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PARTIAL TYPE-SHIFTING AUTOMORPHISMS

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ABSTRACT

We attempt to shed some light on the consistency problem for NF by proving the existence of certain partial type-shifting automorphisms of TST_4-models.

Keywords: Set Theory, New Foundations, N, Simple Type Theory, TST

1. Introduction

Since its conception by Quine (see Quine (1937)), the central question for the theory of New Foundations (NF) has been its consistency relative to ZF. To this day, this question remains unanswered, resisting the attempts of many logicians, who over the years produced a variety of important yet inconclusive results. In this paper, we examine a weaker version of the problem, based on two of these results. The first one, due to Specker, states that a model of NF is basically a model of Simple Type Theory (TST) with a type-shifting automorphism. The second one, proved by Grišin, states that NF is equal to NF_4, the subtheory of NF which is axiomatized by Extensionality plus the 4-stratifiable instances of Comprehension axiom. From these two results, it follows that the consistency of NF is equivalent to the existence of a TST_4-model with a type-shifting automorphism, where TST_4 is the restriction of TST to formulas with variables of type less than 4. In brief, our approach consists of

(i) generalizing the notion of type-shifting automorphism to that of partial type-shifting automorphism,
(ii) introducing the notion of κ-capturing property of a TST_4-model $\mathcal{A}$, which is satisfied if certain partial type-shifting automorphisms of $\mathcal{A}$ exist,
(iii) reducing the consistency of NF to the existence of a TST_4-model $\mathcal{A}$ that satisfies the $|\mathcal{A}|^+$-capturing property, and
(iv) proving that there exist models $\mathcal{A}$ of TST_4 that satisfy the $|\mathcal{A}|$-capturing property.

The paper is divided into 3 sections. Section 2 introduces the definitions, the notation, and the basic theorems that we will use. In section 3 we define the κ-capturing property, and prove our main result. Finally, in

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section 4 we state some interesting open questions that are relevant to our work.

2. Preliminaries

In this introductory section we provide a brief self-contained exposition of TST and NF. For additional information on the subject, we refer the reader to Forster (1995).

The language and axioms of Simple Type Theory. Throughout this paper, we assume that we work in ZF with $\in$ as the membership relation.

The language $\mathcal{L}_{\mathrm{TST}}$ of Simple Type Theory is the many-sorted language of set theory with one binary relation symbol ε, and countably many types indexed by ω. To indicate the type of each variable of $\mathcal{L}_{\mathrm{TST}}$, we use a superscript, i.e., we write x^i to signify that variable x has type i. The formulas of $\mathcal{L}_{\mathrm{TST}}$ are built inductively from the atomic formulas $x^i \mathrel{\varepsilon} y^{i+1}$ and $x^i = y^i$. For $n > 0$, we define $\mathcal{L}_{\mathrm{TST}n}$ to be the language $\mathcal{L}_{\mathrm{TST}}$ restricted to the first n types, i.e., $0, \ldots, n-1$.

Simple Type Theory (TST) consists of the axioms of Extensionality and Comprehension. The *axiom of Extensionality* (Ext) is the set of all the following sentences for each type $i \in \omega$,

$$\forall x^{i+1}\, \forall y^{i+1}\, (x^{i+1} = y^{i+1} \leftrightarrow \forall z^i\, (z^i \mathrel{\varepsilon} x^{i+1} \leftrightarrow z^i \mathrel{\varepsilon} y^{i+1})). \qquad (\mathrm{Ext}^{i+1})$$

The *axiom of Comprehension* (Co) is the set of all the following sentences for each type $i \in \omega$ and formula ϕ of $\mathcal{L}_{\mathrm{TST}}$,

$$\forall u^{-}\, \exists y^{i+1}\, \forall x^i\, (x^i \mathrel{\varepsilon} y^{i+1} \leftrightarrow \phi(x^{i+1}, u^{-})), \qquad (\mathrm{Co}^{i+1})$$

where y^{i+1} is not free in ϕ.

For the rest of this section let us assume that $n \geq 2$. Restricting the above axioms to the language $\mathcal{L}_{\mathrm{TST}n}$ yields a new theory, which we denote by TST_n. We define $\mathrm{TST}_{n(2)}$ to be the subtheory of TST_n that is axiomatized by Extensionality and all the instances of Comprehension that contain variables of only two consecutive types (i.e., the instances described in (Co^{i+1}) in which all variables have type either i or $i+1$). Lastly, we let TST_n^{∞} and $\mathrm{TST}_{n(2)}^{\infty}$ be the theories $\mathrm{TST}_n + \mathrm{Inf}$ and $\mathrm{TST}_{n(2)} + \mathrm{Inf}$ respectively, where (Inf) is the set of sentences

$$\{\exists x_1^0 \ldots \exists x_k^0\, (\textstyle\bigwedge_{i \neq j} x_i^0 \neq x_j^0) : k > 0\}. \qquad (\mathrm{Inf})$$

Structures and isomorphisms. A *structure* $\mathcal{A}$ for the language $\mathcal{L}_{\mathrm{TST}_n}$ is a sequence $(A_0, A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$, where $A_0, A_1, \ldots, A_{n-1}$ are non-empty sets interpreting the n types of $\mathcal{L}_{\mathrm{TST}_n}$, and $\varepsilon^{\mathcal{A}} \subseteq \bigcup_{i<n-1} A_i \times A_{i+1}$ is a binary relation

interpreting the symbol ε. The cardinality of $\mathcal{A}$, denoted by $|\mathcal{A}|$, is defined to be the sum $|A_0| + |A_1| + ... + |A_{n-1}|$.

Note. The definition of $\mathcal{L}_{\mathrm{TST}_n}$-structure above can be perfectly valid even if $\varepsilon^{\mathcal{A}}$ is a binary relation for which $\varepsilon^{\mathcal{A}} \subseteq \bigcup_{i<n-1} A_i \times A_{i+1}$. In such cases, we just assume that ε is interpreted as the restriction of $\varepsilon^{\mathcal{A}}$ on $\bigcup_{i<n-1} A_i \times A_{i+1}$.

In this paper, we will mostly be interested in structures of $\mathcal{L}_{\mathrm{TST}_n}$ that are standard and transitive (see Tzouvaras (2007) or Crabbé (1991)).

Definition 1. Let $\mathcal{A} = (A_0, A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ be an $\mathcal{L}_{\mathrm{TST}_n}$-structure. We say that $\mathcal{A}$ is *standard* if for all $0 \le i < n - 1$, $x \in A_i$, and $y \in A_{i+1}$,

$$x \,\varepsilon^{\mathcal{A}}\, y \Leftrightarrow x \in y.$$

A standard $\mathcal{L}_{\mathrm{TST}n}$-structure $\mathcal{A}$ is called *transitive* if for all $0 \le i < n - 1$,

$$x \in A_{i+1} \Rightarrow x \subseteq A_i.$$

We usually abbreviate "standard and transitive" to "s.t.".

A simple example of an s.t. model of TST_n is the $\mathcal{L}_{\mathrm{TST}_n}$-structure

$$(X, \mathcal{P}(X), \mathcal{P}^2(X), \ldots, \mathcal{P}^{n-1}(X), \in),$$

where X is any non-empty set. Let us now formulate the notion of isomorphism for $\mathcal{L}_{\mathrm{TST}n}$-structures, and show that we can always assume that our structures are standard and transitive.

Definition 2. Let $\mathcal{A} = (A_0, A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ and $\mathcal{B} = (B_0, B_1, \ldots, B_{n-1}, \varepsilon^{\mathcal{B}})$ be two $\mathcal{L}_{\mathrm{TST}_n}$-structures. We say that f is a *partial function* (or *mapping*) *from $\mathcal{A}$ to $\mathcal{B}$*, if f is a sequence $(f_0, f_1, \ldots, f_{n-1})$ of functions such that for all $0 \le i < n$,

$$\mathrm{dom}(f_i) \subseteq A_i \text{ and } \mathrm{ran}(f_i) \subseteq B_i.$$

If every f_i is 1-1 (resp. onto), then we say that f is 1-1 (resp. onto). If $\mathrm{dom}(f_i) = A_i$, for all $0 \le i < n-1$, then f is called a (*total*) *function* (or *mapping*) *from $\mathcal{A}$ to $\mathcal{B}$*.

We say that f is a *partial $\mathcal{L}_{\mathrm{TST}_n}$-isomorphism* from $\mathcal{A}$ to $\mathcal{B}$, if it is 1-1 and for all $0 \le i < n - 1$, $x \in A_i$, and $y \in A_{i+1}$,

$$x \,\varepsilon^{\mathcal{A}}\, y \Leftrightarrow f_i(x) \,\varepsilon^{\mathcal{B}}\, f_{i+1}(y).$$

An *$\mathcal{L}_{\mathrm{TST}_n}$-isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a partial $\mathcal{L}_{\mathrm{TST}_n}$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$, which is onto and total. When such an $\mathcal{L}_{\mathrm{TST}_n}$-isomorphism exists, we say that $\mathcal{A}$ and $\mathcal{B}$ are *$\mathcal{L}_{\mathrm{TST}_n}$-isomorphic*.

Note. We adopt the notation f_i to denote the i-th coordinate function of a partial mapping f.

Collapsing Lemma. Every $\mathcal{L}_{\mathrm{TST}_n}$-structure that satisfies Extensionality is isomorphic to an s.t. $\mathcal{L}_{\mathrm{TST}_n}$-structure.

Proof. Let $\mathcal{A} = (A_0, A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ be an $\mathcal{L}_{\mathrm{TST}n}$-structure that satisfies Extensionality. Let $\mathcal{B}' = (A_0, \mathcal{P}(A_0), \mathcal{P}^2(A_0), \ldots, \mathcal{P}^{n-1}(A_0), \in)$. We let $f : \mathcal{A} \to \mathcal{B}'$ be the function defined by induction on $0 \leq i < n - 1$ such that

$$f_0(a) = a, \qquad \text{for all } a \in A_0, \text{ and}$$
$$f_{i+1}(y) = \{f_i(x) : x \in A_i \wedge x\, \varepsilon^{\mathcal{A}} y\}, \qquad \text{for all } y \in A_{i+1}.$$

For all $0 \leq i < n - 1$, let $B_i = f_i``A_i$, and let $\mathcal{B} = (B_0, B_1, \ldots, \in)$. It is easy to verify that f is an $\mathcal{L}_{\mathrm{TST}_n}$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

We should now make a necessary clarification on the notion of finiteness used in this paper. Let $\mathcal{A} = (A_0, A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ be an $\mathcal{L}_{\mathrm{TST}_n}$-structure. Let also $0 \leq i < n - 1$ and $x \in A_{i+1}$. When we say that "x is finite", we will always mean that x is finite in the metatheory, i.e., that there exist $k \in \omega$ and $x_1, \ldots, x_k \in A_i$ such that $\mathcal{A} \vDash \forall z(z\, \varepsilon\, x \leftrightarrow z = x_1 \vee \ldots \vee z = x_k)$. Furthermore, throughout the paper, we assume that the cardinality $|x|$ of x always refers to the cardinality of x in the metatheory. So, if $\mathcal{A}$ is an s.t. model of $\mathrm{TST}_{n(2)}$, then

$$\text{"}x \text{ is finite" iff } |x| < \omega,$$

and therefore the following lemma holds.

Lemma 1. Let $\mathcal{A} = (A_0, A_1, \ldots, A_{n-1}, \in)$ be an s.t. model of $\mathrm{TST}_{n(2)}$. For all $0 \leq i < n-1$,

$$\mathcal{P}_{\mathrm{fin}}(A_i) \subseteq A_{i+1},$$

where $\mathcal{P}_{\mathrm{fin}}(A_i) = \{u \subseteq A_i\colon |u| < \omega\}$. In other words, if x is a finite subset of A_i, then x is an element of A_{i+1}.

Proof. Just notice that for all $0 \leq i < n - 1$ and $k > 0$, the formula

$$\forall x^i_1 \ldots \forall x^i_k \exists y^{i+1} \forall x^i (x^i\, \varepsilon\, y^{i+1} \leftrightarrow x^i = x^i_1 \vee \ldots \vee x^i = x^i_k).$$

is an instance of (Co) included in $\mathrm{TST}_{n(2)}$. □

New Foundations. The language $\mathcal{L}_{\mathrm{NF}}$ of New Foundations is the (one-sorted) language of set theory $\{\varepsilon\}$ with one binary relation symbol. For every formula ϕ of $\mathcal{L}_{\mathrm{TST}}$ there exists a unique formula ϕ^* of $\mathcal{L}_{\mathrm{NF}}$, which can be obtained by removing all type superscripts from ϕ. A formula ϕ of $\mathcal{L}_{\mathrm{NF}}$ is called *stratified* if there exists a formula ψ of $\mathcal{L}_{\mathrm{TST}}$ such that $\phi = \psi^*$. For any set of formulas Γ of $\mathcal{L}_{\mathrm{TST}}$, we define $\Gamma^* = \{\sigma^* : \sigma \in \Gamma\}$. The theory of *New Foundations* NF is defined by the *axiom of Extensionality*,

$$\forall x \forall y (x = y \leftrightarrow \forall z (z\, \varepsilon\, x \leftrightarrow z\, \varepsilon\, y)),$$

and the *axiom of Stratified Comprehension*,

$$\forall u^{-} \exists y \forall x \, (x \, \varepsilon \, y \leftrightarrow \phi(x, u^{-})),$$

where ϕ is a stratified formula of $\mathcal{L}_{\mathrm{NF}}$ and y is not free in ϕ. Notice that the axioms of Extensionality and Stratified Comprehension for the language of $\mathcal{L}_{\mathrm{NF}}$ are exactly $(\mathrm{Ext})^*$ and $(\mathrm{Co})^*$ respectively. So NF = $(\mathrm{TST})^*$. We define NF_n to be the theory $(\mathrm{TST}_n)^*$. Clearly, NF_n is a subtheory of NF.

Type-shifting automorphisms. Using results by Grišin (see Grišin (1972)) and Specker (see Specker (1958)) we can establish a direct connection between theories NF, NF_4, and TST_4. We first define the notion of type-shifting automorphism.

Definition 3. Let $\mathcal{A} = (A_0, A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ be an $\mathcal{L}_{\mathrm{TST}_n}$-structure. A (resp. partial) $\mathcal{L}_{\mathrm{TST}_{n-1}}$-isomorphism from $(A_0, \ldots, A_{n-2}, \varepsilon^{\mathcal{A}})$ to $(A_1, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ is called a (resp. *partial*) *type-shifting automorphism* of $\mathcal{A}$. We abbreviate the expression "type-shifting automorphism" to "*tsau*".

Theorem 2 (Grišin). NF = NF_4.

Proposition 3 (Specker). If $\mathcal{A}$ is a model of TST_n with a tsau, then there exists an $\mathcal{M}$ model of NF_n such that for every $\mathcal{L}_{\mathrm{TST}}$-sentence σ,

$$\mathcal{A} \models \sigma \Leftrightarrow \mathcal{M} \models \sigma^*. \tag{1}$$

Proof. Let $\mathcal{A} = (A_0, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ be a model of TST_n, and let f be a tsau of $\mathcal{A}$. Let $\mathcal{M} = (M, \varepsilon^{\mathcal{M}})$ such that $M = A_0$ and

$$x \, \varepsilon^{\mathcal{M}} \, y \Leftrightarrow x \, \varepsilon^{\mathcal{A}} f_0(y), \qquad \text{for all } x, y \in M.$$

It can be easily verified that $\mathcal{M}$ satisfies (1). □

As a consequence we get the following corollary.

Corollary 4. If there exists a model of TST_4 with a tsau, then NF is consistent.

The consistency problem for NF can therefore be reduced to that of the existence of a TST_4-model with a tsau. The task of finding such a model is not trivial though. For example, let X be any non-empty set and let $\mathcal{A} = (X, \mathcal{P}(X), \mathcal{P}^2(X), \mathcal{P}^3(X), \in)$. Obviously, we cannot expect to find a tsau of $\mathcal{A}$, since $|X| \neq |\mathcal{P}(X)|$. But what if we let $\mathcal{B}$ be a countable elementary submodel of $\mathcal{A}$? Is there a tsau of $\mathcal{B}$? The answer follows directly from Proposition 3 and the following theorem (proved in Specker (1953)).

Theorem 5 (Specker). The axiom of Choice fails in NF.

In other words, TST_4-models that satisfy choice, i.e., most natural models of TST_4, are not candidates for having a tsau.

3. Partial type-shifting automorphisms

Instead of attacking the consistency problem directly, we will try to tackle a weaker version of it. We have already mentioned that finding a TST_4-model with a tsau is equivalent to proving the consistency of NF. So, constructing tsaus is certainly a very difficult if not impossible task.Below, we examine if the same is true also for partial tsaus.

Definition 4. Let $\mathcal{A} = (A_0, \ldots, A_{n-1}, \varepsilon^{\mathcal{A}})$ be an $\mathcal{L}_{\mathrm{TST}_n}$-structure, where $n \geq 2$, and let κ be an infinite cardinal. We say that $\mathcal{A}$ satisfies the *κ-capturing property* if for all $u_0, \ldots, u_{n-1}$ with

$$u_i \subseteq A_i \text{ and } |u_i| < \kappa, \qquad \text{for } 0 \leq i < n,$$

there exists $p = (p_0, \ldots, p_{n-2})$ partial tsau of $\mathcal{A}$ that *captures* $(u_0, \ldots, u_{n-1})$, i.e., that satisfies

$$u_i \subseteq \mathrm{dom}(p_i) \text{ and } u_{i+1} \subseteq \mathrm{ran}(p_i), \qquad \text{for all } 0 \leq i < n-1,$$

The next proposition follows directly from the definition.

Proposition 6. If there exists a TST_4-model $\mathcal{A}$ that satisfies the $|\mathcal{A}|^+$-capturing property, then NF is consistent.

This now raises the question of whether it is equally difficult to find TST_4-models $\mathcal{A}$ that satisfy the $|\mathcal{A}|$-capturing property. It turns out that finding such models is quite easy.

Capturing Lemma. All models of $\mathrm{TST}^{\infty}_{4\,(2)}$ satisfy the ω-capturing property.
Proof. Let $\mathcal{A} = (A_0, A_1, A_2, A_3, \varepsilon^{\mathcal{A}})$ be a model of $\mathrm{TST}^{\infty}_{4\,(2)}$, and let u_0, u_1, u_2, u_3 be sets such that

$$u_i \subseteq A_i \text{ and } |u_i| < \omega, \qquad \text{for } 0 \leq i \leq 3.$$

We will show that there exists a partial tsau of $\mathcal{A}$ that captures (u_0, u_1, u_2, u_3). By the Collapsing Lemma, we may assume that $\mathcal{A}$ is an s.t. $\mathcal{L}_{\mathrm{TST}_4}$-structure, i.e., that $\varepsilon^{\mathcal{A}}$ is actually $\in$. We may also assume that all u_0, u_1, u_2, u_3 have the same number of elements n, for some $n > 0$ (if they do not, we may just expand them by adding some arbitrary elements). We enumerate each one of these sets and let

$$\begin{aligned} u_0 &= \{a_1, \ldots, a_n\}, \\ u_1 &= \{x_1, \ldots, x_n\}, \\ u_2 &= \{y_{n+1}, \ldots, y_{2n}\}, \text{ and} \\ u_3 &= \{z_{2n+1}, \ldots, z_{3n}\}. \end{aligned}$$

We will define elements

$$\begin{aligned} a_{n+1}, \ldots, a_{3n} &\in A_0, \\ x_{n+1}, \ldots, x_{3n} &\in A_1, \\ y_1, \ldots, y_n, y_{2n+1}, \ldots, y_{3n} &\in A_2, \text{ and} \\ z_1, \ldots, z_{2n} &\in A_3, \end{aligned}$$

such that the partial function $p = (p_0, p_1, p_2)$ defined as

$$a_i \overset{p_0}{\longmapsto} x_i \overset{p_1}{\longmapsto} y_i \overset{p_2}{\longmapsto} z_i,$$

for $0 < i \leq 3n$, is a partial tsau. The picture below shows p at this point.

$$\begin{array}{cccc}
a_1 & x_1 & \bullet & \bullet \\
\vdots & \vdots & \vdots & \vdots \\
a_n & x_n & \bullet & \bullet \\
\bullet & \bullet & y_{n+1} & \bullet \\
\vdots & \vdots & \vdots & \vdots \\
\bullet & \bullet & y_{2n} & \bullet \\
\bullet & \bullet & \bullet & z_{2n+1} \\
\vdots & \vdots & \vdots & \vdots \\
\bullet & \bullet & \bullet & z_{3n}
\end{array}$$

Each row consists of the elements a_i, $p_0(a_i)$, $p_1(p_0(a_i))$, $p_2(p_1(p_0(a_i)))$. We will fill in the missing elements (marked as bullets) by a series of steps.

STEP 1. *Finding candidates for $a_{n+1}, \ldots, a_{2n}$ and $a_{2n+1}, \ldots, a_{3n}$.* Let

$$(i, S) \mapsto a_{i,S}$$

be an 1-1 mapping from $\{n+1, \ldots, 2n\} \times \mathcal{P}(\{n+1, \ldots, 2n\})$ to $A_0 - \{a_1, \ldots, a_n\}$. Such a mapping exists, because A_0 is large enough (infinite). We can think of $a_{i,S}$ as a candidate for a_i, i.e., at some point of our construction we will choose a specific $a_{i,S}$ to be a_i.

To find candidates for $a_{2n+1}, \ldots, a_{3n}$, we first need to fix their relation with $x_1, \ldots, x_n$. Let $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ be such that

$$x_1^{\delta_1} \cap \ldots \cap x_n^{\delta_n} \text{ is infinite,}$$

where $x_i^{\delta_i}$ is x_i when $\delta_k = 1$, and $A_0 - x_i$ when $\delta_k = -1$. The existence of $\delta_1, \ldots, \delta_n$ follows from the fact that A_0 is infinite. For each $2_n < i \leq 3n$ and $S \in \mathcal{P}(\{n+1, \ldots, 3n)\}$ we choose a distinct $a_{i,S} \in x_1^{\delta_1} \cap \ldots \cap x_n^{\delta_n} \subseteq A_0$ such that $a_{i,S} \notin \{a_1, \ldots, a_n\} \cup \{a_{j,S} : n < j \leq 2n \wedge S \in \mathcal{P}(\{n+1, \ldots, 2n\})\}$. Since $x_i^{\delta_i} \cap \ldots \cap x_n^{\delta_n}$ is infinite and we only need a finite number of such elements, this process is well defined. Notice that by definition, for all $0 < j \leq n$, either

$$a_{i,S} \in x_j, \qquad \text{for all } 2n < i \leq 3n \text{ and } S \in \mathcal{P}(\{n+1, \ldots, 3n)\},$$

or

$$a_{i,S} \notin x_j, \qquad \text{for all } 2n < i \leq 3n \text{ and } S \in \mathcal{P}(\{n+1, \ldots, 3n)\},$$

We note that our intention is to use each $a_{i,S}$, defined above, in such a way that the following statement is true,

"$a_{i,S}$ is a candidate for a_i and belongs to any set x_j, for which $j \in S$".

Step 2. *Defining* $a_{n+1}, \ldots, a_{2n}$ *and* $x_{n+1}, \ldots, x_{2n}$. For each $n < i \leq 2n$, let

$$\begin{aligned}
Y_i = \{x \in \mathcal{P}_{\text{fin}}(A_0) - \{x_1, \ldots, x_n\} : \forall_j \in \{1, \ldots, n\}(a_j \in x \leftrightarrow x_j \in y_i)\\
\wedge \forall j \in \{n+1, \ldots, 2n\} \forall S \in \mathcal{P}(\{n+1, \ldots, 2n\})(a_{j,S} \in x \leftrightarrow i \in S)\\
\wedge \forall j \in \{2n+1, \ldots, 3n\} \forall S \in \mathcal{P}(\{n+1, \ldots, 3n\})(a_{j,S} \in x \leftrightarrow i \in S)\}.
\end{aligned}$$

Since $\mathcal{P}_{\text{fin}}(A_0)$ is infinite, the sets $Y_{n+1}, \ldots, Y_{2n}$ are also infinite. We may therefore choose n distinct elements $x_{n+1} \in Y_{n+1}, \ldots, x_{2n} \in Y_{2n}$. By Lemma 1, we have that $x_{n+1}, \ldots, x_{2n} \in A_1$. Now, for each $n < i \leq 2n$, let

$$a_i = a_{i,S},$$

where $S = \{j \in \{n+1, \ldots, 2n\} : x_i \in y_j\}$.

The definitions of $Y_{n+1}, \ldots, Y_{2n}$ and $a_{n+1}, \ldots, a_{2n}$ imply that

$$a_i \in x_j \Leftrightarrow x_i \in y_j, \quad \text{for all } 0 < i \leq 2n \text{ and } n < j \leq 2n. \tag{2}$$

Step 3. *Finding candidates for* $x_{2n+1}, \ldots, x_{3n}$. For each $2n < i \leq 3n$ and $S_1 \in \mathcal{P}(\{1, \ldots, n\})$ let

$$\begin{aligned}
Y'_{i,S_1} = \{x \in \mathcal{P}_{\text{fin}}(A_0) - \{x_1, \ldots, x_{2n}\} : \forall j \in \{1, \ldots, n\}(a_j \in x \leftrightarrow j \in S_1)\\
\wedge \forall j \in \{n+1, \ldots, 2n\}(a_j \in x \leftrightarrow y_j \in z_i)\\
\wedge \forall j \in \{2n+1, \ldots, 3n\} \forall S \in \mathcal{P}(\{n+1, \ldots, 3n\})(a_{j,S} \in x \leftrightarrow i \in S)\}.
\end{aligned}$$

Now, let

$$(i, S_1, S_2) \mapsto x_{i,S_1,S_2}$$

be an 1-1 mapping from $\{2n+1, \ldots, 3n\} \times \mathcal{P}(\{1, \ldots, n\}) \times \mathcal{P}(\{2n+1, \ldots, 3n\})$ to $\mathcal{P}_{\text{fin}}(A_0) - \{x_1, \ldots, x_{2n}\}$ such that

$$x_{i,S_1,S_2} \in Y'_{i,S_1}.$$

Clearly, all sets Y'_{i,S_1} are infinite, so such a mapping exists. By Lemma 1, every x_{i,S_1,S_2} is an element of A_1.

Again, our intention is to use x_{i,S_1,S_2} as a candidate for x_i. The reason we indexed all these sets by S_1 and S_2 is that we want them to eventually satisfy the following,

$$\begin{aligned}
a_j \in x_{i,S_1,S_2} \Leftrightarrow j \in S_1, \quad \text{for } 0 < j \leq n, \text{ and}\\
x_{i,S_1,S_2} \in y_j \Leftrightarrow j \in S_2, \quad \text{for } 2n < j \leq 3n.
\end{aligned}$$

At this point our partially constructed tsau looks like this

$$\begin{array}{cccc}
a_1 & x_1 & \bullet & \bullet\\
\vdots & \vdots & \vdots & \vdots\\
a_n & x_n & \bullet & \bullet\\
a_{n+1} & x_{n+1} & y_{n+1} & \bullet\\
\vdots & \vdots & \vdots & \vdots\\
a_{2n} & x_{2n} & y_{2n} & \bullet\\
a_{2n+1,S} & x_{2n+1,S_1,S_2} & \bullet & z_{2n+1}\\
\vdots & \vdots & \vdots & \vdots\\
a_{3n,S} & x_{3n,S_1,S_2} & \bullet & z_{3n}
\end{array}$$

(we have not decided yet which elements we should choose for $a_{2n+1}, \ldots, a_{3n}$ and $x_{2n+1}, \ldots, x_{3n}$).

STEP 4. *Defining* $y_1, \ldots, y_n$. For each $0 < i \leq n$, we let

$$\begin{aligned} Z_i = \{y \in \mathcal{P}_{\text{fin}}(A_1) - \{y_{n+1}, \ldots, y_{2n}\} : \forall j \in \{1, \ldots, 2n\}(x_j \in y \leftrightarrow a_j \in x_i) \\ \wedge \forall j \in \{2n+1, \ldots, 3n\} \forall S_1 \in \mathcal{P}(\{1, \ldots, n\}) \forall S_2 \in \mathcal{P}(\{2n+1, \ldots, 3n\}) \\ (x_{j,S_1,S_2} \in y \leftrightarrow \exists S \in \mathcal{P}(\{n+1, \ldots, 3n\})(a_{j,S} \in x_i))\}. \end{aligned}$$

Since, all the sets $Z_1, \ldots, Z_n$ are infinite, we can choose n distinct elements $y_1 \in Z_1, \ldots, y_n \in Z_n$.

By the definition of $Z_1, \ldots, Z_n$, we get that

$$a_i \in x_j \Leftrightarrow x_i \in y_j, \quad \text{for all } 0 < i \leq 2n \text{ and } 0 < j \leq n. \tag{3}$$

STEP 5. *Defining* $a_{2n+1}, \ldots, a_{3n}$, $x_{2n+1}, \ldots, x_{3n}$, *and* $y_{2n+1}, \ldots, y_{3n}$. For each $2n < i \leq 3n$, we let

$$\begin{aligned} Z'_i = \{y \in \mathcal{P}_{\text{fin}}(A_1) - \{y_1, \ldots, y_{2n}\} : \forall j \in \{1, \ldots, 2n\}(x_j \in y \leftrightarrow y_j \in z_i) \\ \wedge \forall j \in \{2n+1, \ldots, 3n\} \forall S_1 \in \mathcal{P}(\{1, \ldots, n\}) \forall S_2 \in \mathcal{P}(\{2n+1, \ldots, 3n\}) \\ (x_{j,S_1,S_2} \in y \leftrightarrow i \in S_2)\}. \end{aligned}$$

Clearly, the sets $Z'_{2n+1}, \ldots, Z'_{3n}$ are infinite, so again we may choose n distinct elements $y_{2n+1} \in Z'_{2n+1}, \ldots, y_{3n} \in Z'_{3n}$. By Lemma 1, we know that $y_{2n+1}, \ldots, y_{3n}$ are elements of A_2.

For each $2n < i \leq 3n$, let

$$x_i = x_{j,S_1,S_2},$$

where $S_1 = \{j \in \{1, \ldots, n\} : y_j \in z_i\}$ and $S_2 = \{j \in \{2n+1, \ldots, 3n\} : y_i \in z_j\}$.

Also, for each $2n < i \leq 3n$, let

$$a_i = a_{i,S},$$

where $S = \{j \in \{n+1, \ldots, 3n\} : x_i \in y_j\}$.

The definitions of $a_{2n+1}, \ldots, a_{3n}$, $x_{2n+1}, \ldots, x_{3n}$, and $Z'_{2n+1}, \ldots, Z'_{3n}$ imply that

$$a_i \in x_j \Leftrightarrow x_i \in y_j \Leftrightarrow y_i \in z_j, \quad \text{for all } 0 < i \leq 3n \text{ and } 2n < j \leq 3n. \tag{4}$$

Furthermore, looking back at the definitions of $a_{i,S}$ and $Z_1, \ldots, Z_n$, we get that

$$a_i \in x_j \Leftrightarrow x_i \in y_j, \quad \text{for all } 2n < i \leq 3n \text{ and } 0 < j \leq n. \tag{5}$$

Step 6. Defining $z_1, \ldots, z_{2n}$. Finally, for each $0 < i \leq 2n$, let

$$z_i = \{y_j : x_j \in y_i \wedge 1 \leq j \leq 3n\} \cup \{y'_i\},$$

where $y'_1, \ldots, y'_{2n}$ are any $2n$ distinct elements of $A_2 - \{y_1, \ldots, y_{3n}\}$. Lemma 1 implies that $z_1, \ldots, z_{2n}$ are elements of A_3.

By the definition of $z_1, \ldots, z_{2n}$, we have that

$$x_i \in y_j \Leftrightarrow y_i \in z_j, \quad \text{for all } 0 < i \leq 3n \text{ and } 0 < j \leq 2n. \tag{6}$$

We define $p = (p_0, p_1, p_2)$ to be the mapping

$$u_0 \xrightarrow{p_0} u_1 \xrightarrow{p_1} u_2 \xrightarrow{p_2} u_3$$

where

$$a_i \overset{p_0}{\longmapsto} x_i \overset{p_1}{\longmapsto} y_i \overset{p_2}{\longmapsto} z_i,$$

for all $0 < i \leq 3n$. Notice that the construction above ensures that p is well-defined and 1-1, because we were always careful in choosing distinct new elements. Moreover, (2)–(6) imply that

$$a_i \in x_j \Leftrightarrow x_i \in y_j \Leftrightarrow y_i \in z_j,$$

for all $0 < i, j \leq 3n$. We conclude that p captures (u_0, u_1, u_2, u_3), and satisfies all the requirements for being a partial tsau. □

Corollary 7. There exist models $\mathcal{A}$ of TST_4 that satisfy the $|\mathcal{A}|$-capturing property.

Proof. Let $\mathcal{A} = (A_0, A_1, A_2, A_3, \varepsilon^{\mathcal{A}})$ be any countable model of TST_4^∞, i.e., a model such that $|A_0| = |A_1| = |A_2| = |A_3| = \omega$. The corollary follows directly from the Capturing Lemma. □

4. Conclusion

We believe that our present work reveals an interesting new aspect of the consistency problem for NF, namely its connection with the notions of partial tsau and the "capturing property". In addition, we think that our results raise important issues that can help us gain a better understanding of why it is so hard to construct tsaus of TST_4^∞-models. Below, we state some of these issues, which are probably worth exploring even independently from the problem of consistency for NF.

Question 1. Does Corollary 7 hold for any uncountable cardinality $|\mathcal{A}|$?

Question 2. Notice that the definition of the "capturing property" refers to external cardinality, i.e., cardinality with respect to our metatheory. Can we establish some useful result resembling the Capturing Lemma by altering the definition of the "capturing property" to refer to internal cardinality, i.e., cardinality in the TST_4-model?

Question 3. Can we refine the methods used in the proof of the Capturing Lemma to construct tsaus for Fraenkel-Mostowski models of TST_4^∞? The underlying idea of constructing partial tsaus was to use finite sets to approximate infinite ones. The reason we did this is that unlike infinite sets, finite sets are easy to manipulate. By focusing on Fraenkel-Mostowski models, we have more control on what kind of infinite sets exist in our universe, and therefore it could be easier to find approximations when constructing a tsau.

Question 4. Is there any uncountable model $\mathcal{A} = (A_0, A_1, A_2, A_3, \varepsilon^{\mathcal{A}})$ of TST_4^∞, with $|A_0| = |A_1| = |A_2| = |A_3| = |\mathcal{A}|$, that fails the κ-capturing property for some $\omega < \kappa < |\mathcal{A}|$? These models can be regarded as highly problematic with respect to the consistency of NF.

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References

[1] Crabbé, M. (1991), 'On NFU', *Notre Dame Journal of Formal Logic* **33**(1), 112–119.
[2] Forster, T. E. (1995), *Set theory with a universal set*, Vol. 31 of *Oxford Logic Guides*, second edn, The Clarendon Press Oxford University Press, New York. Exploring an untyped universe, Oxford Science Publications.
[3] Grišin, V. N. (1972), 'Equivalence of Quine's NF system to one of its fragments', *Naučn.-Tehn. Informacija (VINITI) Ser. 2. Informacionnye Processy i Sistemy* (1), 22–24, 32.
[4] Quine, W. V. (1937), 'New Foundations for Mathematical Logic', *Amer. Math. Monthly* **44**(2), 70–80.
[5] Specker, E. (1953), 'The axiom of choice in Quine's New Foundations for Mathematical Logic', *Proc. Nat. Acad. Sci. U.S.A.* **39**, 972–975.
[6] Specker, E. (1958), 'Dualität', *Dialectica* **12**, 451–465.
[7] Tzouvaras, A. (2007), 'A reduction of the NF consistency problem', *J. Symbolic Logic* **72**(1), 285–304.

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