SOME RESULTS IN INTUITIONISTIC MODAL LOGIC

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Several logicians – Fitch, Prior, Bull, Ono(1) – have build systems of intuitionistic modal logic adding to axiomatisations of intuitionistic logic some modal axioms. Bull and Ono have studied the model-theory of these systems.

We propose another approach: we consider that Kripke-models define in a clear and satisfactory way the basic modal notions, but instead of applying to the Kripke-models, as usual, classical logic we apply intuitionistic logic. Therefore we could call our approach "intuitionistic model-theory of Kripke-models". Intuitionistic mathematics starts also with notions taken over from classical mathematics, but uses only intuitionistically valid methods of proof.

In this way we obtain a weak extension to modal intuitionistic logics of the Bernays-Gentzen-Gödel theorem(2) which gives a mapping of classical into intuitionistic logic.

Our modal language has three modal operators: □ necessity, ◇ strong possibility and ⟐ weak possibility(3); contrary to the usual systems of classical modal logic the three are definitionally independent (we could also introduce many other primitive modal operators).

(1) *cf.* the bibliography in Hiroakira Ono: *On some intuitionistic modal logics*, Pub. of the Research Inst. for Math. Sc., Kyoto Univ., vol. 13/2, 1977, pp. 687/722.

(2) We follow Gentzen's proof; *cf. Collected Papers*, Amsterdam 1969. Also Kurt Gödel: *Zur intuitionistischen Arithmetir und Zahlentheorie*, Ergb. eines math. Koll., 4, 1931/32. English translation in Martin Davis, *The undecidable*, New York 1968 and spanish translation in Kurt Gödel, *Obras completas*, edited by Jesús Mosterín, Madrid 1981. Bernays' proof has not been published.

(3) A. N. PRIOR – *cf. Time and Modality*, Oxford 1957, pp. 37/40 – was the first to note this fact. P. N. JOHSON-LAIRD – *cf. The meaning of modality*, Cognitive Science, vol. 2, 1978, pp. 17/26 – has shown the importance of these two possibilities for cognitive psychology.

It contains also the connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$, and sentential variables.

A first-order Kripke-model, KM for short, is a consistent set of first-order formulae with at least one monadic predicate W and two dyadic predicates R and V. R is defined over W^2; the first argument of V is defined over W and the second over all wellformed formulae of our modal language. KM must contain or imply the following formulae:

a) $V(\alpha, \neg F) \leftrightarrow \neg V(\alpha, F)$
b) $V(\alpha, F_1 \wedge F_2) \leftrightarrow [V(\alpha, F_1) \wedge V(\alpha, F_2)]$
c) $V(\alpha, F_1 \vee F_2) \leftrightarrow [V(\alpha, F_1) \vee V(\alpha, F_2)]$
d) $V(\alpha, F_1 \rightarrow F_2) \leftrightarrow [V(\alpha, F_1) \rightarrow V(\alpha, F_2)]$
e) $V(\alpha, \Box F_1) \leftrightarrow \forall\beta(\alpha R \beta \rightarrow V(\beta, F_1))$
f) $V(\alpha, \Diamond F_1) \leftrightarrow \exists\beta(\alpha R \beta \wedge V(\beta, F_1))$
g) $V(\alpha, \Diamond\!\!\cdot F_1) \leftrightarrow \neg \forall\beta \neg(\alpha R \beta \wedge V(\beta, F_1))$

According to the intended meaning W is the set of possible worlds, R the accessibility relation between worlds and V the valuation of all formulae in each possible world. Most of all modal logics are defined by KM's; an exception is the logic G which interpretes necessity as formal deducibility, because its accessibility relation is only second-order definable ([4]). We restrict ourselves to first-order Kripke-models because second-order intuitionistic logic has several open problems.

Every family FAM of KM's defines a modal logic in the following way: starting with $V(\alpha, F)$ and applying a) to g) a finite number of times we obtain for each modal formula F and each world α a non-modal, first-order formula $C_{\alpha, F}$ which describes the conditions on the model under which F is true at α. $C_{\alpha, F}$ contains only the predicates R and V, and the latter with a sentential variable as its second argument.

We say that

1) a modal formula F is true at α iff KM $\vdash_{Cl} C_{\alpha, F}$ i.e. its necessary and sufficient condition in terms of the model KM is deducible in classical logic from KM.
2) a modal formula F is true in KM iff KM $\vdash_{Cl} \forall \alpha \varepsilon W\, C_{\alpha, F}$
3) a modal formula F is valid in a FAM iff it is true in all KM ε FAM.

([4]) *cf.* G. Boolos, *The unprovability of consistency*, London 1979, p. 84.

Because of the use of classical logic we have:

$$\mathrm{KM} \vdash_{\mathrm{Cl}} \forall \alpha, p\,(V(\alpha, p) \vee V(\alpha, \neg p))$$
$$\mathrm{KM} \vdash_{\mathrm{Cl}} \forall \alpha, p\, \neg(V(\alpha, p) \wedge V(\alpha, \neg p))$$

i.e. V is a valuation in the model-theoretic sense. If we replace now first-order classical deducibility $\vdash_{\mathrm{Cl}}$ by first-order intuitionistic deducibility (Heyting deducibility) $\vdash_{\mathrm{Int}}$, then this is no longer always the case. But even if KM is so strong that it entails that V is a valuation, it may still be an open undecidable question whether $\exists \gamma\, V(\gamma, D)$ for a certain formula D. In this way using intuitionistic instead of classical logic we weaken the model an there is now place for a strong and a weak possibility.

Theorem 1: The set of formulae intuitionistically valid in a certain FAM of KM's is a normal modal logic, i.e. it is closed with respect to 1) implication and 2) necessitation.

Proof: 1) $\mathrm{KM} \vdash_{\mathrm{Int}} \forall \alpha\, C_{\alpha, F_1 \to F_2} \to (\forall \alpha\, C_{\alpha, F_1} \to \forall \alpha\, C_{\alpha, F_2})$

2) We must show that I) $\mathrm{KM} \vdash_{\mathrm{Int}} \forall \beta\, C_{\beta, \Box F}$

follows from $\mathrm{KM} \vdash_{\mathrm{Int}} \forall \alpha\, C_{\alpha, F}$. But this is trivial considering that I) is equivalent to

$$\mathrm{KM} \vdash_{\mathrm{Int}} \forall \beta\, \forall \alpha\, (\beta R \alpha \to C_{\alpha, F})$$

In these intuitionistic modal logics we have $\Box A \to \Diamond A$ and $\Diamond A \to \diamond\!\!\!\cdot\, A$ but in general not the converses. This is why we may speak of $\Diamond$ as a strong and $\diamond\!\!\!\cdot$ as a weak possibility.

Theorem 2: (weak extension of the Bernays-Gentzen-Gödel theorem) Let $*$ be a mapping defined by replacing in a formula A – modal or first-order – every disjunction $C \vee D$ by $\neg(\neg C \wedge \neg D)$, every existential quantification $\exists x$ by $\neg \forall x \neg$ and every strong possibility $\Diamond$ by the weak possibility $\diamond\!\!\!\cdot$. Then

$$\mathrm{KM} \vdash_{\mathrm{Cl}} C_{\alpha, F} \Leftrightarrow \mathrm{KM}^* \cup \mathrm{Stab} \vdash_{\mathrm{Int}} C_{\alpha, F^*}$$

(Stab is the set of all formulae $\neg\neg P(x) \to P(x)$ expressing the stability of all predicates occuring in KM and in F).

Proof: let us suppose that

$KM \vdash_{Cl} C_{\alpha, F}$

because of the fact that classically A and A* are equivalent we get

$KM^* \vdash_{Cl} (C_{\alpha, F})^*$ and by the Bernays-Gentzen-Gödel theorem

$KM^* \cup Stab \vdash_{Int} (C_{\alpha, F})^*$

Applying now an induction over the length of F we can prove

$KM^* \cup Stab \vdash_{Int} (C_{\alpha, F})^* \leftrightarrow C_{\alpha, F^*}$

1) $F \equiv p$ trivial
2) $F \equiv \neg F$

$$KM^* \cup Stab \vdash_{Int} (C_{\alpha, \neg F})^* \leftrightarrow (\neg C_{\alpha, F})^* \leftrightarrow \neg (C_{\alpha, F})^*$$

by the inductive hypothesis

$$\leftrightarrow \neg C_{\alpha, F^*}$$
$$\leftrightarrow C_{\alpha, \neg (F^*)}$$
$$\leftrightarrow C_{\alpha, (\neg F)^*}$$

All other cases are proved in the same way.
But then

$KM^* \cup Stab \vdash_{Int} C_{\alpha, F^*}$ q.e.d.

In this way we have reduced the problem of a modal formula F being classically true in a certain KM to the corresponding problem of F*, classically equivalent to F, being intuitionistically true in KM*. The supplementary stability hypothesis, which are classically vacuous, are weaker intuitionistically than the corresponding assertions of decidability. In fact

$$\vdash_{Int} (A \vee \neg A) \to (\neg \neg A \to A)$$

but the converse implication is not valid. KM* may be intuitionistically weaker or stronger than KM; this is why we dont get the full extension of the Bernays-Gentzen-Gödel theorem.

As we pointed above, we can introduce other primitive modal operators; for example $\boxdot$, the necessity induced by weak possibility:

b) $V(\alpha, \boxdot F) \leftrightarrow \neg \neg \forall \beta \neg \neg (\alpha R \beta \wedge V(\beta, F))$

In all intuitionistic modal systems the following formulae are valid, but the implications are not in general reversible:

$$\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$$
$$\boxdot(A \wedge B) \leftrightarrow (\boxdot A \wedge \boxdot B)$$
$$\Diamond A \vee \Diamond B \to \Diamond(A \vee B)$$
$$\dot{\Diamond} A \vee \dot{\Diamond} B \to \dot{\Diamond}(A \vee B)$$

MIPC([5]) is a system of intuitionistic modal logic – an intuitionistic S5 – proposed by Prior and studied by Bull, which is obtained from an axiomatisation of intuitionistic propositional logic by adding the following rules:

$$1)\ \frac{A \to B}{\Box A \to B} \qquad 2)\ \frac{A \to B}{A \to \Box B} \qquad 3)\ \frac{A \to B}{A \to \Diamond B} \qquad 4)\ \frac{A \to B}{\Diamond A \to B}$$

(A in 2) and B in 4) must be fully modalised)

Let us call MCPC the system with the same four rules but based on an axiomatisation of classical propositional logic.

Theorem 3 (extension of the Bernays-Gentzen-Gödel theorem): Let + be a mapping defined by replacing in a formula of the language of MIPC every disjuntion $C \vee D$ by $\neg(\neg C \wedge \neg D)$, every porpositional variable p by $\neg\neg p$ and every possibility $\Diamond$ by $\neg\Box\neg$.

Then,

$$\vdash_{MCPC} A \Leftrightarrow \vdash_{MIPC} A^{+}$$

Proof: a) from right to left, trivial; b) from left to right. We use induction on the number of modal rules employed in the proof in MCPC of A. As $\neg\neg p$ is stable and stability is inherited by negation, conjunction and implication, we only need to show that it is also inherited by necessitation. But

$$\dfrac{\dfrac{\dfrac{A \to A}{\Box A \to A}\ \text{rule 1)}}{\neg\neg\Box A \to \neg\neg A} \qquad \neg\neg A \to A}{\dfrac{\neg\neg\Box A \to A}{\neg\neg\Box A \to \Box A}\ \text{rule 2)}}$$

([5]) R. A. BULL, *MIPC as the formalisation of an intuitionist concept of modality*, Jour. of Sym. Logic, 31/4, 1966, pp. 609/616.

If to get an intuitionistic S4 we change the rules 2) and 4) – A in 2) must by begin a necessity and B in 4) by a possibility –, then theorem 3 is no longer provable because stability fails to be inherited by necessitation.

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