ON DEDUCTIBILITY OF THE AXIOM OF CHOICE FROM COMPREHENSION AXIOMS WHICH CONTAIN $\mathscr{E}$-OPERATOR

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There is a close connection between the use of Hilbert's $\mathscr{E}$-operator in certain axiomatic systems and the deductibility of the axiom of choice in these systems. This connection was founded by R. Carnap (1), who proved the following theorem.

The axiom of choice is deductible from Zermelo's Scheme of axioms of subsets by means of classical predicate calculus with equality and $\mathscr{E}$-operator, if it is allowed to consider as particular cases of this Scheme all formulae having corresponding form and containing quantifiers, which bind variables occuring inside $\mathscr{E}$-terms. It is possible to extend this result as follows.

Theorem 1. Let us suppose that an axiomatic system $T_{\mathscr{E}}$ satisfies following four conditions.

1) The only non-logical constant in the language of $T_{\mathscr{E}}$ is binary predicate symbol $\in$.

2) The list of logical constants of the language of $T_{\mathscr{E}}$ contains usual propositional connectives, quantifiers and $\mathscr{E}$-operator. Definitions of well-formed term and formula are standard.

3) All axioms of classical prdicate calculus are axioms of $T_{\mathscr{E}}$. Rules of inferences of $T_{\mathscr{E}}$ are the same as in classical predicate calculus. The list of logical axioms of $T_{\mathscr{E}}$ contains also all formulae having a form

$$\exists x F(x) \supset F(\mathscr{E}_x F(x)$$

where F(x) is an arbitrary formula of $T_{\mathscr{E}}$ and x is a variable.

4) Specific axioms of $T_{\mathscr{E}}$ are two following axioms of comprehension:

$$\forall a\, \exists u\, \forall x\, [x \in u \equiv \exists y\, [x = \mathscr{E}_z(z \in y)\ \&\ y \in a]\ \&\ \exists y\, [x \in y\ \&\ y \in a]] \quad (1)$$

$$\forall a\, \exists y\, \forall x[x \in y \equiv x = a \,\&\, \mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in a)] \tag{2}$$

where $x = y$ is the abbreviation of $\forall z[z \in x \equiv z \in y] \,\&\, \forall z[x \in z \equiv y \in z]$

In that case the axiom of choice is a theorem of $T_{\mathscr{E}}$.

Proof. Let us consider an arbitrary set a, which satisfies conditions:

$$\forall x[x \in a \supset \exists z[z \in x]] \tag{A}$$

$$\forall x\, \forall y\, [x \in a \,\&\, y \in a \,\&\, x \neq y \supset \neg\exists z[z \in x \,\&\, z \in y]] \tag{B}$$

In order to prove theorem 1 it is sufficient to deduce the formula

$$\exists u\, \forall x[x \in a \supset \exists v[v \in x \,\&\, v \in u \,\&\, \forall t[t \in x \,\&\, t \in u \supset t = v]]]$$

from (A) and (B). The axiom (1) implies that there exists the set u such that

$$\forall x[x \in u \equiv \exists y[x = \mathscr{E}_z(z \in y) \,\&\, y \in a] \,\&\, \exists y[x \in y \,\&\, y \in a]]$$

This set is u to be found. The last assertion can be proved as follows. Let us consider an arbitrary member x of a, $x \in a$. It is sufficient for our aim to deduce in $T_{\mathscr{E}}$ from (A), (B) and $x \in a$ the formula

$$\exists v[v \in x \,\&\, v \in u \,\&\, \forall t[t \in x \,\&\, t \in u \supset t = v]]$$

In order to achieve this aim it is sufficient to found v, which satisfies the following condition; $v \in x \,\&\, v \in u \,\&\, \forall t[t \in \,\&\, t \in u \supset t = v]$ One can take as v the term $\mathscr{E}_z(z \in x)$. Let us prove this assertion. The assumption (A) implies that x is non-empty, i.e. $\exists z[z \in x]$. From this fact and from the axiom $\exists z[z \in x] \supset \mathscr{E}_z(z \in x) \in x$ one can draw conclusion that $\mathscr{E}_z(z \in x) \in x$. Since $x \in a$, then $\exists y[\mathscr{E}_z(z \in x) \in y \,\&\, y \in a]$. Formula $\mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in x)$ is a theorem of $T_{\mathscr{E}}$. It is easy to prove formulae $\mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in x) \,\&\, x \in a$ and $\exists y[\mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in y) \,\&\, y \in a]$. Hence $\mathscr{E}_z(z \in x) \in u$. Now it is necessary to prove that $\forall t[t \in x \,\&\, t \in u \supset t = \mathscr{E}_z(z \in x)]$. Suppose t simultaneously belongs to x and u, i.e. $t \in x$ and $t \in u$. Since $t \in u$, there exists a set y such that $t = \mathscr{E}_z(z \in y)$ and $y \in a$. Two cases are possible.

Case 1: y and x are not equal, $x \neq y$. It is easy to prove that $\mathscr{E}_z(z \in y) \in y$. As $\mathscr{E}_z(z \in y)$ and t are equal, then t belongs to y too, $t \in y$. In the same time t belongs to x. But is was assumed earlier that x and y have not any common member (assumption (B)). These statements contradict each other and this contradiction implies that $t = \mathscr{E}_z(z \in x)$.

Case 2: y and x are equal, $x = y$. Let $\bar{y}$ be such that $\forall \bar{x} [\bar{x} \in \bar{y} \equiv \bar{x} = x \,\&\, \mathscr{E}_z(z \in \bar{x}) = \mathscr{E}_z(z \in x)]$. Both $x = x$ and $\mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in x)$ are theorems of classical predicate calculus. So $x = x \,\&\, \mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in x)$ is its theorem too. Then $x \in \bar{y}$. Since $x = y$ and $x \in \bar{y}$, then (by definition of equality) one can get $y \in \bar{y}$. But $y \in \bar{y} \equiv y = x \,\&\, \mathscr{E}_z(z \in y) = \mathscr{E}_z(z \in x)$ and therefore $y = x \,\&\, \mathscr{E}_z(z \in y) = \mathscr{E}_z(z \in x)$. Hence $\mathscr{E}_z(z \in y) = \mathscr{E}_z(z \in x)$. So (by definition of equality) $\mathscr{E}_z(z \in x) = \mathscr{E}_z(z \in y)$. Hence $t = \mathscr{E}_z(z \in x)$. Therefore the last formula is the consequence of both assumptions $y = x$ and $y \neq x$, which was to be demonstrated.

Now our aim is achieved. The proof of the theorem 1 is finished.

The theorem 1 has some important consequences.

Firstly, the axiom of choice is a theorem of any modification of axiomatic systems Z and ZF, which logic coincides with the logic of $T_{\mathscr{E}}$, if it is allowed to consider as particular cases of the Scheme of axioms of subsets all formulae having corresponding form and containing quantifiers, which bind variables occuring inside $\mathscr{E}$-terms. It is true as all axioms of $T_{\mathscr{E}}$, which where explicitly used in the proof of the theorem 1, are axioms of such modifications of z and zF.

Secondly, axioms of choice for different types are theorems of the simple type theory, which logic coincides with the logic of $T_{\mathscr{E}}$ and which specific axioms are all axioms of comprehension written in the extended language of this theory. The proof of this assertion can be obtained from the proof of the theorem 1 by means of restoring of the indices of type, where it is need.

Thirdly, inasmuch as both axioms (1) and (2) are stratified, it is possible to prove the axiom of choice in the axiomatic system $NF_{\mathscr{E}}$, which language and logic coincide correspondingly with the language and logic of $T_{\mathscr{E}}$ and which specific axioms are the axiom of extensionality and all stratified axioms of comprehension written in the language of $T_{\mathscr{E}}$. As the system $NF_{\mathscr{E}}$ is the extension of well-known Quine's system NF, the negation of the axiom of choice is also a theorem of $NF_{\mathscr{E}}$. Actually, in his well-known article (2) Specker demonstrated that in NF is provable the negation of the statement: each set can be well- ordered. But this statement can be deduced in NF from the axiom of choice as it was demonstrated by Rosser (3).

This means that the negation of the axiom of choice is provable in NF. Thus the negation of the axiom of choice is a theorem of any extension of NF, in particular of $NF_{\mathcal{E}}$. So $NF_{\mathcal{E}}$ is inconsistent.

LITERATURE

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