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"*STRENGE*" ARITHMETICS

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Abstract
In *Entailment*, Anderson and Belnap motivated their modification E of Ackermann's *strenge Implikation* Π' as a logic of relevance *and* necessity. The kindred system R was seen as *relevant* but *not* as *modal*. Our systems of *Peano arithmetic* R# and *omega arithmetic* R## were based on R to avoid *fallacies of relevance*. But problems arose as to which arithmetic sentences were (relevantly) *true*. Here we base analogous systems on E to solve those problems. Central to motivating E is the rejection of *fallacies of modality*. Our slogan here for this is, "No *diamonds* entail any *boxes*." Form the *strenge Peano arithmetic* E# like R#, adding appropriate forms of the Peano axioms to Ackermann's $E^{\forall x}$. Extend E# to the *strenge omega arithmetic* E## by adding the ω-rule $A(0), A(1), ... \Rightarrow \forall x A(x)$. E# and E## make *explicit* a rejection of "fallacies of modality" *implicit* in R#, where already "equations" work like boxes and "unequations" like diamonds. (And no unequations relevantly imply any equations.) The R# theory of *secondary formulas* extends straightforwardly to our *strenge* arithmetics. Finally *metavaluing* E## yields the *strenge true arithmetic* TE#. TE# treats truth-functions and quantifiers truth-functionally, settling sentences like $0 = 2 \to 0 = 1$ by *affirming their negations* (as Belnap once suggested).

1. *Introduction*

Restall objected to Meyer's claim in [1] that the system R## of that paper is "true" relevant arithmetic. "How can that be," he wanted to know, "when there are sentences A of R## such that neither A nor $\sim$A is a theorem?" (An example is $0 = 2 \to 0 = 1$; see [1].) "We can fix that up," retorted Meyer, "by applying *metavaluations* to R##." But, noted Restall, that doesn't work either. For R## requires that $\sim$A be equivalent to $A \to 0 \neq 0$, whereas this may not happen on a metavaluation. O.K., let's switch from R to E, suggested Meyer. The result is this paper.

We have discussed formulating arithmetic using a *relevant* logic in a number of places; see [2] for an ABD survey and for references.[1] We have most often chosen R as that relevant logic, as Meyer did in [3].[2] But see Restall's [4, 14] for arithmetics developed on a wide choice of substructural logics.[3] True, Meyer did bid "farewell to entailment" in [5]. (Should *this* paper be called "Hello again"?) But Meyer was aware even while writing [3, 5] that there is an odd resonance in R# of the "fallacies of modality" story that Anderson and Belnap used to motivate E in [6]. For just as, in E, no negated entailment entails an entailment, just so in R# and R## no negated equation entails an equation.[4]

It will be our purpose here to base relevant arithmetic on E and related systems. This will produce systems E# and E## analogous to R# and R##. More accurately, the systems proposed here will be systems of *strenge* arithmetic, since we formulate E in the manner of Ackermann's [7].[5] This means that we make explicit Ackermann's rules γ and δ, which ABD chopped. We then extend E## to a system TE# of *strenge true arithmetic*. Let there be definitions, axioms, and rules, which follow a brief interlude on *modal fallacies*.

2. *Ackermann, Anderson, Belnap and fallacies of modality*

Prominent in early relevant polemics were the identification and condemnation of some classes of fallacies. Of these, purported *fallacies of relevance* drew the most ink.[6] But so-called *fallacies of modality* were also chastised in [7], [6] and elsewhere. The root of this chastisement was the thought that, from necessary propositions, what follows is further necessary stuff. But the

[1] 'ABD' stands, here and henceforth, for 'Anderson, Belnap & Dunn'.

[2] We did consider basing arithmetic on E in [5], for some of the reasons viewed here as conclusive.

[3] [4] was a Ph.D. thesis on logics without the contraction principle $(A \to (A \to B)) \to (A \to B)$, which R has.

[4] For proof of the R facts, see Appendix 1 of this paper.

[5] See [2], esp. pp. 129–141, for more on the relation between E and Ackermann's original systems.

[6] See [6] and the prior Anderson-Belnap papers cited there for more on fallacies of relevance.

root went to the *top* and the tree grew *upside down*. And the purported thesis became something like,

FoM1. Unnecessary stuff does *not* entail necessary stuff.

Alas, *FoM1* is clearly false, as even [6] came to concede.[7] We now make contact with the jaw-breaking terminology of [6] (for which it was suitably contrite), where what we shall call a *box* was identified on pp. 36ff. as a *necessitive*. For the record,

DB. A is a *box* if it is demonstrably equivalent to some $\Box$B.

DD. A is a *diamond* if it is demonstrably equivalent to some $\Diamond$B.

Having abandoned *FoM1*, [6] decided that what it had had in mind was that *boxes* were choosy about the sorts of propositions that they followed from. In particular, [6] agreed with [7] that

FoM2. p $\to$ (A $\to$ B) is *never* valid, for a propositional variable p and any formulas A, B. Reason: A $\to$ B is itself a necessitive (as [6] sees it). And boxes do *not* follow from variables. *FoM2* is inapplicable in this paper, since nothing in the formal theories of arithmetic that we shall be examining works like a variable p. But not far off is the further E-metatheorem

FoM3. $\Diamond$A $\to$ $\Box$B is *never* valid. I.e., *diamonds* don't entail *boxes*.

Note that *FoM2* follows quickly from *FoM3*. Consider the following argument:

Given:	1) No diamonds entail any boxes. (*FoM3*)
Assumption:	2) p $\to$ (A $\to$ B) is nonetheless E-valid, for some p, A, B. (For *reductio*)

Of course the rule of uniform substitution for propositional variables like p is admissible for the *logic* E. Letting A$'$ and B$'$ be the result of substituting $\Diamond$p for p in A and B we get

Conclusion:	3) $\Diamond$p $\to$ (A$'$ $\to$ B$'$) is a theorem of E.

But, according to E, the *entailment* A$'$ $\to$ B$'$ is already a *box*, whence 3) contradicts 1). Moral: the *reductio* assumption 2) is false, whence *FoM2* is established.

That fallacies of modality are *bad* is not yet widely accepted (even by us) as *good philosophy*. Imagine our surprise, accordingly, when some formulas of relevant arithmetic (like anything of the form u = v) started *acting like boxes*. To complete the shock, their negations behaved like *diamonds*. Even (the advertised as non-modal) R# and R## respect *FoM3*, it would seem. We adapt all this to the E environment here, where *FoM3* holds *ab initio*.

What *should* we think, philosophically, of the *FoM3* prohibition against $\Diamond$A *ever* entailing $\Box$B? An off-the-cuff thought is that it makes good sense, since diamonds regularly come from boxes. But who extracts boxes from diamonds? Nor is it unreasonable to let E speak for itself on the point. "I

[7] See it and [5] for discussion and references, mainly to Sylvan and Plumwood, formerly Routley & Routley.

am *not* the sort of logic," E might say, "to permit $\Box$B to follow from *any old thing*. I particularly object when the old thing is a diamond. On the recently fashionable Kripke semantics for modal logics, $\Diamond$A is true at a "world" w just in case w *sees* some world a such that A is true at a. But $\Box$B is true at w iff B is true at every world b that w *sees*. It sounds like a quantifier mix-up to me. Why should w's looking in one direction, say to the northeast, and seeing A true thataway, ever lead us to suppose that B is true *in every direction?* Is this not a fallacy based on a 'Come one, come all' maxim?"

At this point many readers —maybe even Kripke himself— will want to quarrel with E. "What," they may interject, "of the case when B is itself a logical truth, and is accordingly true *everywhere?*" E has a quick rejoinder, since on the semantics of relevant logics not even the logical truths are true everywhere.[8] Renewing the attack S5'ers may point to their thesis $\Diamond\Box A \rightarrow \Box A$, a diamond entailing a box if ever there was one.[9]

"So much the worse," E will respond, "for S5. I always preferred S4 myself."[10]

3. *Axioms for strenge arithmetics*

Our systems are formulated in a traditional arithmetical vocabulary, with *terms* built up from the constant 0 and individual variables x, y, z, etc., using the successor operation $'$ and the dyadic function symbols $\times$ and $+$. Atomic formulas are of the form $t = u$, where t and u are terms. *Formulas* A, B, C, etc. are then built up as usual from the atomic ones under &, $\vee$, $\sim$, and $\rightarrow$, together with the *universal quantifier* $\forall$. *Sentences* shall be formulas in which no variable occurs free. We enter the following additional definitions:

[8] The chief tool for refuting paradoxes of implication in relevant semantics lies precisely in admitting points in frames at which even theorems of logic can be falsified. Cf. [16]. For how this works for the logic E, see [17]. Or [2], for an older Meyer plan.

[9] This thesis has been invoked in a (purportedly valid) version of the Ontological Argument for the Existence of God. Dunn has quipped that S5 must be false, since one can prove therein that God exists. We remark in rebuttal that an *invalid* argument to a *true conclusion* is best replaced by a *valid* one. Cf. [18], owed in part to Putnam, which *proves* the true conclusion.

[10] Sharing this preference was A. R. Anderson, who identified S4 as the *true one* among *many modal logics*.

(D$\supset$)	$A \supset B =df \sim A \vee B$
(D$\leftrightarrow$)	$A \leftrightarrow B =df (A \to B) \& (B \to A)$
(D$\exists$)	$\exists xA =df \sim \forall x{\sim}A$
(Dt)	$t =df\ 0 = 0$
(Df)	$f =df \sim t$
(D$\neq$)	$u \neq v =df \sim (u = v)$
(D$\Box$)	$\Box A =df\ t \to A$
(D$\Diamond$)	$\Diamond A =df \sim \Box \sim A$
(D1)	$1 = df\ 0'$
(D2)	$2 = df\ 1'$

etc. In particular, we take $0, 1, 2, 3...$ as the *numerals*, each of them to be thought of as the *name* of the corresponding natural number. These definitions give the items defined their usual properties in systems of *strenge* implication. $0 \neq 0$, which is f, will have the properties of [7]'s *das Absurde*. We largely follow [7] (rather than ABD) in our choices of axioms and rules.

We divide the axioms of E# and E## into three parts (like Gaul).[11]

(S) Propositional axioms (of E)[12]

AxI	$A \to A$
AxB	$(B \to C) \to (A \to B) \to A \to C$
AxB$'$	$(A \to B) \to (B \to C) \to A \to C$
AxW	$(A \to A \to B) \to A \to B$
Ax&E	$A\&B \to A$ and $A\&B \to B$
Ax$\to$&I	$(A \to B)\&(A \to C) \to A \to B\&C$
Ax$\vee$I	$A \to A \vee B$ and $B \to A \vee B$
Ax$\to\vee$E	$(A \to C)\&(B \to C) \to A \vee B \to C$
AxDist	$A\&(B \vee C) \to B \vee A\&C$
AxTranspos	$(A \to B) \to \sim B \to \sim A$
AxCounterex	$A\& \sim B \to \sim (A \to B)$
Ax$\sim\sim$E	$\sim\sim A \to A$
Ax$\sim\sim$I	$A \to \sim\sim A$

[11] "*Gallia est omnis divisa in partes tres*," said Caesar.

[12] For the time being, we allow free variables in theorems here. We rank binary connectives thus in order of *increasing* scope: &, o, $\vee$, $\supset$, $\to$, $\leftrightarrow$. We break with standard relevant practice by (i) eschewing dots as parentheses & (ii) associating $\to$ to the *right*. As usual, $\times$ precedes $+$ among term-forming operators, and we may drop $\times$ for simple juxtaposition. Unary operators and quantifiers have minimal scope. Our only (binary) predicate is $=$.

(Q) Quantifier axioms (of $E^{\forall x}$).[13] We follow [2] in writing Ax for a formula in which x *may* (but need *not*) occur free; and Au shall be the result of *proper* substitution of the term u for the individual variable x in Ax.
The quantifier axioms, then, will be the following:

Ax∀E	$\forall x Ax \to Au$, u any term
Ax→∀I	$\forall x(A \to Bx) \to A \to \forall x Bx$, x not free in A
Ax∀∨	$\forall x(A \lor Bx) \to A \lor \forall x Bx$, x not free in A
Ax∀→	$\forall x(A \to B) \to \forall x A \to \forall x B$
Ax∀&	$\forall x A \,\&\, \forall x B \to \forall x(A\&B)$

These are most of the quantifier axioms for $E^{\forall x}$ in [2], with a few notational and other inessential changes. One change is reflected in ∀E axioms, which take Au as the result of properly substituting u for x in Ax, when u is free for x.[14] Another nominal change is that we now permit terms to be *complex*, in view of the additional term-forming operators $'$, $\times$, and $+$; while ABD state their axioms (on p. 72 of [2]) only for the case where u is another individual variable.[15]

The arithmetical particles and proper axioms of E# are stolen from those of [1] for R#. The same goes for the relation between E## and R##. Here they are.

(N) Arithmetical axioms of E# and E##

E#1	$x = y \to x' = y'$
E#2	$x = y \to x = z \to y = z$
E#3	$x + 0 = x$
E#4	$x + y' = (x + y)'$
E#5	$x \times 0 = 0$
E#6	$x \times y' = xy + x$
E#7	$x' = y' \to x = y$
E#8	$x' \neq 0$

We now require some rules. For the *strenge P-arithmetic* E# we choose the following:[16]

[13] We follow [2] in replacing the old name EQ by $E^{\forall x}$ for first-order E. Since we have defined ∃ as the (DeMorgan) dual of ∀ by (D∃) above, we do *not* follow [2] in having explicit axioms governing ∃ (since these are proved using dual ∀ theorems).

[14] The dual ∃I axiom, explicit in [2], is $Au \to \exists x Ax$. This is by definitions a theorem scheme. See the preceding footnote.

[15] But ABD clearly *intend* the more general forms of the axioms.

[16] *P* is for *Peano*. $\Rightarrow$ is a *metalogical* "if". Thus read α below as the →E rule, β as &I, and γ as ⊃E.

α	$A \rightarrow B \Rightarrow (A \Rightarrow B)$	$\rightarrow$E
β	A and B $\Rightarrow$ A & B	&I
γ	$A \supset B$ and A $\Rightarrow$ B	$\supset$E
δ	$A \rightarrow (B \rightarrow C)$ and B $\Rightarrow A \rightarrow C$	
$\forall$I	$A \Rightarrow \forall xA$	
RMI	$\forall x(Ax \rightarrow Ax')$ and $A0 \Rightarrow \forall xAx$	

In E#, as in R#, we may replace RMI (the Rule of Mathematical Induction) by its deductive equivalent

E#9 $\quad \forall x(Ax \rightarrow Ax')\&A0 \rightarrow \forall xAx$

But RMI makes sense for a wider class of formal arithmetics than does the axiom scheme E#9.[17] We extend E# to the *strenge ω-arithmetic* E## by adding the well-known ω-rule

$\omega \quad$ A0 and A1 and ... and An and ... $\Rightarrow \forall xAx$

I.e., the *premisses* of ω are the An for *every numeral* n, and its *conclusion* is $\forall xAx$. Note also that, given ω, we can drop RMI as primitive; for RMI is easily shown admissible anyway by induction in the metatheory of E##. Other rules, including δ, remain primitive for E##.

4. *Elementary consequences of the axioms*

In our previous work in relevant arithmetic, we have recalled that the natural numbers are built up from the fundamental number 0 by adding 1's. Just so, we have claimed, *propositions* about these numbers ought reasonably to be taken as following from some fundamental true proposition t, to be interpreted (following ABD) as the *conjunction* of all such fundamental truths. The t that we have previously chosen for this role is $0 = 0$; we choose it again, motivating Dt. But it is not so clear in the E# case that this t will play the role that we have assigned to it. Specifically, we shall want as a theorem (from E#8)

E#8t. $\quad t \rightarrow \forall x(x' \neq 0)$

But to *get* this theorem requires some care. We follow Ackermann and restore the primitive rule δ of [7]. For it is easy to see (and to prove) that we have

(1) $\quad 0 = 0 \rightarrow (A \rightarrow A)$

as a theorem scheme of E#, by structural induction on A. And we then get E#8t from (1) by applying rule δ to (1) and the E# theorem $\forall x(x' \neq 0)$, detaching a *second* antecedent in (1). So,

Fact 1. $A \Rightarrow \Box A$ is an admissible rule of E# and of E##.

[17] See [3, 4, 14]. Dunn suggested E#9 as an axiom scheme of mathematical induction. RMI is ours.

Proof. Use (1) and δ as just above to show $t \to A$ for all theorems A, ending the proof.

There was, in R#, an interesting theory of what we called *secondary formulas* in [3]. A version of this theory passes over to the *strenge* arithmetics E# and E##. We observe first

(2)	$x = y \to (x = y \to y = y)$	E#2 (Symmetry and transitivity of $=$)
(3)	$x = y \to y = y$	AxW, (2), $\to$E
(4)	$y = y \to 0 = 0$	Subtraction (Hint: use E#7, RMI)[18]
(5)	$x = y \to 0 = 0$	(3), (4), AxB$'$, $\to$E

Thus by (5) and Dt, all *equations* entail t. Let us accordingly call *any formula* A of E# which entails t a *secondary equation*. We call a *negated equation* an *unequation*. By transposition in (5) it is evident that f entails every *unequation*. Generalizing again, any formula B of E# which is provably entailed therein by f shall be a *secondary unequation*. Finally C is a *secondary formula* iff C is either a secondary equation or a secondary unequation. We have now

Fact 2. All $\to$-free formulas of E# are secondary formulas, and they are provable in E# iff provable in classical Peano arithmetic P#.
Proof. We have noted that both equations and unequations are secondary formulas; to show that this property is preserved under truth-functional combination and quantification is by a straightforward induction. (Note that it is *not* in general preserved under combination by $\to$.) As for the final claim, on direct translation E# is evidently a subsystem of P#. A good exercise, which we commend to readers, is to show that the axioms of P# (in the truth-functional vocabulary, with $\supset$ for $\to$) are theorems of E#. Whence because E# is closed under the rules of P# (in particular under γ, by fiat), any classical proof of a P# theorem is (near enough) an E# proof.[19] Q.E.D.

[18] Alternatively, follow Dunn by multiplying both sides by 0, invoking E#5. Cf. [2], p. 437.

[19] Contrast the R# situation, which does *not* have Ackermann's γ as a primitive or even as an admissible rule. For (thanks to Friedman) we refuted γ by producing in [8] a theorem QRF of P# which was not (even truth-functionally) a theorem of R#. However we think it no great virtue of E# that it delivers P# so simply. In relevant theories we prefer to *prove* γ, not to *impose* it by fiat. The contrasting and more interesting result for R# is that secondary *unequations* are provable in R# iff provable in P#. This leads in [3] to a direct homomorphic *exact* translation from P# to R#, preserving both theorems and non-theorems.

5. *The modal structure of E#*

We have decided to make something of the modal distinctions of E. So it is time to draw some. First, we wish to show that our identification back in R# of equations with boxes and unequations with diamonds holds in a structured way in E# (and so in its super-systems E## and TE#). Here is our

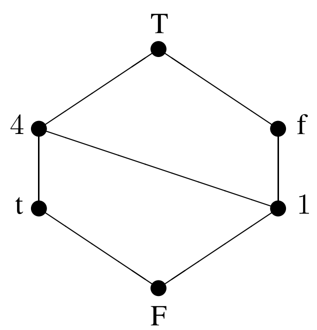
Fact 3. Among the theorem schemes of E# are the following:

E#10.	$x = y \leftrightarrow \Box(x = y)$
E#11.	$x \neq y \leftrightarrow \Diamond(x \neq y)$
E#12.	$A \to B \leftrightarrow \Box(A \to B)$
E#13.	$t \to (A \to A)$
E#14.	$\Box A \to A$
E#15.	$t \to x = x$

Proof. Recall that we have defined $\Box A$ as $0 = 0 \to A$. E#14 then follows immediately from $\Box A \to \Box A$ and an application of δ, "detaching" $0 = 0$. Easy application of Fact 1 produces both E#13 and E#15. E#11 follows from E#10 by transposition and D$\Diamond$. We conclude the verification of Fact 3 by showing E#10, E#12 from left to right. For the former, note $x = y \to (x = x \to x = y)$ by transitivity of $=$, after which apply E#15 and D$\Box$. Similarly, we get E#12 from L to R by E#13, from the instance $(A \to B) \to ((A \to A) \to (A \to B))$ of AxB. Q.E.D.

Among the boxes of E# (and super-theories like E##) are all the $u = v$ and all the $A \to B$. (Apply E#10, E#12.) If A and B are *both* boxes then A&B and A$\vee$B are boxes. (Apply Ax$\to$&I, Ax$\to\vee$E.) If A is a box then $\forall$xA and $\exists$xA are boxes. (Apply Ax$\to\forall$I.) Dually, negations of boxes such as the $u \neq v$ and $\sim (A \to B)$ are diamonds. The class of diamonds is closed likewise under the lattice connectives & and $\vee$ and quantifiers $\forall$ and $\exists$.

To show that E# (and its super-systems) reject modal fallacies, we recall Ackermann's ⑥ ([2], p. 136). Its Hasse diagram and $\to$ table are as follows:



The Ackermann matrix ⑥ for E[20]

$\to$	F	1	f	t	4	T			$\sim$
F	t	t	t	t	t	t		F	T
1	F	t	t	F	t	t		1	4
f	F	F	t	F	F	t		f	t
*t	F	F	F	t	t	t		*t	f
*4	F	F	F	F	t	t		*4	1
*T	F	F	F	F	F	t		*T	F

Inspection of ⑥ shows that the *box* values are t (for *true* $\to$ statements) and F (for *False* ones). Consulting the $\sim$ table the corresponding *diamond* values are f and T. Thus we can turn ⑥ into a matrix for all of *strenge* arithmetic on the interpretation I that assigns t to correct[21] equational sentences[22] $u = v$ and F to incorrect ones. We have set out the Ackermann $\to$ and $\sim$ tables for ⑥; otherwise, as ⑥ (being finite) is a complete distributive lattice, the values to be assigned to *arbitrary* sentences are determined homomorphically. We lay it down that the homomorphic determination of the value of a sentence $\forall xAx$ on interpretation I in ⑥ is just the *meet* of $\{I(An) : n \text{ is a numeral}\}$; otherwise $I(A \vee B) = I(A) \vee I(B)$, etc. It is evident that all closed theorems A of E# (and indeed of E##) are *true* on our suggested interpretation I, in the sense that $I(A)$ takes one of the (starred) *designated* values t, 4, T. But then

Ackermann theorem for E# and E##. No fallacies of modality hold in *strenge* arithmetic; specifically, no diamonds entail boxes.
Proof. It is clear that diamonds take one of the values f, T on our suggested interpretation of sentences of arithmetic in ⑥. By contrast boxes are restricted to the values t, F. Inspection of the $\leq$ relation of our Hasse diagram makes it clear that if $a \in \{f, T\}$ and $b \in \{t, F\}$ then it is *not* the case that $a \leq b$; or, what comes to the same thing, $a \to b$ is in all such cases the *undesignated* value F. So ⑥ rejects all candidate theorems of E## of the form $\Diamond A \to \Box B$.

[20] Identify F, 1, f, t, 4, T respectively with the $0, 1, 2, 3, 4, 5$ of p. 136 of [2]. Set $\sim b = 5 - b$. Designate $3, 4, 5$.

[21] Being *number* terms, each of u,v denotes a unique natural number in virtue of the algorithms that you learned by 3rd grade. And $u = v$ is *correct* if both of u,v denote the *same* number, else it is *incorrect*.

[22] Recall that a formula is a *sentence* if it contains *no* free variables.

6. *Metavaluing E## to get TE#*

It is time to keep our promise to make ω-arithmetic into a *true* arithmetic, by specifying that exactly one out of each pair of sentences A, ∼A shall be a theorem. We may take E## as reformulated so that *only sentences* shall count as theorems (like our presentation in [1] of R##). We may achieve this by substituting for each axiom *all its universal closures* (and counting, if the reader wishes, open formulas as theorems iff their universal closures are). The rules remain the same (except that they now apply only to *sentences*), while the ω-rule in particular is available to take up any slack. We now define a *metavaluation* V of E##, specifying a set TR of truths, as follows on all *sentences* A, B, C of the arithmetical vocabulary:

VAt	If A is an atomic sentence u = v, then A ∈ TR iff A is *arithmetically correct*
V∼	∼B ∈ TR iff B ∉ TR
V&	B&C ∈ TR iff, B ∈ TR and C ∈ TR
V∨	B ∨ C ∈ TR iff, B ∈ TR or C ∈ TR
V∀	∀xBx ∈ TR iff, for all numerals n, Bn ∈ TR
V→	B → C ∈ TR iff both (i) E## ⊢ B → C and (ii) B ∈ TR ⇒ C ∈ TR

We have, more or less, reverted to our *original* characterization of a metavaluation in [9], as a valuation that is truth-functional on intended truth-functional particles, while satisfying a more intricate condition (here, V→) on the *non-truth-functional* →. Let now TE# be the system TR of *true sentences* on V. The true sentences are evidently closed under Ackermann's rules α, β and γ. But they are *not closed* under δ.

Soundness theorem for E##. E# ⊆ E## ⊆ TE#

Proof. We know already that E## is a super-system of E#. So it will suffice to complete this proof to show that all (closures of) theorems of E## are true on our metavaluation V above. This involves a straightforward deductive induction, verifying the axioms of E## on V, and showing also that the rules preserve truth on V *for theorems of E##*. We do some cases, and leave the rest to the reader. We carry out the inductive argument only for *sentences*. It then suffices in all cases to examine an arbitrary *numerical* instance of an axiom or rule, leaving it to the ω-rule and its mate V∀ to deliver universal closures.

Ad AxB. You need to show $(B \to C) \to (A \to B) \to A \to C \in$ TR, where all of A, B, C are sentences. It's a pleasant exercise, mixing appeals to (i) and (ii) of V→. Enough, *ad* AxB!

Ad Rule δ. Suppose $A \to (B \to C)$ and B are closed theorems of E##, which on inductive hypothesis both belong to TR. By δ we have $A \to C$ as a theorem of E##. To show $A \to C \in TR$ this is (i) of V→; for (ii) assume $A \in TR$ and show $C \in TR$. But now we may use (ii) of V→; since A and $A \to (B \to C)$ are both true, we have by (ii) that $B \to C \in TR$ as well; but then, since $B \in TR$ we have again that $C \in TR$, which ends the verification that $A \to C \in TR$.

Ad Rule ω. Suppose An is a *true* theorem of E## for each numeral n. By ω, we have E## $\vdash \forall xAx$. Then by V∀ we have $\forall xAx \in TR$, which suffices.

Ad Ax∀∨. We verify (i) and (ii) under V→ for any *sentence* $\forall x(A \vee Bx) \to A \vee \forall xBx$. (i) of V→ is immediate; as an axiom, the sentence is an E## theorem. For (ii), assume $\forall x(A \vee Bx) \in TR$. Then for each numeral n either A is true or Bn is true, by V∀, V∨. If $A \in TR$ then $A \vee \forall xBx \in TR$, sufficing for (ii) by V∨. Otherwise $Bn \in TR$ for all n, whence $\forall xBx \in TR$ by V∀ and $A \vee \forall xBx \in TR$ by V∨.

Ad E#8. Note that $n' = 0$ is *always* incorrect; whence by V∀ we have $\forall x(x' \neq 0) \in TR$ by V∀. We rest our cases, leaving others to the reader, and declare the soundness theorem *proved*.

Observe, if you will, the *delicacy* of our argument *ad* the rule δ in the argument just concluded. We do *not* say that δ preserves *truth* on V when applied to *arbitrary* members $A \to (B \to C)$ and B of TR. It does *not*. To show $A \to C \in TR$, we must by (i) of V→ *prove* $A \to C$ in E##. For this we need not merely that B is true on V, but that it is a *true theorem* of E##. With that information δ *can be applied* in E## —and the difficulty disappears.[23]

Anyway, our goal of having specified the *strenge* truths is now accomplished. For in view of V∼ above and our understanding of intuitive 'not', TE# has split arithmetic *sentences* univocally into the truths and their negations, namely the falsehoods. But what, the reader may wonder, has happened to our old counterexample, $0 = 2 \to 0 = 1$? Easy —it's *false*. For as a non-theorem of E##, it fails part (i) of our truth-condition on →; as that suffices for $0 = 2 \to 0 = 1 \notin TR$, we need look no further for its refutation. And so much, by V∼, assures TE# $\vdash\sim (0 = 2 \to 0 = 1)$. By the same argument, whenever A and B are *sentences* such that $A \to B$ is unprovable in E##, then TE# $\vdash\sim (A \to B)$.

All of this returns us to a point made by Belnap when we told him of our initial work on R#. "It would seem," he observed, "that propositions like $0 = 2 \to 0 = 1$ should *fail*." We pass, at least for now, from relevant implication to *entailment* to make that observation stick. As, we suppose,

[23] Another way to make the same point is the following: E#13 says that E# $\vdash t \to (A \to A)$. So this is *true*, by the soundness theorem. What Restall saw was that, if we allowed ourselves unlimited appeal to δ on members of TR, we should have $A \in TR \Rightarrow \Box A \in TR$. That is *not the idea* from our present perspective.

Anderson and Belnap might have advised from the outset!

Appendix 1. How *fallacies of modality* showed up *rejected* in R# and R##.

For proof that diamonds (e.g., unequations) imply *no* boxes (e.g., equations) in arithmetics based on R, consider the chain ④ $= \langle \mathrm{F, t, f, T} \rangle = \langle 0, 1, 2, 3 \rangle$, totally ordered as usual. ④ is a *DeMorgan lattice*, on Dunn's definition in [6], if we set $b \wedge c = \min(b,c)$ and $b \vee c = \max(b,c)$; moreover, define negation[24] by setting $\sim b = b \rightarrow f$ on the *implicative extension* to be introduced immediately, with the following → table (due originally to Church):

→ table for the DeMorgan monoid ④

→	F	t	f	T
F	T	T	T	T
*t	F	t	f	T
*f	F	F	t	T
*T	F	F	F	T

④ is now not merely a DeMorgan lattice but a *DeMorgan monoid* in the sense of [6], taking t as the monoid identity and *fusion* o as defined by $b \circ c = \sim (b \rightarrow \sim c)$. DeMorgan monoids *verify* all theorems of R, since for each theorem B and algebraic interpretation I, $t \leq I(B)$. We may verify also all the theorems of R##, on an *extremely* simple-minded plan. Just assign t to each atomic formula $v = w$ and let $\forall x A$ have the value of A. But as identities are assigned t their negations will be assigned f; whence $s \neq u \rightarrow v = w$ will get in ④ the value $f \rightarrow t = F$, for all terms s, u, v, w. So in this sense no "fallacies of modality" are theorems of R## (or of its subsystem R#). We express again our shock. R is *not supposed* to have any doctrine of modality. In spite of itself, it *does*. And all this is grist to the mill of E, which is formulated to avoid modal fallacies.

Appendix 2. What are some *other* modal arithmetics to which these ideas apply?

We have set out arithmetics based on E in some detail. But save for the *interplay* between relevance and modality at which E aims, there is nothing *that* special about our choice. Still in the relevant ballpark, for example, we might have preferred the system NR of [11] as our vehicle to axiomatize arithmetic.[25] We might still avoid fallacies of modality as in section 4

[24] Numerically, this means that $\sim b = 3 - b$ for all b in $\{0, 1, 2, 3\}$.

[25] On the conventions of [2] NR is called $R^{\Box}$.

above. This would produce a set of truths of TNR#, applying a *metavaluation* to an NR## as we did to E## in section 5. Going even further afield we might follow Shapiro [12] in metavaluing what we'd call S4## to get a TS4#.

Appendix 3. What crazy modal notions does *strenge* arithmetic enjoin?

There is an *old view* of how modality enters our understanding of the world. This view (to be traced to Plato, Leibniz, Hume, Kant, and the gang) says that *necessary truth* is the unique province of *mathematics* and of *logic*; whatever is true in these areas is true *of necessity*; other truths (e.g., about the *world*) just *happen* (more or less) to be true.

We mention all this just because our research into *strenge* arithmetic does *not* seem to confirm it. It is perfectly conceivable, we have said above, that there are *truths* of TE# which are *not necessary*.[26] It is probably not surprising that our paradigm instances of these truths are about entailments that *fail*. An idea that we could have, perhaps, is that $0 = 2$ *might have entailed* $0 = 1$; but, in sober fact, it doesn't. Anyway, we enjoin our readers to develop a Complete System of the World, starting from TE#.

Meanwhile, there is another set of modal notions —those that accompany the so-called "logics of provability" in [10] and elsewhere— that it is interesting to attempt to wed to relevant logics. As a prominent place is reserved in these logics for "Löb's Rule" (which is the inference from $\Box A \to A$ to A), readers may be pardoned if they can contain their enthusiasm for the idea. Still, as provability logics derive much of their motivation from the arithmetization of metamathematics, we would like to know how these things work out in the context of (say) E#. We know already that they work out *classically*; as E# contains P# as its truth-functional part, it also contains classical provability logic, classically expressed. "But what," we hear you say, "is relevant about that?" We'd like to know, too. Something more *sui generis*, appealing to properties of a relevant $\to$, would be more interesting still. Having looked with Mares into a relevant provability logic based on R in [15], we challenge you to advance the subject.

Appendix 4. Why do we insist on Ackermann's Rule γ?

Following earlier work by Anderson and Belnap, [2] takes some pleasure in chopping away at Ackermann's rules. [2] chops specifically the rule $\supset$E, which is Ackermann's γ. So when R# was first formulated in [3] it also did without γ, in the hope that (as in our work with Dunn on R and E, etc., and in much that Meyer has pursued since) this rule would turn out *admissible* anyway for R#. But that hope was too sanguine, in view of the Friedman

[26] It is certainly the case that there are *rules* of E## that are *inadmissible* in TE#. Cf. our discussion of rule δ above.

counterexample QRF of [8]. (Mints suggested in conversation that perhaps we should simply add γ also to R#, as we added it above in formulating E#. Well, maybe.) For as the "disjunctive syllogism" strikes us as a generally OK inference (against views such as those expressed in [2] that it is relevantistically awful), we have followed Ackermann here in restoring it to the primitive logical equipment. But we do so with a definite lack of enthusiasm. For (like Gentzen's cut, as Dunn said for us in [13]) we take it as a sign of the stability (and niceness) of a system that γ should be admissible therein without being primitive. As the argument of [1] can be adapted for E##, we don't need a *primitive* γ for strenge ω-arithmetic; for γ is anyway *admissible*, whence E## will *still* be stable and nice. But we cannot so drop γ from E#. 'Tis a pity.[27]

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