MATERIAL IMPLICATION: A VARIANT OF THE DALE DEFENCE

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This paper contains two things.

The first is a variant of Dale's defence of material implication. I claim that my version has somewhat greater generality than his.

The second is a sequence of comments on the original dispute between us, and on the relation between Dale's defence and my version of it.

I.

First of all, my variant of Dale's defence.

The lattice-theoretic variant

Propositional logics are lattices. A lattice is a partially-ordered set closed under a pair of binary operations *meet* ($\wedge$) and *join* ($\vee$). The partial ordering relation $\leqslant$ corresponds to *implication* or *entailment*, and the lattice operations meet and join correspond to *and* and *or*. One can define an *equivalence* ($=$) on the lattice in the following way.

$$a = b \quad =_{\text{Df.}} \quad a \leqslant b \ \& \ b \leqslant a$$

A complemented lattice has in addition a unary operation *complementation* ($\sim$) whose properties reflect those of negation.

In any lattice

(1) $a \leqslant a \vee b$
(2) $b \leqslant a \vee b$
(3) $a \leqslant c \ \& \ b \leqslant c$ only if $a \vee b \leqslant c$

The *principle of duality* holds. That is, given any theorem concerning a lattice, one obtains another by interchanging $\leqslant$ and $\geqslant$, $\vee$ and $\wedge$ [Curry (1977), p. 134].

From (1) — (3) it follows that

(4) $a \leqslant b$ only if $c \vee a \leqslant c \vee b$.

In his defense of material implication Dale assumes classical conjunction, disjunction and negation. In lattice-theoretic terms this means that we are dealing with a lattice that is distributive uniquely complemented. That is, that

(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

and its dual holds; and that there are unique maximal and minimal elements 1 and 0 such that the complement $\sim a$ is defined by

(6i) $a \vee \sim a = a$
(6ii) $a \wedge \sim a = 0$

and is unique for each a.

It follows from (2) and (3) and the principle of duality that

(7i) $1 \wedge a = a$
(7ii) $0 \vee a = a$.

And it follows from (2) that

(8) $a \vee b \leqslant c$ only if $b \leqslant c$.

One can introduce a further binary operation ($\rightarrow$), the *conditional,* which has the following two properties

(MP) $a \wedge (a \rightarrow b) \leqslant b$
(ENT) $a \leqslant b$ if and only if $(a \rightarrow b) = 1$.

Our variant of Dale's defense then takes the form of the following theorem.

THEOREM: If (EXP) $(a \wedge b) \rightarrow c \leqslant a \rightarrow (b \rightarrow c)$ holds,
then $(a \rightarrow b) = (\sim a \vee b)$

(A) If (EXP) holds, $(a \rightarrow b) \leqslant (\sim a \vee b)$

$$a \wedge (a \rightarrow b) \leqslant b \qquad \text{(MP)}$$
$$\sim a \vee (a \wedge (a \rightarrow b)) \leqslant \sim a \vee b \qquad (4)$$
$$(\sim a \vee a) \wedge (\sim a \vee (a \rightarrow b)) \leqslant \sim a \vee b \qquad (5)$$
$$1 \wedge (\sim a \vee (a \rightarrow b)) \leqslant \sim a \vee b \qquad \text{(6i)}$$
$$(\sim a \vee (a \rightarrow b) \leqslant \sim a \vee b \qquad \text{(7i)}$$
$$(a \rightarrow b) \leqslant \sim a \vee b \qquad \text{(8)}.$$

(B) If (EXP) holds, $(\sim a \vee b) \leqslant (a \rightarrow b)$

$$a \wedge b \leqslant b \qquad (1 + \text{duality})$$
$$b \wedge a \leqslant b \qquad (2 + \text{duality})$$
$$O \vee (b \wedge a) \leqslant b \qquad \text{(7ii)}$$
$$(\sim a \wedge a) \vee (b \wedge a) \leqslant b \qquad \text{(6ii)}$$
$$(\sim a \vee b) \wedge a \leqslant b \qquad (5)$$
$$[((\sim a \vee b) \wedge a) \rightarrow b] = 1 \qquad \text{(ENT)}$$

but $((\sim a \vee b) \wedge a) \rightarrow b \leqslant (\sim a \vee b) \rightarrow (a \rightarrow b)$ (EXP)

therefore

$$[(\sim a \vee b) \rightarrow (a \rightarrow b)] = 1$$

and

$$(\sim a \vee b) \leqslant (a \rightarrow b) \qquad \text{(ENT)}$$

Note that (A) doesn't depend upon either (ENT) or (EXP), as you would expect, though (B) depends upon both. Put another way, material implication is the logically weakest conditional consistent with (MP), or modus ponens.

II.

In what way is this a defense of material implication ?
In what way is it more general than Dale's defence ?

The result is: given exportation

$$(a \rightarrow b) = (\sim a \vee b)$$

It was derived from the structure of the lattice (whose operations are $\wedge$, $\vee$, and $\sim$) and from a couple of constraints, (MP) and (ENT), on the conditional operation.

The difference between Dale's defence and mine lies in my (direct) assumption of (MP) and his assumption of the *sufficiency condition*. The two are related in the following way. Given bivalence (i.e. that each proposition has one of the two truth-values T and F), the assumptions of (MP) and the sufficiency condition are equivalent.

However, suppose bivalence fails. I take it that this case is at least worth considering. Then the Dale move from the sufficiency condition to his modus ponens is blocked. For the argument goes like this.

Modus ponens:
$$\frac{P \rightarrow Q, \quad P}{Q.}$$

Suppose Q is false. Then if P is true, it follows from the sufficiency condition that $P \rightarrow Q$ is false. Therefore whenever both P and $P \rightarrow Q$ are true, Q is true. But if bivalence fails the sufficiency condition doesn't rule out the possibility that both P and $P \rightarrow Q$ are true and Q is neither true nor false. (Though a generalised sufficiency condition would: that is, that $P \rightarrow Q$ is less than true if P is more true than Q.)

In this (possibly picayune) sense my defence is more general than Dale's. It applies to a many-valued logic with Boolean conjunction, disjunction and negation.

As a reply I can imagine it being said that material implication just *is* defined by the classical bivalent truth-table rather than by the equivalence proved in the lattice. My reply is that my variant is more general in that it is a defence of the truth functionality of the conditional for many-valued logics and that therefore material implication comes out as a special case.

The original dispute

My original attack took the following form. I claimed (Gibbins, 1979) that in addition to the sufficiency condition, Dale's defense required a further assumption, along the lines of the rule of inference *conditional proof* and that this rule is, or ought to be, controversial since it leads straight to some counter-factual fallacies.

Dale now points out rightly (Dale, *this journal*) that even viewing the matter as I did, the further assumption he needs is a weakened form of conditional proof which even iconoclasts like Anderson and Belnap flinch from denying.

I naturally concede this point. (The assumption of (ENT) in the proof above is equivalent to it.)

But to it I should add a conjecture. Perhaps it isn't totally wild to doubt restricted conditional proof

$$\frac{A \vdash B}{\vdash A \rightarrow B}.$$

at least for the case where A is a contradiction. Perhaps one can derive anything from a contradiction without its being the case that the corresponding conditional is true, and that this singularity shows that «$A \vdash B$» and «$\vdash A \rightarrow B$» aren't equivalent. Such an assumption blocks the derivation (B) at line 6 for the case where $a \leqslant \sim b$, or equivalently $A \vdash \sim B$.

Therefore given (ENT) anyone who thinks that counterfactual conditionals are sometimes false has to reject the logical truth of exportation (as for example the accounts of Lewis and Stalnaker do).

So I take the defence of material implication to be an attack on exportation. But that leaves me with one puzzle. Why is it so difficult to come up with plausible counter-examples to exportation from ordinary talk ? Is it because we systematically avoid nested conditionals in everyday life ?

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