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QUANTUM EXPERIMENTS AND THE LATTICE OF ORTHOMODULAR LOGICS*

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1. *Introduction*

In [36] Birkhoff and von Neumann construct a logical system able to avoid inconsistences coming from paradoxes of quantum mechanics. Due to its philosophical roots all of quantum deductive systems should be considered a paraconsistent logic. The aim of this paper is to show how to construct an algebraic semantics for classical and quantum propositional logics starting from some physical experiments and to then investigate the obtained systems in a purely algebraic way. The notions and results presented in the first part of this paper come from C. Piron [64], [77], (see also M. Majewski [78]). In the remaining parts we present some properties of logical systems determined by experiments of linear polarization of light. In particular, in the second part we present some advanced algebraic results describing properties of modular ortholattices determined by the lattices MOn (see the picture 2). This part of the paper forms a base for the last section, where we show some logical and philosophical consequences of the algebraic results.

The third part of the paper contains some basic notions and results of the theory of logical consequence. From the point of view of our considerations, most important among them is the notion of the lattice of strengthenings of a given logic (logics stronger than a given logic). Our main aim here is to link technically sophisticated results from the second paragraph with philosophical notions important for logic. Theorem 2 links the algebraic notion of quasivariety with, very important from the logical point of view, binary relation determining a deductive strength of a given logical system. The results presented here are known in mathematics. There is an extensive literature about their generalizations and consequences (see for example Blok, Font, Pigozzi [2000]). However their philosophical consequences still remain underestimated. The fourth part is more philosophical. It is devoted to a detailed presentation of the logical and philosophical consequences of

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results from the second part. We make use there of the result on duality between universal algebras and operations of logical consequences.

2. *Physics: quantum experiments*

Let's take after C. Piron [77] the realistic point of view:

> We take the realistic point of view. The system is and it is what it is. It has different properties and whether these properties are known or not by somebody does not change anything to the reality itself. We have to describe these properties and not to explain how the physicist can increase his information about the system. In fact when the physicist believes that the system possesses a certain property he checks up by performing an experiment. If he obtains what he expects he is reinforced in his beliefs but if he obtains something else he must admit his error and change his belief. We shall define a property by the corresponding test.

A physicist investigates a physical system. The properties of the system don't depend on the fact that they are known for someone or not —they are real. By performing experiments physicists try to enlarge their knowledge of the system. Experiment brings new information about the properties of the system. We will consider exclusively experiments which confirm or reject some hypothesis posed in advance. As a matter of fact they could be considered as a questions posed to Nature, questions which admit only two possible answers —"yes" and "no". Any experiment of this form will be called after Piron *a question*. We will say that a question is *true*, if its result (as experiment) is positive and *false*, if it is negative.

Let 1 and 0 denote constant questions, which should be answered respectively "yes" and "no". Let's assume moreover that for any questions α and β, $\alpha \leq \beta$ if and only if the question β has the answer "yes", when α has an answer "yes". The question α and β are called *equivalent* $\alpha \equiv \beta$ if an only if $\alpha \leq \beta$ and $\beta \leq \alpha$. As one can easily observe, equivalent questions have the same answers. The relation $\equiv$ is a congruence on the set of all questions. Its equivalence classes $a = [\alpha] = \{\beta : \alpha \equiv \beta\}$, will be called *propositions*. A proposition is *true*, if some (or equivalently any) question corresponding to it is true. Otherwise it is *false*. Propositions which are true in a given system correspond to the actual properties of the system, other proposition describe potential properties of the system. By a *negation* of the question α we will mean a question $\neg\alpha$ such that $\neg\alpha$ is true if an only if α is false. By negation of a proposition $a = [\alpha]$ we mean a proposition $a' = [\neg\alpha]$.

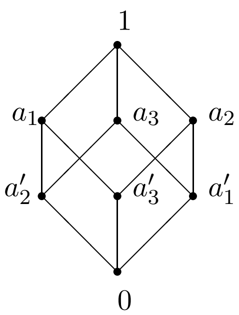
Theorem 1. (Piron [78]) *The relation $\leq$ determines on the set of all propositions a structure of complete lattice L. We will call it a lattice of system. Let $\wedge$ and $\vee$ denote respectively the greatest lower bound and the last lower bound. Then:*
a) $a \wedge b$ is true if an only if a is true and b is true.
b) $a \vee b$ is true if a is true or b is true.
c) The distributivity of L is the necessary and sufficient condition to satisfy the following: $a \vee b$ is true if and only if a is true or b is true.
d) If for any a a is true or a' is true then L is complete and atomic Boolean algebra.

Example 1. Classical system —weighting. We weight the objects to classify them into one of three categories, light (less than 1 kg.), middle, weight (heavier than 10 kg.) Let's consider following question-experiments:
α_1: Is a given object light? α_2: Is it middle? α_3: Is it heavy?

Let's observe that propositions determined by the question above satisfy the following conditions:

$$a_1 \wedge a_2 \wedge a_3 \equiv 0, \qquad a_1 \vee a_2 \vee a_3 \equiv 1, \qquad (a_1 \wedge a_2)' \equiv a_3.$$

The lattice of this system is the eight-element Boolean algebra (see picture 1). As a consequence of theorem 1 (a, c, d) the lattice operation of the least upper bound and the greatest lower bound correspond to the classical connectives of conjunction and disjunction. The complementation correspond to classical negation. Hence $a \wedge \beta$ has an answer "yes" if and only if both α and β have an answer "yes". $\alpha \vee \beta$ has an answer "yes" if an only if at least one of propositions α, β has an answer "yes".



pic. 1

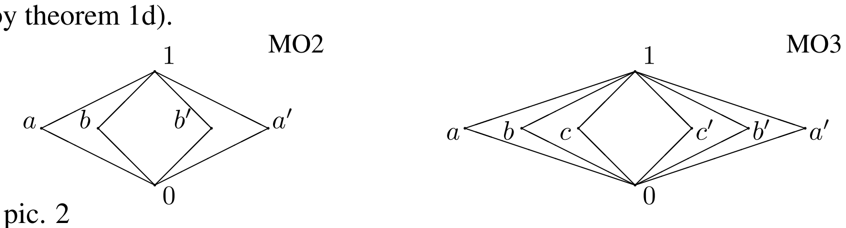
Example 2. Quantum system —linear polarization of light. We send a stream of photons in direction of the polarizator inclined with the gradient α with respect to the polarization plane. Dependently on the gradient α, a given

photon get through the polarizator or not. $Z \subseteq [0, \pi)$ denotes the set of all admissible gradients. Hence, Z can denote the set of all numbers of the interval $[0, \pi)$ or any of its subsets (also finite subset). The questions for this system are:
α_ϕ: Did a photon get through the polarizator inclined with the gradient ϕ, $(\phi \in Z)$.

The experiment allows us to set it up so that it is impossible to get a series of photons which get through a polarizator inclined with the gradient ϕ as well as through a polarizator inclined with another gradient. By the theorem 1b) for $\phi \neq \psi$ we have $\alpha_\phi \wedge \alpha_\psi = 0$. It is also easy to show that $\neg\alpha_\phi = \alpha_{\phi+\frac{\pi}{2}}$

In the lattice of the system an operation of complementation is defined as $\alpha'_\phi = \alpha_{\phi+\frac{\pi}{2}}$. The final form of the lattice depends on the cardinal of the set Z. If it is two-element, the lattice of the system is the lattice MO2 depicted below. If it has n elements it determines a lattice MO2n (for n=3 it is a lattice MO3 from the picture below). If Z contains all the numbers from the interval $[0, \pi)$ the lattice of the system contains continuously many atoms which are coatoms at the same time.

Let's observe that all those lattices are not distributive. As a consequence an operation of the least upper bound doesn't correspond to the classical alternative. We have here only one-sided implication (theorem 1b,c). Also the operation of complementation doesn't correspond to classical negation (by theorem 1d).



pic. 2

3. *Algebra: orthomodular lattices*

An algebra $A = (A, \vee, \wedge, ')$ with two binary operations $\vee$ and $\wedge$ and one unary operation $'$ will be called an *ortholattice* if and only if $(A, \vee, \wedge)$ is a bounded lattice (i.e. a lattice with the smallest element 0 and the greatest one 1), and $'$ is anti-monotonic operator on A (i.e. an operator satisfying the condition $a \leq b$ if and only if $b' \leq a'$. This condition is equivalent to any of the de Morgan laws:

$$(a \wedge b)' = a' \vee b'$$

$$(a \vee b)' = a' \wedge b'$$

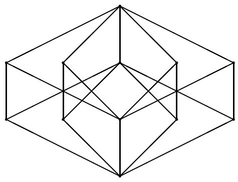
An ortholattice satisfying the identity $x \vee (x' \wedge (x \vee y)) = x \vee y$ will be called an *orthomodular lattice*. The family of all orthomodular lattices is definable by means of a set of identities hence it is a variety. For the basic notions and results about varieties we refer to any monograph in Universal Algebra see for example S. Burris R. Shankapanavar [81]. For the aims of this paper it is only important that varieties are some special classes of algebras connected with logical notions by the results presented in the next paragraph. The variety of all orthomodular lattices will be denoted by OML.

A lattice is modular if and only if it satisfies the following identity: $x \leq y$, $x \vee (y \wedge z) = y \wedge (x \vee z)$. It is easy to show that any modular ortholattice satisfies the law of orthomodularity and hence it is an orthomodular lattice. The class of all modular ortholattices will be denoted my MOL. Thus any Boolean algebra is a modular ortholattice and any modular ortholattice is an orthomodular lattice.

For the complete study of the subject we refer to Beran [84] and Kalmbach [83]. Let's here recall the mathematical origins of the notions defined above. The measurable properties (observables) of the physical system closely correspond to closed subspaces of a Hilbert space. This is why the structure of closed subspaces of a Hilbert space is crucial for the description of a physical system. They form a lattice with respect to set-theoretic meet as the greatest lower bound and the closed subspace spanned on the given closed subspaces as the least upper bound. It is easy to show that for one or two dimensional Hilbert space the lattice of closed subspace forms a Boolean algebra (two element one and four-element one respectively). For higher (but still finitely) dimensional Hilbert space, the lattice of all closed subspace is not distributive but it still forms a modular ortholattice. For an infinitely dimensional Hilbert space the set of all closed subspaces is not modular, however it still satisfies the law of orthomodularity as far as the Hilbert space is separable. For nonseparable Hilbert spaces the lattice of closed subspaces is not orthomodular but it still always forms an ortholattice.

All the lattices of the system of linear polarization of light described in the first paragraph are modular ortholattices. They appear to be especially important from point of view of algebraic investigations. Let's recall: MOn for $n \in \omega$ and $MO\omega$ denotes respectively modular ortholattices with $2n$ (respectively ω- infinitely countably many) pairwise incomparable elements with added zero and unit elements.

Gudrun Kalmbach in [74] has proved that any nontrivial (i.e. larger than the class of Boolean algebras) variety of orthomodular lattices contains the lattice $MO2 \times MO1$ (see picture 3). Greechie has proved (see Kalmbach [83]), that no variety lies between the variety of Boolean algebras and the variety determined by $MO2$.



$MO2 \times 2$

pic. 3

In [86] Roddy proved that the varieties determined by ortholattices MOn for $n \in \{0, 1, ..., \omega\}$ form an initial chain in the lattice of varieties of orthomodular lattices. A stronger result is presented in J. Malinowski [90] where a full description of the lattice of sub-quasivarieties of $MO\omega$ is given. It appears that its structure is relatively clear. And all of its elements are generated by the Cartesian product of two different lattices of the form MOn. For the sake of completeness this lattice is depicted below. For clarity's sake some shorthand has been introduced there. Single numbers $0, 1, 2, ...$ denote quasivarieties determined by respective lattices: 0 denotes the quasivariety determined by $MO0$ —the one-element lattice. 1 denotes the quasivariety determined by $MO1$ —four-element Boolean algebra. It is just a class of all Boolean algebras. 2 denotes quasivariety determined by the lattice $MO2$ and so on. Generally in the whole picture, the higher is a given quasivariety located, the larger (in the sense of inclusion) is a given quasivariety. Let's pay attention that the quasivarieties described by Roddy [86] are just $0, 1, 2, ...$ The lattice contains however many more other quasivarieties which are denoted with pairs n, m of numbers. Each of them is determined by the Cartesian product of the lattices MOn and MOm (a lattice from picture 3 shows a Cartesian product of lattices $MO2$ and $MO1$. The description below allows us to tell how big this lattice is. For example the lattice of quasivarieties of containing in quasivariety determined by MOn has exactly $\frac{n(n+1)}{2}$ elements.

4. *Logic: consequence operation*

The approach to logic and the results presented in this section form an important part of the heritage of Polish School of Logic. They come from many authors. Due to lack of space we are going to present here neither the full history of them nor all the references. For them we refer to the most complete monograph on the subject —the book Wójcicki [87]. The most important of the results presented below comes from J. Czelakowski [81] (see also J. Malinowski [89]). Let L denotes a sentential language with connectives $\vee$, $\wedge$, $'$ and sentential variables p, q, r,.... Algebraically oriented logicians often define a sentential language as an absolutely free algebra. This is not just a sophisticated manner of expressing relatively simple notion. As a consequence of such a definition we can properly describe some very important properties of the language. Most important of them is that by defining a sentential language as an absolutely free algebra we can easily prove that two sentences are identical just if they are identical as a sequences of symbols. This way the very technical notion of free algebra could be intuitively understood as an algebra which is "free" from any irrelevant connections between its elements. Other definitions of the sentential language, although often used in the literature don't explicitly guarantee this property of the language. There are also other important consequences of such a definition of the language. Such an approach to the language allows us to define the valuation as the unique extension of a given function defined on the set of sentential variables. Having in mind all the restrictions above, for the sake of simplicity we will consider a language just as a set of well formed sentences.

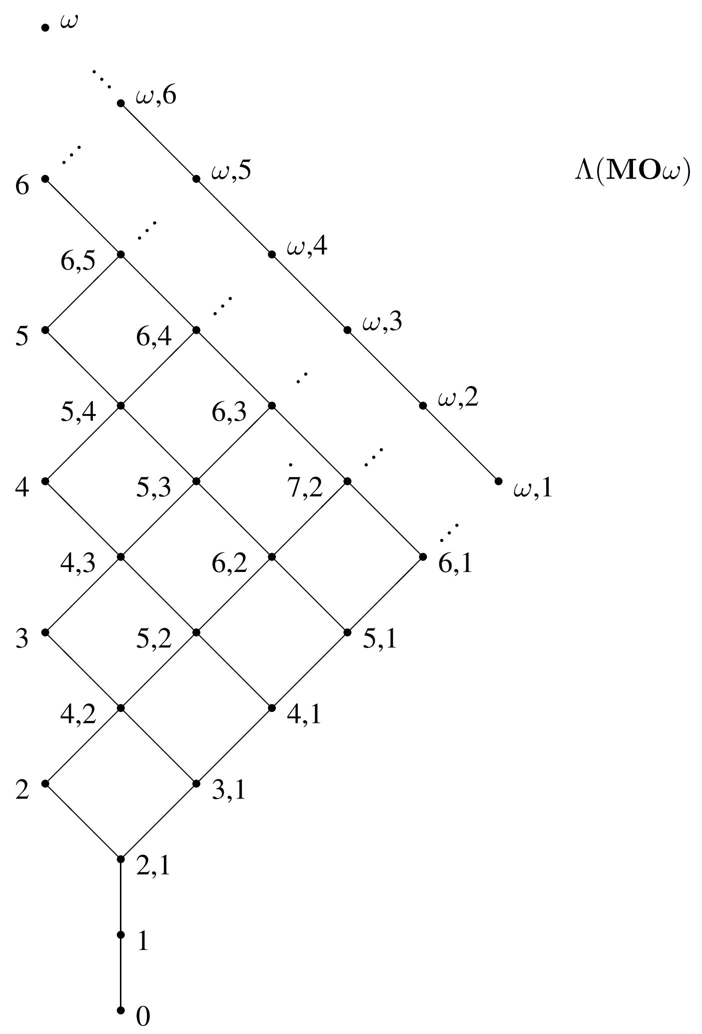
An operation which correspond sets of sentences to set of sentences satisfying conditions:

$X \subseteq C(X)$,
if $X \subseteq Y$, then $C(X) \subseteq C(Y)$,
$CC(X) = C(X)$

will be called a *consequence operation*. If C satisfies moreover the following structurality principle: $e(C(X)) \subseteq C(e(X))$ for any substitution e (i.e. any homomorphizm of the language into itself.) then C is called a structural consequence operation or a *logic*. A consequence operation C is *finitary*, if and only if for any set of sentences $X \subseteq S$ and sentence α if $\alpha \in C(X)$ there exists a finite set Y of X such that $\alpha \in C(Y)$.

The notion of logic as a structural consequence operation is one of most important logical notions. It gives a general framework which allows to investigate the general properties of logical systems. The investigations within of this framework belong to the heritage of the Lvov-Warsaw school of logic. This approach allows the formulation of problems and the investigation of subjects which seem impossible to formulate in other approaches to logic. Let's recall some examples of questions of this form: What could we tell about the properties of logics in which we can define an implication connective? The complete analysis of this problem is given in H. Rasiowa [74]. Other important question concern the characterization of the logics allowing a connective of equivalence —just the equivalential logics (J. Czelakowski [81], J. Malinowski [89]). Just this class of logic is important for considerations of this paper.

A set of sentences X will be called a *C-theory* if and only if $X = C(X)$. The set of all theories C-theories will be denoted by Th_C.



pic. 4

It is worth here to pay attention to two competing ways of understanding logic. If contemporary logic is the only concern then, historically, the first approach was to consider logic as a set of logically valid sentences. A paradigmatic example is here a set of classical tautologies. According to this approach a logical system is a set of sentences closed under substitution and some rules of inference. A logic considered as a logical consequence operation (or relation) is presenting by the competing approach. It formalizes not the set of logically valid sentences but just the general principles of reasoning. Both the approaches are not equivalent to each other. Starting from the logical consequence we will uniquely obtain the set of logically valid sentences as a set of consequence of the empty set of premises. On the other hand starting from the set of logically valid sentences we cannot uniquely determine the consequence operation. Usually for a given logical system there exist a number of consequence operations having that system as a set of consequences of the empty set. Thus logical validity doesn't determine rules of reasoning.

Let's also pay some attention to the distinction between C-theory and a logical system. In particular cases a logical system is usually defined by means of axioms and rules of inference. The rules allow proving that some sentences are logically true without being axioms. Similarly a structural consequence operation is often defined by means of some set of axioms and rules of inference. Given C-theory is then the set of sentences derivable from a given set of premises. Despite superficial similarities the logical system and C-theory have quite different status. The logical system contains exclusively logically valid sentences, C-theory contains also contingent sentences (i.e. such sentences which depending on interpretation can be true or false). Rules of inference have in both cases quite different character and cannot be identified. An especially suggestive example is the rule of necessitation in the modal consequence operation. "From α infer $\Box\alpha$". This rule is commonly accepted in all normal modal systems. It allows getting logical truth from logical truth. However it cannot be applied to contingent sentences. In such cases it gives paradoxical consequences. For example from "It is raining" we would get "It is necessary that it is raining" which is rather impossible to justify.

A logic C' is called a *strengthening* of a logic C, (in symbols $C \leq C'$), if for any set of sentences $X \subseteq S$, we have $C(X) \subseteq C(Y)$. The relation $\leq$ defined this way on the set of strengthenings of a given logic C is a relation of partial order, moreover the family of all strenghtenings of C form a complete lattice with respect of this order we will call it a *lattice of strengthenings* of the logic C. If $C \leq C'$, then we will tell that C is *weaker* than C', and C' is *stronger* than C. The stronger a given logic is, the fewer theories it has because for any logics C_1 and C_2 for $C_1 \leq C_2$ it is necessary and sufficient that $Th_{C_2} \subseteq Th_{C_1}$.

Thus a logic is stronger than another logic if and only if it allows to deduce more from a given set of premises. Of course it happens quite often that two logics are incomparable in this respect. As the historically first example of investigations of the lattice of strengthenings one should recognize the theorem on maximality of classical logic. Thus classical sentential logic has the property that the only logic stronger from it is trivially inconsistent. The lattice of strengthenings of classical logic has then just two elements. We will present this example in more detail in the last part of this paper where also much more complicated lattices of strengthenings will be presented.

A logical matrix is a pair: $M = (A, D)$, where A is an algebra (with the operations corresponding to the connectives of considered language), and D a subset of A its elements are called *designated* elements. The language and matrix are linked by means of valuation —just the homomorphisms of the language into the algebra A.

For any class of matrices K an operation Cn_K is defined in the following way:

> $\alpha \in Cn_K(X)$ if and only if for any matrix from K and any valuation if all the sentences from the set X take designated value, then α also takes a designated value.

is a logic, i.e. structural consequence operation. One can prove that any logic can be represented by means of some class of matrices.

Let C be a logic, a matrix M will be called *C-matrix*, if and only if $C \leq Cn_M$. A matrix M is called *simple* if and only if the identity relation is the only matrix congruence on it. Let C be a logic, the class of all simple C-matrices will be denoted by $Matr^{\star}(C)$. Any logic C is uniquely determined by the class $Matr^{\star}(C)$. The logic is finitely equivalential if some finite set of sentences satisfies in this logic the natural properties of equivalence: reflexivity, symmetry, detachment and substitution (see Malinowski [1989] for details).

Theorem 2. Let C be a standard finitely equivalential logic. The lattice of all standard strengthenings of C is dually isomorphic with the lattice of all sub-quasivarieties of $Matr^{\star}(C)$.

The theorem above forms a bridge between a logic and universal algebra. It allows one to reduce an investigation of the lattice of strengthenings of a given logic to the investigation of respective lattice of quasivarieties (and otherwise). After reversing the order, the largest quasivariety corresponds to the weakest logic, but the structures of both the lattices are identical. The least upper bound of two logics corresponds to the meet of corresponding varieties, the greatest lower bound correspond to the least upper bound of two varieties. The facts presented above allow us to use the results of the previous paragraph for description of the lattice of orthomodular logics.

5. *Philosophical consequences*

Any orthomodular lattice, and hence any lattice of physical experiments can be considered a logical matrix. Let's assume that the unique designated is 1 —the greatest element of the lattice. Any class of orthomodular lattices determines a logic in a way described in the previous paragraph. What's very important here, any of those logics is finitely equivalential, and hence the assumptions of theorem 2 are satisfied. As a consequence the structure of the lattice of quasi-varieties described above mirrors the structure of the lattice of logics.

Let's come back to picture 4, considering it now as a lattice of strengthenings of logic determined by the lattice $MO\omega$. Its reverse order causes that the smallest element 0 corresponds to the strongest logic —trivial logic. 1 corresponds to classical logic and the fact that there are no logics between them expresses nothing else but the well known maximality theorem mentioned in the previous paragraph. As an illustration of the duality between the lattice of logics and the lattice of quasivarieties we will pay here more attention to this theorem.

Let $Taut$ denote the set of all tautologies of the classical sentential calculus. Let Z denote a set closed under substitution and derivation, which at the same time is essentially larger than $Taut$. There exists then a sentence α, belonging to Z without being a tautology. Thus for some valuation V the sentence α gives logical value 0. Let α' be a sentence appearing by a substituting $p \wedge \neg p$ instead of any sentential variables for which valuation v takes value 0 and $p \vee \neg p$ instead of variables for which it takes value 1. It is easy to show that the sentence α' is a countertautology —it takes value 0 for any valuation. Let β denote any sentence. The sentence $\alpha' \to \beta$ takes the value 1 under any valuation, because the precedent of this implication takes always the value 0. As a consequence $\alpha' \to \beta$ is a tautology and hence it also belongs to Z. Applying the rule of modus ponens to α' and $\alpha' \to \beta$ we easily get $\beta \in Z$. β is by assumption any sentence, this entails that every sentence belongs to Z. Z is then the set of all sentences.

The reasoning above has been elaborated for a classical logic considered as a set of tautologies. It would be similar if classical logic were considered as a structural consequence operation Cl. Assuming that a logic C is essentially stronger than Cl one can prove that C is trivially inconsistent. The lattice of strenghtenings of the classical logic is then quite simple. It has only two elements: classical logics and trivially inconsistent logics. It forms an initial fragment of the lattice under consideration.

The next (in order from weaker logics to stronger ones) is the logic determined by the lattice $MO2 \times 2$. Here the lattice loses the linearity. Above incomparable logics appear. For example logic determined by $MO2$ and the one determined by $MO3 \times 2$. Above them the lattice gets much more complicated.

Which one of the logics from picture 4 properly mirrors logical principles of quantum mechanics? This question still remains open. Possibly it lies outside of the picture being incomparable with $MO\omega$? No logical investigations could answer those questions. However, the investigation of lattices of concrete quantum experiments could bring us closer to an answer.

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