DROPPING A FEW WORLDS

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Most people think that logic is primarily concerned with the analysis of *arguments* : call them the *argument-supporters*. But that is not the only view. There are people who claim instead that the primary concern of logic is (or should be) logically true *sentences* : call them the *sentence-supporters*.(1) Though more numerous, the argument-supporters do not have any substantial argument (!) for their position. Indeed, whatever little arguments are available in the area seem to go against them. For example, one might want to require a certain level of decidability for something to count as a logic: say recursive enumerability. In that case, it would certainly be of some comfort for the sentence-supporters that there are logical semantics (for example, the supervaluational semantics for free quantification theory) in which the set of logically true sentences is recursively enumerable but the set of valid arguments is not.(2) At any rate, if we disregard such fairly esoteric (and to some extent deviant) contexts, and concentrate on the standard core of logic, on logic as is taught in standard introductory courses, it is not clear how either view *could* be argued for. For in that standard area there is a canonical association of arguments with (conditional) sentences,(3) and a canonical associa-

(1) I have found both attitudes well represented among my logic colleagues and in logic textbooks. But in a way their being represented is beside the point: my concern here is theoretical, not historical.

(2) The non-recursive enumerability of valid arguments in supervaluational semantics has been recently proved by Peter Woodruff, in a paper forthcoming in the *Journal of Symbolic Logic*.

(3) With valid arguments having infinitely many premises, this association requires compactness (which of course holds in the area we are talking about), as well as some canonical way of singling out exactly one finite subset of premises that still entails the conclusion (an alphabetical ordering of sentences would do).

tion of sentences with (premiseless) arguments, such that an argument is valid if and only if the associated sentence is logically true, and a sentence is logically true if and only if the associated argument is valid. In view of these equivalences, it really looks like it is indifferent to logic whether one should consider arguments or sentences to be logic's primary concern. From the equivalences in question, one might want to conclude that the matter can only be settled (if at all) in a larger context, or maybe that it should be left entirely to personal taste.

In the present note, I propose an argument in favor of the argument-supporters. The argument is well within philosophy of logic: it does not depend on taking any position on larger issues. Essentially, the argument shows that taking arguments to be logic's primary concern does justice to our intuitions about logic in a way in which taking sentences to be logic's primary concern does not. The argument does not depend on any esoteric or deviant conception of logic either: indeed, I will formulate it with exclusive reference to the most standard of logical systems, classical propositional logic. And finally, though the argument is somewhat technical in nature, I am not claiming here any new technical result. It is likely that the technical facts I will point out have been noticed before, but to my knowledge they have never been put to the *use* I suggest here. And of course, it is this use that I am presently concerned with.

So take classical propositional logic. Its primitive vocabulary includes an infinite list of sentence letters (among which p, q, r, ...), truth-functional connectives, and parentheses. Its sentences include things like

(1) $(p \& q) \supset r$,

and its arguments things like

(2) $p \vee q$
$\underline{\neg p}$
$\therefore q$

To determine what sentences are logically true and what arguments are valid, the most basic intuitive notion, here as everywhere else in logic, is the notion of a possible world. A sentence is logically true if and only if it is true in all possible worlds. An argument is valid if and only if in all the possible worlds in which its premises are jointly true, its conclusion is true as well. To move from these intuitions to a formal definition of validity and logical truth, however, we need a formal counterpart for the notion of a possible world. What, then, is to count as such a counterpart in classical propositional logic?

There is a very natural answer to this question. Given that the only elementary non-logical expressions here are sentence letters, and that the truth-value of a complex sentence in a world is entirely determined by the truth-values of its components (and hence ultimately of its atomic components) in that world, a possible world only counts to the extent to which it assigns certain truth-values to sentence letters. If two possible worlds assign the same truth-values to all sentence letters, no harm will result in this context from thinking of them as one and the same world. Thus in classical propositional logic, possible worlds may be *identified* with assignments of truth-values to sentence letters. Once this identification is made, a formal definition of logical truth and validity is a very simple matter, and it becomes possible to give a precise formulation of the two positions we are presently discussing. According to the argument-supporters, one should define validity first, and then possibly define logical truth in its terms; according to the sentence-supporters, one should go the other way around.

So far so good. But now suppose you simply *drop* some possible world, say the possible world a assigning T to p and F to all other sentence letters. That is, suppose you decide that all assignments *except* a count as possible worlds. You would expect this change in the notion of a possible world to make some difference in your logic, and indeed in some sense it does. The argument whose conclusion is $\neg p$ and whose premises are the negations of all other sentence letters (call it argument (S)) was not valid when all assignments were possible worlds, but is valid now because the only world that could count as a counterexample to it has been dropped. However, it is interesting to notice that dropping a has no effect on the set of logically true sentences. This set remains exactly the same!

To prove the last claim, it is enough to show that

(3) If a sentence A was not logically true before dropping a, it remains not logically true after dropping a.

The converse of (3) is trivial. As for the proof of (3), it goes as follows. Let $p_1, \ldots, p_n$ be all the sentence letters occurring in A, and let q be a sentence letter not occurring in A. There must be such a q because A is finitely long and there are infinitely many sentence letters. Also, let the truth-value of A in a be F (briefly, let $v_a(A) = \mathrm{F}$); if it's not, the proof is trivial. Now there must be an assignment a' such that $a'(p_1) = a(p_1), \ldots, a'(p_n) = a(p_n)$ and $a'(q) \neq a(q)$. It is easy to prove by induction on the complexity of A that $v_{a'}(A) = v_a(A)$, and hence that $v_{a'}(A) = \mathrm{F}$. But then, even if we drop a, there still is at least one counterexample to the logical truth of A.

Indeed, this result can be strengthened. By using essentially the same argument, we can prove that

(4) The set of logically true sentences remains the same if we drop any finitely many possible worlds.

And we can also prove that

(5) The set of logically true sentences remains the same if we drop any denumerably many possible worlds.

To see that (5) is the case, it is enough to observe that

(6) For any distribution of truth-values to finitely many sentence letters, there are uncountably many worlds that agree on that distribution.

So if we drop denumerably many worlds, (6) will still be true, and essentially the same argument as before will establish (5).

Finally, we can prove that, if we choose judiciously, we can even drop uncountably many worlds and still have the same set of logically true sentences. For example, we could drop all possible worlds except those that assign T to finitely many sentence letters and still be able to use the argument above.

So much for the technical facts. It is time now to draw our philosophical moral. If we think of logic as primarily concerned with determining the set of logically true sentences, it turns out that logic is essentially incapable of discriminating among an infinite range of alternative construals of logical possibility. [4] At one extreme of this range there is the set of *all* possible worlds, our "intended model" of logical possibility if you wish. At the other extreme, there are sets that contain only denumerably many worlds, and lack uncountably many of them. In the middle, there are all sorts of intermediate sets, which lack one, two, ... or denumerably many worlds. The situation reminds one of formal arithmetic, where the first-order theory describes not only the intended model but also $2^{\aleph_0}$ nonstandard alternatives, and just as in that case one is led to think here that one has not really "captured" the intuitive notion one was trying to capture (in that case arithmetical truth, in this case logical possibility).

On the other hand, if we think of logic as primarily concerned with arguments, it turns out that even a minimal variation in the set of possible worlds is detected by the theory. Dropping world *a* turns argument (S) from invalid to valid, and any other variation would have similar effects. So when seen in terms of arguments, the theory appears to reproduce our logical intuitions much more faithfully, and to describe *only* the intended model of logical possibility. This, I think, may be a convincing argument for taking arguments (and not sentences) to be logic's primary concern, even in presence of equivalences as the ones I mentioned on p. 1.

These equivalences obscure the primacy of arguments, for they show that the propositional notion of a valid argument is *in fact* just as finitary as (and reducible to) the propositional notion of a logically true sentence. But consideration of alternative possibilities shows that with arguments this factual truth does not represent a theoretical limitation: even if propositional valid arguments are in fact finitary, and hence can be exhaustively dealt with by a finitary proof theory, they *could* be infinitary, and this possibility makes their finitary character more than just a triviality, it makes it informative. To put it otherwise, the problem is not so much that *as things* (that is, as our logical intuitions) *stand* we may express our logic in terms of sentences as well as in terms of arguments. The problem is rather that by using arguments we can specify *how things stand*, and how they

(4) Of course, I am not suggesting that all these alternative construals are "sensible" (though a number of them could turn out to be, for think of non-standard analysis).

could be different, to a larger extent and with greater precision than we can by using sentences.(5)

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(5) It could be suggested that the asymmetry I pointed out disappears if we allow for infinitely long sentences. This is certainly true, but has no bearing on my argument here. For my point is: the asymmetry exists in as central and fundamental a system as classical propositional logic, and hence if we take sentences to be logic's primary concern we will run into trouble (at least) there. The fact that we might not run into trouble elsewhere is irrelevant, unless of course we want to use this fact to argue that classical propositional logic is somehow a defective logical system.