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A LOGIC FOR WEAK ESSENCE AND STRONG ACCIDENT

Tianqun Pan and Chungui Yang

Abstract

Two new metaphysical notions 'weak essence' and 'strong accident' are introduced. A proposition φ is weakly essential if once φ is true, φ is possibly true; and the proposition φ is strongly accidental if φ is true but necessarily false. Under the condition that the frame is serial, if φ is essential, then φ is weakly essential, and if φ is strongly accidental, then φ is accidental. The relation between weak essence and strong accident, as that between essence and accident, is that one is the negation of the other. A logical system that describes weak essence and strong accident is established, and the system is shown to be sound and complete with respect to serial frames using possible worlds semantics.

Keywords: Strong accident; Weak essence; Nonstandard modal logic; Possible worlds semantics.

1. Introduction

Accident is a modal notion that is distinct from contingency. 'An accidental proposition is one that is the case, but could have been otherwise. An essential proposition is one that, whenever it enjoys a true status, it does it per force' (Marcos 2005). Thus, a proposition φ is accidental if φ is true, but φ could be false. Essence is a negative conception of accident. A proposition φ is essential if, whenever φ is true, φ is necessarily true. Several logical systems for essence and accident have been presented and informatively discussed by logicians (Marcos 2005, Steinsvold 2008, 2011) following the creative work of Humberstone (1995).

The modality of the world is discussed by metaphysicians because the world could differ from that of the present. The actuality of the world may not be necessary. Some facts or phenomena that are accidental might not have occurred. Both accident and essence can help people to understand the world more favourably. Suppose that actuality may not be possible; consequently, there must be some phenomena that are actual but impossible. This type of fact or phenomenon is 'absolutely accidental' or 'strongly accidental'. Hence, we define strong accident as follows: 'φ is strongly accidental' is defined as 'φ is true but φ is necessarily false'. Accordingly, weak essence as the negation of strong accident is defined as 'φ is weakly

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essential', which means that 'if φ is true, φ is possibly true'. According to the terms of possible worlds semantics, because the actual world often differs from its possible world(s) and the truth value of a proposition can be assigned different values in the actual world and its possible world(s), strong accident is not a formally contradictory concept, and weak essence is not a trivial concept.

Strong accident and weak essence are two metaphysical notions which can be used to, in a non-empirical way, discuss how the world or facts could have been or would be. The two notions are appropriately introduced because the following relations hold that, provided the frame is serial, if φ is strongly accidental, φ must be accidental; and if φ is essential, φ must be weakly essential.

In this study we constructed a system for weak essence and strong accident. This paper presents its soundness and completeness using possible worlds semantics.

2. A logical system, WS, and its soundness and completeness

In this paper, '$\odot\varphi$' means 'φ is strongly accidental', and '$\circledast\varphi$' means 'φ is weakly essential'. The formal language *L* is defined as follows in BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \odot\varphi \mid \circledast\varphi$$

The other connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined by $\neg$ and $\wedge$ as standard.

The language in *L* is interpreted using the standard possible worlds semantics.

Definition 2.1 [Frames, Models, and Satisfaction]. A Kripke Frame F=<W,R> is a tuple, where W is a set of possible worlds, and $R \subseteq W \times W$ is an accessibility relation. A Kripke Model M=<F, π> is a tuple, where F is a Kripke frame and $\pi: P \rightarrow 2^W$ is an interpretation of a set of propositional variables P. A formula φ is true in a model M at world w if:

$M,w \vDash p$ iff $w \in \pi(p)$,

$M,w \vDash \neg\varphi$ iff it is not the case that $M,w \vDash \varphi$,

$M,w \vDash \varphi\wedge\psi$ iff $M,w \vDash \varphi$ and $M,w \vDash \psi$,

$M,w \vDash \circledast\varphi$ iff $M,w \vDash \neg\varphi$, or for some w' with Rww', $M,w' \vDash \varphi$,

$M,w \vDash \odot\varphi$ iff $M,w \vDash \varphi$ and for any w' with Rww', $M,w' \nvDash \varphi$.

The operators $\circledast$ and $\odot$ are interdefinable as the essence operator and the accident operator. The formula is

$\circledast\varphi \leftrightarrow \neg\odot\varphi$.

In the following, we use $\odot$ as the primitive operator.

Well-established logic systems exist for necessity $\Box$ or possibility $\Diamond$, and the formula $\odot\varphi$ here is semantically equivalent to $\varphi\wedge\Box\neg\varphi$ and $\circledast\varphi$ to $\varphi\rightarrow\Diamond\varphi$. It seems possible to establish the logic on the basis of the equivalences. This is an alluring shortcut. However, the shortcut is infeasible because although the equivalences $\odot\varphi\equiv(\varphi\wedge\Box\neg\varphi)$ and $\circledast\varphi\equiv(\varphi\rightarrow\Diamond\varphi)$ can be obtained, it is impossible to define the necessity operator $\Box$ or the possibility operator $\Diamond$ using the operator $\odot$ or $\circledast$. A simple reason for the impossibility is that the value of $\Box\varphi$ or $\Diamond\varphi$ at a certain world w depends only on the value of φ at the w's accessible worlds while $\odot\varphi$ and $\circledast\varphi$ contain a requirement of the value of φ at the world w. Logicians addressing other concerns, such as contingency (or non-contingency) and accident (or essence), have encountered the same difficulty. The method that these logicians (Humberstone 1995, Kuhn 1995, Marcos 2005, Steinsvold 2008) presented is feasible, although somewhat complex.

Definition 2.2. System WS comprises the following axioms and transformation rules:

AX0	All tautologies of propositional logic
AX1	$\vdash_{WS}\odot\varphi\rightarrow\varphi$
AX2	$\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow\odot(\varphi\vee\psi)$
MP	$\vdash_{WS}\varphi, \vdash_{WS}\varphi\rightarrow\psi \Rightarrow \vdash_{WS}\psi$
RE	$\vdash_{WS}\varphi\leftrightarrow\psi \Rightarrow \vdash_{WS}\odot\varphi\leftrightarrow\odot\psi$
RC1	$\vdash_{WS}\varphi\rightarrow\psi \Rightarrow \vdash_{WS}(\odot\psi\wedge\varphi)\rightarrow\odot\varphi$
RC2	$\vdash_{WS}\varphi \Rightarrow \vdash_{WS}\neg\odot\varphi$

With RC1 and AX1 a derived rule can be obtained whose form exhibits a perfect symmetry, although it is not used in the following argument.

$$\text{RC1'}\ \vdash_{WS}\varphi\rightarrow\psi \Rightarrow \vdash_{WS}(\odot\psi\wedge\varphi)\rightarrow(\psi\wedge\odot\varphi)$$

Theorem 2.1. WS is sound with respect to serial frames.

A serial frame is a frame in which each world has at least one accessible world. Formally, a Frame F=<W,R> is serial if, for every w∈W, there is some w'∈W such that Rww'.

Proof: The proof of the validity of propositional logic (i.e., the validity of AX0 and the validity-preserving MP) is not provided here.

Suppose that M is a model based on a serial frame F, and w is a world in M.

Regarding AX1, assume M,w⊨$\odot\varphi$. According to the definition of $\odot$ in Definition 2.1, M,w⊨φ. Thus, AX1 holds.

Regarding AX2, assume that $M,w\vDash\odot\varphi\wedge\odot\psi$. Consequently, $M,w\vDash\odot\varphi$ and $M,w\vDash\odot\psi$. According to Definition 2.1, from $M,w\vDash\odot\varphi$ we have $M,w\vDash\varphi$ and for any w' such that Rww', $M,w'\vDash\neg\varphi$; and from $M,w\vDash\odot\psi$ we have $M,w\vDash\psi$ and for any w' such that Rww', $M,w'\vDash\neg\psi$. Therefore, we have (a) $M,w\vDash\varphi\wedge\psi$, and (b) for any w' such that Rww', $M,w'\vDash\neg\varphi\wedge\neg\psi$. According to (a), because of the tautology $(\varphi\wedge\psi)\rightarrow(\varphi\vee\psi)$, (c) $M,w\vDash\varphi\vee\psi$ is obtained. According to (b), because of the tautology $(\neg\varphi\wedge\neg\psi)\rightarrow\neg(\varphi\vee\psi)$, (d) $M,w'\vDash\neg(\varphi\vee\psi)$ is obtained. According to (c), (d), and the definition of $\odot$, we have $M,w\vDash\odot(\varphi\vee\psi)$. Thus, AX2 holds.

To prove the correctness of RE, suppose that $\vDash\varphi\leftrightarrow\psi$. Consequently, $M,w\vDash\varphi\leftrightarrow\psi$, and for any w' such that Rww', $M,w'\vDash\varphi\leftrightarrow\psi$. (a) Assume $M,w\vDash\odot\varphi$. According to the definition of $\odot$, $M,w\vDash\varphi$, and for any w' such that Rww', $M,w'\vDash\neg\varphi$. Hence, according to $M,w\vDash\varphi\leftrightarrow\psi$ and $M,w'\vDash\varphi\leftrightarrow\psi$, we have $M,w\vDash\psi$ and $M,w'\vDash\neg\psi$. Therefore, according to the definition of $\odot$, $M,w\vDash\odot\psi$ is obtained. (b) Assume $M,w\vDash\odot\psi$, the same reason as that in (a) ensures that $M,w\vDash\odot\varphi$. Thus by (a) and (b), $M,w\vDash\odot\varphi\leftrightarrow\odot\psi$.

To prove the correctness of RC1, suppose that $\vDash\varphi\rightarrow\psi$. Consequently, $M,w\vDash\varphi\rightarrow\psi$, and for any w' such that Rww', $M,w'\vDash\varphi\rightarrow\psi$. Assume $M,w\vDash\odot\psi\wedge\varphi$. Therefore, (a) $M,w\vDash\odot\psi$, and (b) $M,w\vDash\varphi$. According to (a) and the definition of $\odot$, $M,w\vDash\psi$, and for any w' such that Rww', $M,w'\vDash\neg\psi$. Hence, by $M,w'\vDash\varphi\rightarrow\psi$, $M,w'\vDash\neg\psi$ and the tautology $((\varphi\rightarrow\psi)\wedge\neg\psi)\rightarrow\neg\varphi$, $M,w'\vDash\neg\varphi$ is obtained. Thus, according to the definition of $\odot$, by (b) and $M,w'\vDash\neg\varphi$, we have $M,w\vDash\odot\varphi$.

To prove the correctness of RC2, suppose that $\vDash\varphi$. Consequently, $M,w\vDash\varphi$, and for any w' such that Rww', $M,w'\vDash\varphi$. As the frame is serial, there must be an extra w' such that Rww' and $M,w'\vDash\varphi$. Thus, according to Definition 2.1, $\odot\varphi$ must be false at w; namely, $M,w\vDash\neg\odot\varphi$. ■

Notice that if the rule RC2 in WS is dropped, the system WS_0 is obtained. If the axiom $\neg\odot\top$ is added to WS_0 as an extra axiom, the resulting system WS' is a deductively equivalent one with the system WS. The reasons why the two systems WS and WS' are deductively equivalent are: On the one hand, suppose $\vdash_{WS'}\varphi$. Then, $\vdash_{WS'}\varphi\leftrightarrow\top$. By RE, $\vdash_{WS'}\odot\varphi\leftrightarrow\odot\top$. Hence, according to $\vdash_{WS'}\neg\odot\top$, we have $\vdash_{WS'}\neg\odot\varphi$. On the other hand, according to $\vdash_{WS}\top$ and RC2, we have $\vdash_{WS}\neg\odot\top$.

In the proving of Theorem 2.1 the serial property of the frames was used only in proving the correctness of RC2. Thus we have the following theorem:

Theorem 2.2. WS_0 is sound with respect to arbitrary frames.

The system WS_0, which is weaker than WS, is a system for $\odot$ and $\circledast$ over K. However, WS_0 is not discussed in detail in the paper because the relations between strong accident and accident, and essence and weak essence, which we presented in the introduction, do not hold in the system WS_0.

Theorem 2.3.

(1) $\vdash_{WS}\odot\varphi\leftrightarrow\odot\odot\varphi$

(2) $\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow\odot(\varphi\wedge\psi)$

(3) $\vdash_{WS}\odot(\varphi\vee\psi)\rightarrow(\odot\varphi\vee\odot\psi)$

(4) $\vdash_{WS}(\psi\wedge\odot(\varphi\rightarrow\psi))\rightarrow\odot\psi$[1]

The details of the proof are presented in Appendix 1.

Because of Theorem 2.3(1) and the rule RE, the iterated $\odot$ operators that appear in the formulae in WS can be simplified into one operator.

The formulae in Theorem 2.3 are valid with respect to all frames. This is evidenced by the fact that we did not use RC2 in proving Theorem 2.3. (Appendix 1) This means that the formulae in Theorem 2.3 are provable in WS_0.

However, the formula $\odot(\varphi\wedge\psi)\rightarrow(\odot\varphi\wedge\odot\psi)$ is not a theorem of WS. Consider the following counter-model: $W = \{w_1,w_2\}$, Rw_1w_2, Rw_2w_2, and let φ be true at w_1 and w_2, and ψ be true at w_1 but false at w_2. The model is serial. According to the definition of $\odot$, $\odot(\varphi\wedge\psi)$ and $\odot\psi$ are true at w_1, but $\odot\varphi$ is false at w_1. Thus, $\odot(\varphi\wedge\psi)\rightarrow(\odot\varphi\wedge\odot\psi)$ is false at w_1.

To prove the completeness of WS, we build maximally WS-consistent sets of formulae and show that the canonical model for the logic WS can be built on these sets.

Definition 2.3. A set Γ of well-formed formulae is maximally consistent with respect to a system S, if and only if for every formula α, either $\alpha\in\Gamma$ or $\neg\alpha\in\Gamma$, and there is no finite collection $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}\subseteq\Gamma$, such that $\vdash_S\neg(\alpha_1\wedge\alpha_2\wedge\ldots\wedge\alpha_n)$.

Notice that a set Γ which is maximally consistent with respect to a system S is simply called that Γ is maximally S-consistent.

For any maximally S-consistent set Γ, according to Definition 2.3, if $\vdash_S\alpha$, $\{\neg\alpha\}$ is S-inconsistent. Hence $\neg\alpha$ cannot be in Γ. And α must be in Γ. Thus, if $\vdash_S\alpha$, then $\alpha\in\Gamma$.

Lemma 2.4 [Lindenbaum's lemma]. Every set of S-consistent formulae can be extended to a maximally S-consistent set of formulae.

The proof is routine and its process can be found in logic textbooks. (Blackburn, P., de Rijke M., Venema, Y., 2002, p. 199).

The key point in using the canonical model method is to define the successor of a maximally consistent set of WS in order to eliminate the strongly accidental operator $\odot$.

[1] We thank one of the anonymous referees for providing the strong formula (4) in Theorem 2.3. According to Theorem 2.3(4) and $\vdash_{WS}(\varphi\wedge\odot(\varphi\rightarrow\psi))\rightarrow(\psi\wedge\odot(\varphi\rightarrow\psi))$, which holds true obviously, we have $\vdash_{WS}(\varphi\wedge\odot(\varphi\rightarrow\psi))\rightarrow\odot\psi$.

Definition 2.4. Let Γ be a maximally WS-consistent set of formulae. The successor of Γ, $D(\Gamma)$, is defined as $D(\Gamma) = \{\alpha \mid \odot\neg\alpha \in \Gamma\}$.

Lemma 2.5. For a maximally WS-consistent set Γ of formulae, the successor $D(\Gamma)$, is closed under conjunction.

Proof: Let Γ be a maximally WS-consistent set, and $D(\Gamma)$ be the successor of Γ. Suppose $\alpha \in D(\Gamma)$ and $\beta \in D(\Gamma)$. By the construction of $D(\Gamma)$, $\odot\neg\alpha \in \Gamma$ and $\odot\neg\beta \in \Gamma$. By AX2, $\vdash_{WS}(\odot\neg\alpha \wedge \odot\neg\beta) \to \odot(\neg\alpha \vee \neg\beta)$. Because Γ is maximally WS-consistent, $\odot(\neg\alpha \vee \neg\beta) \in \Gamma$. By RE, $\vdash_{WS}\odot(\neg\alpha \vee \neg\beta) \leftrightarrow \odot\neg(\alpha \wedge \beta)$. Hence, $\odot\neg(\alpha \wedge \beta) \in \Gamma$. Thus $\alpha \wedge \beta \in D(\Gamma)$. ■

Lemma 2.6. Let Γ be a maximally WS-consistent set of formulae. Assume that for some β, $\neg\odot\beta \wedge \beta \in \Gamma$. Then, $\{\beta\} \cup D(\Gamma)$ is WS-consistent.

Proof: Suppose that Γ contains $\beta \wedge \neg\odot\beta$, but $D(\Gamma) \cup \{\beta\}$ is not WS-consistent. Consequently, there must be $\psi_1, \ldots, \psi_n \in D(\Gamma)$ such that:

(a) $\vdash_{WS}\neg(\psi_1 \wedge \ldots \wedge \psi_n \wedge \beta)$.

Let us write ψ for $\psi_1 \wedge \ldots \wedge \psi_n$. Then (a) becomes

(b) $\vdash_{WS}\neg(\psi \wedge \beta)$.

According to $\psi_1, \ldots, \psi_n \in D(\Gamma)$ and Lemma 2.5,

(c) $\psi \in D(\Gamma)$.

According to (c) and the construction of $D(\Gamma)$,

(d) $\odot\neg\psi \in \Gamma$.

According to AX0, from (b):

(e) $\vdash_{WS}\beta \to \neg\psi$.

According to RC1, from (e):

(f) $\vdash_{WS}(\odot\neg\psi \wedge \beta) \to \odot\beta$.

We assumed that $\neg\odot\beta \wedge \beta \in \Gamma$. As Γ is maximally WS-consistent, the assumption leads to

(g) $\beta \in \Gamma$,

and

(h) $\neg\odot\beta \in \Gamma$.

According to (d), (f) and (g), $\odot\beta$ must be contained in Γ. Therefore $\neg\odot\beta \notin \Gamma$. However, this contradicts with (h). ■

Definition 2.5 [Canonical Model]. The canonical model $M^C = <W^C, R^C, \pi^C>$ for the logic WS is defined as follows: (1)W^C is the set of all maximally WS-consistent sets of formulae, (2) $R^C \subseteq W^C \times W^C$ is defined by $R^C\Gamma\Gamma_1$ iff $\Gamma_1 \supseteq D(\Gamma)$, (3) $p \in \pi^C(\Gamma)$ iff $p \in \Gamma$.

Lemma 2.7 [Fundamental Theorem]. Let $M^C = <W^C, R^C, \pi^C>$ be the canonical model for WS. For all formulae φ and all maximally WS-consistent sets Γ, $M^C, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma$.

Proof: The theorem can be proven by induction on the structure of φ. Here we prove the case of $\odot\varphi$ only:

$$M^C, \Gamma \vDash \odot\varphi \Leftrightarrow \odot\varphi \in \Gamma$$

We assume that the theorem holds for φ and for all Γ: $M^C, \Gamma \vDash \varphi \Leftrightarrow \varphi \in \Gamma$.

Suppose that $\odot\varphi \notin \Gamma$. Consequently, as Γ is maximally WS-consistent, $\neg\odot\varphi \in \Gamma$, and either $\neg\varphi \in \Gamma$ or $\varphi \in \Gamma$. (1) If $\neg\varphi \in \Gamma$, $\varphi \notin \Gamma$. Therefore, by the assumption that the theorem holds for φ and for all Γ, we have $M^C, \Gamma \nvDash \varphi$. According to the definition of the truth value of $\odot\varphi$, we directly obtain $M^C, \Gamma \nvDash \odot\varphi$. (2) If $\varphi \in \Gamma$, we obtain $\neg\odot\varphi \wedge \varphi \in \Gamma$. According to $\neg\odot\varphi \wedge \varphi \in \Gamma$ and Lemma 2.6, $\{\varphi\} \cup D(\Gamma)$ is WS-consistent. According to Lemma 2.4 and the definition of W^C, there is a certain $\Gamma_1 \in W^C$ such that $\{\varphi\} \cup D(\Gamma) \subseteq \Gamma_1$; therefore, (a) $D(\Gamma) \subseteq \Gamma_1$ and (b) $\varphi \in \Gamma_1$. By (a) we have $R^C\Gamma\Gamma_1$; and by (b) and the assumption that the theorem holds for φ and for all Γ, we have $M^C, \Gamma_1 \vDash \varphi$. Thus, according to the definition of the truth value of $\odot\varphi$, $M^C, \Gamma \nvDash \odot\varphi$.

Suppose that $\odot\varphi \in \Gamma$. By RE, $\odot\varphi \leftrightarrow \odot\neg\neg\varphi$. Consequently, $\odot\neg\neg\varphi \in \Gamma$. By the construction of $D(\Gamma)$, $\neg\varphi \in D(\Gamma)$. According to the definition of the canonical model, if $R^C\Gamma\Gamma_1$, then $\neg\varphi \in \Gamma_1$. According to $\odot\varphi \in \Gamma$ and $\odot\varphi \rightarrow \varphi \in \Gamma$(AX1), we have $\varphi \in \Gamma$. Therefore, according to the definition of the truth value of $\odot\varphi$, $M^C, \Gamma \vDash \odot\varphi$. ■

Theorem 2.8. In the canonical model of WS, RC is serial.

Proof: Suppose that $\vdash_{WS}\varphi$. Consequently, according to RC2, $\vdash_{WS}\neg\odot\varphi$. Therefore, for all Γ in the canonical model of WS, $\neg\odot\varphi \in \Gamma$ and $\varphi \in \Gamma$. By Lemma 2.7, $M^C, \Gamma \vDash \neg\odot\varphi$ and $M^C, \Gamma \vDash \varphi$. Hence, there must be a certain Γ_1 with $R^C\Gamma\Gamma_1$ and $M^C, \Gamma_1 \vDash \varphi$. Thus, R^C is required to be serial. ■

Theorem 2.9 [Completeness] Given the system WS, for any formula φ, we have $\vdash_{WS}\varphi \Leftrightarrow \vDash \varphi$ with respect to serial frames.

Proof: Soundness, i.e., $\vdash_{WS}\varphi \Rightarrow \vDash \varphi$, was shown in Theorem 2.1.

Completeness follows in the usual way. Suppose that $\nvdash_{ws}\varphi$. Consequently, $\{\neg\varphi\}$ is WS-consistent, and so, by Lemma 2.4, there is some maximally WS-consistent set Γ in W^C such that $\neg\varphi \in \Gamma$. By Lemma 2.7, $M^C, \Gamma \vDash \neg\varphi$. Thus, $\nvDash \varphi$. ■

Notice that since RC2 is not employed in the Lemmas 2.5-2.7, the Lemmas 2.5-2.7 hold for WS_0-consistent sets of formulae. Hence, R^C in the canonical model for WS_0 is arbitrary. Therefore it can be proved that the system WS_0 is complete with respect to arbitrary frames. Together with Theorem 2.2, we have that WS_0 is sound and complete with respect to arbitrary frames.

3. Conclusions and remarks

Numerous papers have examined the logic of contingency (and non-contingency) (Routley 1966, 1969, Humberstone 1995, Kuhn 1995, Zolin 1999, 2002). Some papers have addressed the logic of essence and accident. In this study, using references to essence and accident, we defined weak essence and strong accident and established a logical system for them.

Six remarks summarise this work.

First, the logical system WS is sound and complete with respect to serial frames in which the axiom D is valid. According to the axiom D, $\Box\varphi\rightarrow\Diamond\varphi$, and the propositional calculus, we have $(\varphi \wedge \Box \neg \varphi)\rightarrow(\varphi \wedge \Diamond \neg \varphi)$ and $(\varphi\rightarrow\Box\varphi)\rightarrow(\varphi\rightarrow\Diamond\varphi)$, where $\varphi\wedge\Diamond\neg\varphi$ and $\varphi\rightarrow\Box\varphi$ are accident and essence in terms of the possibility operator and the necessity operator. At the beginning of the section 2, we mentioned that $\odot\varphi$ is semantically equivalent to $\varphi\wedge\Box\neg\varphi$ and $\circledast\varphi$ to $\varphi\rightarrow\Diamond\varphi$. Thus, if A and E are used to denote the accident and essence operators respectively, we have

$$\odot\varphi\rightarrow A\varphi$$

$$E\varphi\rightarrow\circledast\varphi.$$

This means that in serial frames if φ is strongly accidental, φ must be accidental; and if φ is essential, φ must be weakly essential. In other words, in serial frames, $\odot$ is stronger than A, and E is stronger than $\circledast$.

Second, the notions of strong accident and weak essence introduced here are new notions that differ from other modal notions. Marcos (2005) devoted a long and relevant discussion to the relations between the notions of accident and contingency. Strong accident and contingency are formally distinct. Though strong accident and weak essence are based on and related to accident and essence, strong accident is distinct from accident, and weak essence from essence. Since the logics of accident can be regarded as logics for truths of fact because, as Leibniz stated, the opposite of truths of fact is possible, the logic for strong accident (and weak essence) presented here is that for truths of such a specific type of fact that their opposite is necessary. Such a fact can be regarded as a 'pure appearance'. A pure appearance occurs only in the actual world, whereas if something called a weak essence is determined to occur, it must also occur in some other world. If a certain strongly accidental phenomenon, or pure appearance, is supposed to exist in our world, the dogma that whatever is necessary must be actual would be challenged. Or if we maintain that what is true in the world is either necessarily true or possibly true, the strongly accidental phenomenon occurs only in the possible worlds that parallel to our world, and whether there is some strongly accidental phenomenon constitutes a criterion to distinguish our world from its possible worlds.

Third, the system WS_0 for the operators $\odot$ and $\circledast$ defined in Definition 2.1 is sound and complete with respect to all frames, or K-frames. WS can be regarded as a system by adding RC2 to WS_0. In addition, an equivalent system to WS can be obtained by adding the axiom $\neg\odot\top$ instead of the rule RC2 to WS_0. WS_0 is not discussed here in detail because the relations between strong accident and accident, and essence and weak essence, which we presented in the introduction, do not hold in the system WS_0.

Fourth, some extensional systems can be obtained by adding axioms to WS. There is a trivial axiom $\neg\odot\varphi$, to the system WS. If $\neg\odot\varphi$ is added to WS, the axioms AX1 and AX2, and the rules RE, RC1 and RC2 become trivial, and the resulting system will collapse into the propositional logic. The frame that is semantically required by $\neg\odot\varphi$ is reflexive. While WS_0 is extended, it may also encounter a trivial axiom, $\varphi\rightarrow\odot\varphi$. In the system WS_0 plus $\varphi\rightarrow\odot\varphi$, the operator $\odot$ would merely 'idle' because by $\varphi\rightarrow\odot\varphi$ and AX1, we have $\odot\varphi\leftrightarrow\varphi$. The frame of the trivial axiom $\varphi\rightarrow\odot\varphi$ is a frame of dead ends.

Fifth, if the logics in which the modal operator is not interdefinable with a necessity operator are regarded as one kind of nonstandard modal logic, the logic for strong accident and weak essence presented in the paper is a type of this kind of nonstandard modal logic. Numerous philosophical issues (for example, knowledge, obligation, tense) are addressed using modal logics in which the modal operators are semantically interdefinable with 'necessity'. Numerous philosophical issues require nonstandard modal logic to analyse them. The logic of contingency is a type of the nonstandard modal logic that can be used to address issues such as ignorance (van der Hoek and Lomuscio 2004) and a type of propositions that, with their negations, are unprovable. An additional type of the nonstandard modal logic is that of 'essence and accident' (Marcos 2005, Steinsvold 2008, 2011). The proposition 'φ is accidental' is equivalent to 'φ is true but φ could be false'. Hence, the logic of essence and accident can be used to reveal the structure of truths of fact, and the structure of the unprovable truths called the Gödel propositions (Kushida 2010). Likewise, through interpreting necessity in strong accident (and weak essence) as other senses, such as deontic necessity and provability, we would establish some new conceptions and logics for them. Thus, the logic for weak essence and strong accident has a wide range of applications.

Sixth, a formally crucial point is that the canonical model used in this study is special; it is simple, but it has a defect. The definition of the canonical model in Definition 2.5 presupposes that the frame is irreflexive, but that the axioms of WS are valid in serial frames including reflexive frames. The semantics does not completely match the syntax. Similar defects are associated with the establishment of logics of essence and accident (e.g., Steinsvold 2008). Such defects must be realized.

Appendix 1

Theorem 2.3.

(1) $\vdash_{WS}\odot\varphi\leftrightarrow\odot\odot\varphi$

(2) $\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow\odot(\varphi\wedge\psi)$

(3) $\vdash_{WS}\odot(\varphi\vee\psi)\rightarrow(\odot\varphi\vee\odot\psi)$

(4) $\vdash_{WS}(\psi\wedge\odot(\varphi\rightarrow\psi))\rightarrow\odot\psi$

Proof: Regarding 2.3(1),

(a) $\vdash_{WS}\odot\varphi\rightarrow\varphi$	AX1
(b) $\vdash_{WS}(\odot\varphi\wedge\odot\varphi)\rightarrow\odot\odot\varphi$	(a), RC1
(c) $\vdash_{WS}\odot\varphi\rightarrow\odot\odot\varphi$	(b), AX0
(d) $\vdash_{WS}\odot\odot\varphi\rightarrow\odot\varphi$	AX1
(e) $\vdash_{WS}\odot\varphi\leftrightarrow\odot\odot\varphi$	(c), (d), AX0

Regarding 2.3(2),

(a) $\vdash_{WS}(\varphi\wedge\psi)\rightarrow(\varphi\vee\psi)$	AX0
(b) $\vdash_{WS}(\odot(\varphi\vee\psi)\wedge(\varphi\wedge\psi))\rightarrow\odot(\varphi\wedge\psi)$	(a), RC1
(c) $\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow\odot(\varphi\vee\psi)$	AX2
(d) $\vdash_{WS}\odot\varphi\rightarrow\varphi$	AX1
(e) $\vdash_{WS}\odot\psi\rightarrow\psi$	AX1
(f) $\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow(\varphi\wedge\psi)$	(d), (e), AX0
(g) $\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow(\odot(\varphi\vee\psi)\wedge(\varphi\wedge\psi))$	(c), (f), AX0
(h) $\vdash_{WS}(\odot\varphi\wedge\odot\psi)\rightarrow\odot(\varphi\wedge\psi)$	(g), (b), AX0

Regarding 2.3(3),

(a) $\vdash_{WS}\varphi\rightarrow(\varphi\vee\psi)$ AX0

(b) $\vdash_{WS}(\odot(\varphi\vee\psi)\wedge\varphi)\rightarrow\odot\varphi$ (a), RC1

(c) $\vdash_{WS}(\odot(\varphi\vee\psi)\wedge\varphi)\rightarrow(\odot\varphi\vee\odot\psi)$ (b), AX0

(d) $\vdash_{WS}\psi\rightarrow(\varphi\vee\psi)$ AX0

(e) $\vdash_{WS}(\odot(\varphi\vee\psi)\wedge\psi)\rightarrow\odot\psi$ (d), RC1

(f) $\vdash_{WS}(\odot(\varphi\vee\psi)\wedge\psi)\rightarrow(\odot\varphi\vee\odot\psi)$ (e), AX0

(g) $\vdash_{WS}(\odot(\varphi\vee\psi)\wedge(\varphi\vee\psi))\rightarrow(\odot\varphi\vee\odot\psi)$ (c), (f), AX0

(h) $\vdash_{WS}(\odot(\varphi\vee\psi)\rightarrow(\varphi\vee\psi))\rightarrow(\odot(\varphi\vee\psi)\rightarrow(\odot\varphi\vee\odot\psi))$ (g), AX0

(i) $\vdash_{WS}\odot(\varphi\vee\psi)\rightarrow(\varphi\vee\psi)$ AX1

(j) $\vdash_{WS}\odot(\varphi\vee\psi)\rightarrow(\odot\varphi\vee\odot\psi)$ (i), (h), MP

Regarding 2.3(4),

(a) $\vdash_{WS}\psi\rightarrow(\varphi\rightarrow\psi)$ AX0

(b) $\vdash_{WS}(\odot(\varphi\rightarrow\psi)\wedge\psi)\rightarrow\odot\psi$ (a), RC1

(c) $\vdash_{WS}(\psi\wedge\odot(\varphi\rightarrow\psi))\rightarrow\odot\psi$ (b), AX0 ■

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Tianqun Pan
Department of Philosophy,
Nanjing University, Nanjing 210023.
tqpan@126.com

Chungui Yang
School of Economics and Management,
Nanjing University of information Science and Technology,
Nanjing 210044.
sprose521@163.com