BOUND VARIABLES AND SCHEMATIC LETTERS

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Quine says: '... if we extend truth function theory by introducing quantifiers "(p)", "(q)", "$(\exists p)$", etc., we can no longer dismiss the statement letters as schematic. Instead we must view them as variables taking appropriate entities as values...'. ([1]), p. 118). Quine would certainly agree that this claim is not true if the introduced quantifiers are understood substitutionally. For elsewhere he points out that if the language is extensional and if quantification in the language is substitutional, then that quantification is virtual ([2], pp. 74-75). Quine's point is that if '(p)', '(q)', etc. are not understood substitutionally, the only alternative is to understand 'p', 'q', etc. as taking entities as values.

We argue against Quine on this point. First we describe a language T that results from extending truth function theory by introducing sentence letter quantification. Next we describe a semantics for this language and argue that its account of sentence letter quantification is neither substitutional nor requires viewing the statement letters as taking entities as values.

I

Syntax

1. The vocabulary of T consists of these signs:

   $\pi, \mathrm{N}, \mathrm{C}, \mathrm{p}, \mathrm{P}, '$

2. The sentential variables of T:

   $\mathrm{p}, \mathrm{p}', \mathrm{p}'', \ldots$

3. The sentential constants of T:

   $\mathrm{P}, \mathrm{P}', \mathrm{P}'', \ldots$

4. The formulas of *T*:

(*i*) all propositional constants and variables;
(*ii*) $N\Psi$, if Ψ is a formula;
(*iii*) $C\Psi_1\Psi_2$, if Ψ_1 and Ψ_2 are formulas;
(*iv*) $\pi\alpha\Psi$, if α is a variable and Ψ is a formula;
(*v*) nothing else.

5. We assume the usual definitions of free and bound occurrences of variables and define a sentence of *T* as a formula in which no variable occurs free.

In the rest of the paper we shall use

α, β, Ψ

to range over, respectively, variables, constants and formulas of *T*;

ϕ, ϕ_1, ϕ_2

will range over sentences of *T*; while

Γ, Δ

will range over sets of sentences of *T*. In addition, '$\Psi\alpha/\beta$' means 'the result of replacing each free occurrence of α in Ψ by β'.

6. The derivability relation $\vdash$ between a set of sentences Γ and a sentence ϕ is defined inductively by the following clauses:

P: $\{\phi\} \vdash \phi$.
MP: If $\Gamma \vdash C\phi_1\phi_2$ and $\Delta \vdash \phi_1$, then $\Gamma \cup \Delta \vdash \phi_2$.
MT: If $\Gamma \vdash CN\phi_1 N\phi_2$ and $\Delta \vdash \phi_2$ then $\Gamma \cup \Delta \vdash \phi_1$.
C: If $\Gamma \vdash \phi_2$ then $\Gamma \vdash C\phi_1\phi_2$;
If $\Gamma \vdash \phi_2$ then $\Gamma - \{\phi_1\} \vdash C\phi_1\phi_2$.
US: If $\Gamma \vdash \pi\ \alpha\ \Psi$ then $\Gamma \vdash \Psi\ \alpha/\beta$
UG: If $\Gamma \vdash \Psi\ \alpha/\beta$ then $\Gamma \vdash \pi\ \alpha\ \Psi$provided that β is not in Ψ nor in any sentence in Γ.

7. ϕ is derivable from Γ if and only if there is a finite subset Γ' of Γ such that $\Gamma' \vdash \phi$.

8. ϕ is a thesis of *T* if and only if ϕ is derivable from the null set.

Semantics

9. A model for T is an ordered triple $<D, S, V>$ such that D is any non-empty set, S maps the set of constants into D, V maps the elements of D into $\{0, 1\}$.

10. Truth in a model is defined inductively:

   (*i*) if $\phi = \beta$, for some constant β, ϕ is true in $<D, S, V>$ iff $V(S(\phi)) = 1$;
   (*ii*) if $\phi = N\phi_1$, for some sentence ϕ_1, then ϕ is true in $<D, S, V>$ iff ϕ_1 is not true in $<D, S, V>$;
   (*iii*) if $\phi = C\phi_1\phi_2$, for some sentences ϕ_1 and ϕ_2, ϕ is true in $<D, S, V>$ iff ϕ_1 is not true in $<D, S, V>$ or ϕ_2 is true in $<D, S, V>$;
   (*iv*) if $\phi = \pi\, \alpha\, \Psi$, for some variable α and formula Ψ, ϕ is true in $<D, S, V>$ iff $\Psi\, \alpha/\beta$ is true in $<D, S', V>$ for any function S' which differs at most from S in what is assigned to β, and β is the first constant not occuring in Ψ.

11. A sentence ϕ is a consequence of a set of sentences Γ iff ϕ is true in any model in which each sentence of Γ is true; ϕ is valid iff ϕ is a consequence of the null set.

The proofs of the soundness and completeness of T are analogous to the corresponding proofs for first order logic.

II

Since (by 10 (*iv*) above) the constants of T are the substituends for the variables of T, quantification in T is not substitutional if the biconditional

(1) $\Pi\, \alpha\, \Psi$ is true in $<D, S, V>$ iff, for every constant β, $\Psi\, \alpha/\beta$ is true in $<D, S, V>$

is false for some variable α, formula Ψ and model $<D, S, V>$.
To see that (1) is sometimes false just pick D in such a way that S: $\{$'P', 'P''....$\} \rightarrow D$ is not onto; then let V satisfy
(2) $V(x) = 1$ if x is in the range of S; $V(x) = 0$ if x is not in the range of S.

Then, e.g., 'πpp' will be false in such a model, while each of its instances 'P', 'P′', ... are true.

III

Recall Quine's view that if we extend truth-functional quantification by binding the sentential variables and do not construe the resulting quantification as substitutional, then the variables lose their schematic statues and we must view them as taking entities as values. *T* is truth functional logic with sentence quantification added; the treatment of 'π' is not substitutional. So if Quine is right, the variables should lose their schematic status and take entities as values. In view of 1-11 these two claims must come to this: (*a*) the sentential variables take elements of *D*j in a given <*D, S, V*>, as values; and (*b*) the sentential constants are names of the elements of *D assigned to them by S*. With regard to a system like *T*, which contains sentential constants as admissable substituends of the variables, we can see no logical difference between these claims.

First, a very simple but (we think) important point. From the fact that *D* is a non-empty set and *S* ('P') ε *D* it does not follow that 'P' names that element. A similar point can be made with regard to first order logic, where a model is an ordered couple <*D, V*> and *V* assigns the usual things to the constants, predicates and sentences. The same function *V* whose value for the individual constant 'a' as argument is the object 32 may have the number 1 as its value for the sentence 'Fa' as argument. Clearly enough, 'Fa' is not to be construed as naming 1. More generally, from the fact that for function f, f(x) = y, it does not follow that x names or denotes or has any semantic relation to y. Thus, from the mere fact that *V*('a') = 32 it does not follow that 'a' names 32. So also, from the mere fact that our function *S* associates a certain object with a sentential constant, it does not follow that that constant names that object.

Now, what shows whether or not a particular expression *names* the object which some semantic function associates with it? Basically, this seems to be an extra-formal matter: Formal syntactical and semantical considerations force no decision upon us. Indeed, all of the formal work can be, and typically is, carried through independently of the use of 'names' or any such term.

Next, in addition to sentential constants we also utilize sentence letters as bindable variables. But we do not think this point sufficient to establish (*a*) and (*b*). If it were sufficient, then it would have to follow from rules 9-10 that the quantified variables take entities of a particular D as values. Rule 9 defines a certain correspondence $S : D \rightarrow \{1, 0\}$. Rule 10 defines truth in a model in terms of these correspondences. It is only if one can infer that the first correspondence confers a naming status to the constants 'P', 'P′', ... that one could get this conclusion. And, as we have argued in the previous two paragraphs, this cannot be inferred.

So far we have argued that the semantics of T, formally construed, does not establish (*a*) or (*b*). Now consider the following extra-formal points accepted by Quine. First, sentences are not names. Second, connectives are not predicates. Third, the position of a variable is referential only if adjoined to a predicate. From these three points it follows that the quantification in T is not referential.

Quine could challenge this conclusion only by urging that the so-called sentential constants of T are not sentences, but names; and that the so-called connectives of T are not connectives, but predicates. But so far as we can see these conclusions would require the premise that non-substitutional quantification must be referential, and the point of this paper has been precisely to bring that premise into question.

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[2] Quine, W.V.O., *Philosophy of Logic*, Prentice-Hall, Englewood Cliffs, N.J., 1970.