A USER-FRIENDLY QUANTUM LOGIC

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Quantum logic, a naturally user-hostile system, can benefit from being given a user-friendly and user-familiar look. In this paper I make quantum logic look as similar as possible to Lemmon's well-known version of the classical propositional calculus. (1)

Lemmon's system has ten rules of inference and a definition of the material equivalence connective. In this version of quantum logic we restrict three of the rules, those that discharge assumptions: conditional proof, vel-elimination and reductio ad absurdum. To compensate we add a second definition, that for material implication. One advantage of the system is that we can give some justification for restricting the rules that we restrict. The justification comes from simple facts about quantum systems.

We depart from Lemmon in our treatment of what wffs are. We give a BNF formalism for the context-free language which wffs should form. This formalism includes Lemmon's as a special case. No binding conventions for eliminating brackets are needed. As a bonus it leads to a simple recursive descent parser for wffhood. This is entirely standard, and I incorporated such a parser into a proof-checker for Lemmon-like quantum logic written in PASCAL.

Wffhood

Wffhood is a matter of convention but some conventions are happier than others. Traditionally, logicians take something like Lemmon's line. The class WFFS of wffs is specified recursively from a basis of *atoms* or *propositional variables*. Brackets are forced to proliferate but conventions are supplied which allow one to omit brackets from one's representation of a wff. Thus Lemmon says that (2)

(1) See Lemmon (1971).

(2) LEMMON (1971 pp. 44-48.

$$(P \rightarrow Q) \vee -Q \leftrightarrow --P \,\&\, Q$$

is not itself a wff but may be used as an abbreviated representation of the wff

$$(((P \rightarrow Q) \vee -Q) \leftrightarrow (--P \,\&\, Q)).$$

Lemmon's definition of wffhood is:

(1) any propositional variable (atom) is a wff;
(2) if α is a wff, so is $-\alpha$;
(3) if α and β are wffs, so are $(\alpha \,\&\, \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$ and $(\alpha \leftrightarrow \beta)$;
(4) there are no other wffs.

In a BNF formalism this amounts to

<WFF> : : = *Atom* | - <Wff> | (<Wff> <2op> <Wff>)
<2op> : : = ↔ | → | ∨ | &

Brackets are then omitted in practice in accordance with a convention on the binding of the two-place operators.

The BNF formalism for Lemmon's convention shows just how unstructured it is. Better then to define wffhood so that no bracket-dropping conventions are needed; better then to define WFFS via an operator precedence grammar in BNF form as follows.

<Wff> : : = <Cterm> | <Cterm> ↔ <Wff>
<Cterm> : : = <Dterm> | <Dterm> → <Cterm>
<Dterm> : : = <Kterm> | <Kterm> ∨ <Dterm>
<Kterm> : : = <Nterm> | <Nterm> & <Kterm>
<Nterm> : : = <Factor> | - <Nterm>
<Factor> : : = *Atom* | (<Wff>)

The terminal symbols are the atoms, the logical connectives and the brackets. It is easy to see that whatever is a wff according to Lemmon is a wff according to this definition. It is less easy to see, though nevertheless true, that Lemmon's conventions on binding and bracket-dropping are automatically satisfied.

Sequents, Proofs and the Rules

In the matters of sequent, proof and rule of inference we follow

Lemmon. Thus our proved items are sequents, objects of the form $\Gamma \vdash \alpha$ where Γ is a finite set of wffs and α is a wff. A proof is a finite, non-empty sequence of sequents in which each sequent is derived using one of the rules of inference together with a sequent or sequents appearing earlier in the proof (except in the case of the rule of assumptions). We represent a proof as a sequence of triples, each consisting of a list of premise-names (usually numerals) naming the elements of Γ, a wff and the justification for the step in the proof. The rules are, for us, the important things and we give them as inferences from sets of sequents to sequents. As usual, we let α, β, γ etc. range over wffs, and γ_1, γ_2, etc. range over sets of wffs.

The Rule of Assumptions (A)
Infer $\alpha \vdash \alpha$.

Modus Ponens (MP)
From $\Gamma_1 \vdash \alpha$ and $\Gamma_2 \vdash \alpha \rightarrow \beta$
Infer $\Gamma_1, \Gamma_2 \vdash \beta$.

Modus Tollens (MT)
From $\Gamma_1 \vdash \alpha \rightarrow \beta$ and $\Gamma_2 \vdash -\beta$
Infer $\Gamma_1, \Gamma_2 \vdash -\alpha$.

Double Negation (DN)
From $\Gamma \vdash \alpha$
Infer $\Gamma \vdash --\alpha$.
or
From $\Gamma \vdash --\alpha$
Infer $\Gamma \vdash \alpha$.

Conditional Proof (CP)
From $\alpha \vdash \beta$
Infer $\vdash \alpha \rightarrow \beta$.

&-Introduction (&I)
From $\Gamma_1 \vdash \alpha$ and $\Gamma_2 \vdash \beta$
Infer $\Gamma_1, \Gamma_2 \vdash \alpha \,\&\, \beta$.

&-Elimination (&E)
From $\Gamma \vdash \alpha \,\&\, \beta$
Infer *either* $\Gamma \vdash \alpha$
or $\Gamma \vdash \beta$.

vel-Introduction ($\vee I$)
From *either* $\Gamma \vdash \alpha$
or $\Gamma \vdash \beta$
Infer $\Gamma \vdash \alpha \vee \beta$.

vel-Elimination ($\vee E$)
From $\Gamma \vdash \alpha \vee \beta$; $\alpha \vdash \gamma$ and $\beta \vdash \gamma$
Infer $\Gamma \vdash \gamma$.

Reductio as Absurdum (RAA)
From $\alpha \vdash -\beta \,\&\, \beta$
Infer $\vdash -\alpha$.

Lemmon has a single definition Df. $\leftrightarrow$:

$$\alpha \leftrightarrow \beta =_{\text{df.}} (\alpha \rightarrow \beta) \,\&\, (\beta \rightarrow \alpha)$$

To this we are required to add a definition for material implication Df. $\rightarrow$

$$\alpha \rightarrow \beta =_{\text{df.}} -\alpha \vee (\alpha \,\&\, \beta)$$

Our material implication is none other than the well-known Sasaki hook.

$$\mathscr{L} = \langle L, \leq, 1, 0, \wedge, \vee, \perp \rangle$$

For all a, b, c $\in$ L

(1)			$a \leq a$
(2)	$a \leq b$ and $b \leq a$	$\Rightarrow$	$a = b$
(3)	$a \leq b$ and $b \leq c$	$\Rightarrow$	$a \leq c$
(4)			$a \wedge b \leq a$
(5)			$a \wedge b \leq b$
(6)	$a \leq b$ and $a \leq c$	$\Rightarrow$	$a \leq b \wedge c$
(7)			$a \leq a \vee b$
(8)			$b \leq a \vee b$
(9)	$a \leq c$ and $b \leq c$	$\Rightarrow$	$a \vee b \leq c$
(10)			$a \vee a^{\perp} = 1$
(11)			$a \wedge a^{\perp} = 0$
(12)			$a = (a^{\perp})^{\perp}$
(13)	$a \leq b$	$\Rightarrow$	$b^{\perp} \leq a^{\perp}$
(14) (OM)	$a \leq b$	$\Rightarrow$	$b \wedge (b^{\perp} \vee a) \leq a$.

Fig. 1 The general orthomodular lattice.

We use the usual abbreviations: $\vdash \alpha$ means $\phi \vdash \alpha$; $\alpha \vdash \beta$ means $\{\alpha\} \vdash \beta$; $\Gamma, \alpha \vdash \beta$ means $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma_1, \Gamma_2 \vdash \alpha$ means $\Gamma_1 \cup \Gamma_2 \vdash \alpha$.

Call the resulting natural deduction system for quantum logic NDQL. We now prove the soundness and completeness of NDQL in the following sense: we show that the Lindenbaum algebra of NDQL is a general orthomodular lattice.

Let $\alpha \dashv\vdash \beta =_{df.} \alpha \vdash \beta$ and $\beta \vdash \alpha$.

It is easy to see that $\dashv\vdash$ is an equivalence relation on WFFS. The soundness and completeness proof for NDQL then amounts to a proof that the structure

$<\mathrm{WFFS}/\|\!\!\vdash, \leq, \mathit{1}, \mathit{0}, \wedge, \vee, \perp>$

where for all $\alpha, \beta \in$ WFFS

$$
\begin{aligned}
[\alpha] \wedge [\beta] &= [\alpha \,\&\, \beta] \\
[\alpha] \vee [\beta] &= [\alpha \vee \beta] \\
\mathit{1} &= [\alpha \vee -\alpha] \\
\mathit{0} &= [\alpha \,\&\, -\alpha] \\
[\alpha]^{\perp} &= [-\alpha]
\end{aligned}
$$

is a general orthomodular lattice, for which see Fig. 1.

Soundness and Completeness

As a simple preliminary lemma we note the &-regularity of NDQL.

Lemma 0 (The &-regularity of NDQL)

$\Gamma \vdash \alpha$ iff $\& \Gamma \vdash \alpha$

where $\& \Gamma$ can refer to any of the conjunctions of all the formulae in Γ. When Γ is empty, $\& \Gamma$ means the empty set.

Proof

By induction on the size of Γ. The base case, for empty Γ, holds by identity. Assuming $\Gamma \vdash \alpha$, $\& \Gamma \vdash \alpha$ follows by repeated applications of

&E. Assuming $\&\Gamma \vdash \alpha$, $\Gamma \vdash \alpha$ follows by CP, &I and MP. Both induction steps naturally use A.

The &-regularity of NDQL enables us to infer

$$\Gamma \vdash \alpha \text{ iff } [\&\Gamma] \leq [\alpha].$$

Theorem 1 (The soundness of NDQL)

$$\alpha \vdash \beta \text{ implies } [\alpha] \leq [\beta]$$

We write $a = [\alpha]$ etc.

Proof

By induction on the length of the proof of $\alpha \vdash \beta$.

For the most part the steps are routine.

Thus, if the length of the proof is one, it must consist of an application of the rule of assumptions A. But $a \leq a$, so the base case is established.

For the induction step we show that any proof of $\alpha \vdash \beta$ may be transformed into a proof concerning the lattice. We give the cases for MP, MT and CP as examples. In each we must use the definition of $\rightarrow$.

MP

We must prove $c_1 \leq a^\perp \vee (a \wedge b)$ and $c_2 \leq a$ imply $c_1 \wedge c_2 \leq b$.

From condition (OM) $a \leq b$ implies $b \wedge (b^\perp \vee a) \leq a$.

But $a \wedge b \leq a$. Hence in (OM), substituting $a \wedge b$ for a, and b for a, one has

$a \wedge b \leq a$ implies $a \wedge (a^\perp \vee (a \wedge b)) \leq b$,

so $a \wedge (a^\perp \vee (a \wedge b)) \leq b$.

Hence by isotony[3]

$c_1 \wedge c_2 \leq a \wedge (a^\perp \vee (a \wedge b)) \leq b$

which is the required result.

MT

We must prove $c_1 \leq a^\perp \vee (a \wedge b)$ and $c_2 \leq b^\perp$ imply $c_1 \wedge c_2 \leq a^\perp$

In (OM), substituting $a^\perp$ for a, and $(a \wedge b)^\perp$ for b we have

[3] Isotony is that lattice property that
$a \leq b$ and $c \leq d$ together imply $a \wedge c \leq b \wedge d$
and $a \vee c \leq b \vee d$.