{IAA}} A$ *or* $\vdash_{\mathrm{IAA}} \neg A$.

Proof. Consider the collapsed model M with domain $\mathcal{D} = \{[0], [1], \ldots, [K], [L, L', \ldots]\}$ that has CLuNs as its underlying logic. Clearly M is a minimally abnormal model (ACLuNs2-model) of IAA. Remark that the domain $\mathcal{D}$ has exactly L elements, that $[0]$ is the only element of $\mathcal{D}$ that is not the successor of any other element, that L is the only element that is its own successor, and that the addition and multiplication functions behave as expected.
It is easily seen that any minimally abnormal model M' of IAA is isomorphic to M, and hence is equivalent to M (verifies and falsifies the same formulas). It follows that $\{A | \mathsf{v}_{\mathsf{M}}(A) = 1\} = \{A | \mathsf{v}_{\mathsf{M}'}(A) = 1\}$. But then in view of Theorem 10, the set of formulas verified by all ACLuNs2-models of IAA

is maximally non-trivial. In view of the completeness of ACLuNs2 with respect to its models, $Cn_{\mathrm{ACLuNs2}}(\mathrm{IAA})$ is maximally non-trivial. □

This theorem proves IAA to be complete in the sense of Batens' [1] as stated in section 2.

Secondly, there is no need for the elaborate list of axioms that were needed to state the inequalities true in CA: axioms AX1 and AX2^{CP} ensure that for every two numbers n and m, if $\vdash_{\mathrm{CA}} \sim n = m$ then $\vdash_{\mathrm{IAA}} \sim n = m$.

And thirdly, by the construction of ACLuNs2, with its minimal abnormality strategy, we know that there are no models $\mathsf{M} \in \mathcal{M}_{\mathrm{IAA}}$ in which there are more true contradictions than required by IAA.

6. *Closing remarks*

What has happened in this paper? The domain of the IAA-models is the same as that of the $\mathrm{IA_C}$-models and so are the functions (successor, addition and multiplication) defined on the domain. Even the assignment functions of both types of models are in a sense equivalent. The central change concerns the logic: LP was replaced by ACLuNs2. As a result, the three objections against $\mathrm{IA_C}$ are avoided. First, IAA is complete in Gödel's sense: for each closed formula A, either A or $\neg A$ is an IAA-theorem. Next, IAA has an axiomatization that is sensible with respect to IAA in that the number of axioms is IAA-significant. Finally, IAA has no models in which some number that is classically smaller than L is its own successor.

IAA is different from $\mathrm{IA_C}$. So, the point I tried to make in this paper does not concern the literal system presented by Van Bendegem and Priest, but rather its intuitive interpretation. The properties that seem to make $\mathrm{IA_C}$ attractive stem precisely from this interpretation. For example, the $\mathrm{IA_C}$-models are not attractive because, notwithstanding the collapse from L on, $1 = 0$ is *false* in them, but because it is *false only* in them. This fact is warranted by IAA, thanks to the underlying logic ACLuNs2, but cannot be warranted by $\mathrm{IA_C}$ because of its underlying logic LP.

E-mail: timothy.vermeir@planetinternet.be

REFERENCES

[1] Diderik Batens. The demise of rich finitism. A study in the limitations of paraconsistency. To appear.

[2] Diderik Batens. Inconsistency–adaptive logics. In Ewa Orłowska, editor, *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*, pages 445–472. Physica Verlag (Springer), Heidelberg, New York, 1999.

[3] Diderik Batens. Minimally abnormal models in some adaptive logics. *Synthese*, 125:5–18, 2000.

[4] Diderik Batens. In defence of a programme for handling inconsistencies. In Joke Meheus, editor, *Inconsistency in Science*. Kluwer, Dordrecht, 200x. To appear.

[5] Diderik Batens and Kristof De Clerq. A rich paraconsistent extension of a full positive logic. To appear.

[6] Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8(2):219–241, 1979.

[7] Graham Priest. Minimally inconsistent LP. *Studia Logica*, 50(2):321–331, 1991.

[8] Graham Priest. Is arithmetic consistent? *Mind*, 103:337–349, 1994.

[9] Graham Priest. What could the least inconsistent number be? *Logique et Analyse*, 145:3–12, 1994.

[10] Jean Paul Van Bendegem. Strict, yet rich finitism. In Z.W. Wolkowski, editor, *First International Symposium on Gödels Theorems*. World Scientific, 1993.

[11] Jean Paul Van Bendegem. Strict finitism as a viable alternative in the foundations of mathematics. *Logique et Analyse*, 145:23–40, 1994.