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THE COMPLEMENTARY FACES OF MATHEMATICAL BEAUTY[1]

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ABSTRACT

This article focuses on the writings of Hardy, Poincaré, Birkhoff, and Whitehead, in order to substantiate the claim that mathematicians experience a mathematical proof as beautiful when it offers a maximum of insight while demanding a minimum of effort. In other words, it claims that the study of the aesthetic success of theorem-proofs can benefit from the analogy with the economic success of a business, which involves maximizing return on investment. On the other hand, the article also draws on Le Lionnais and Whitehead (again) in order to show that, whereas the kind of aesthetic delight offered by beautiful proofs is typical for well-established branches of mathematics, a romantic and adventurous spirit that goes beyond the search for classical aesthetic delights is needed when the exploration of new mathematics is at stake. The history of mathematics is not only a story of feelings of beauty invoked by perfect products, but also a survey of sublime periods of creative production. No account of mathematical beauty can be complete if it does not complement the classical product aesthetics with a romantic creation aesthetics.

1. Introduction

Since antiquity philosophers have stressed that beauty is all about the emergence of unity amid diversity, and of the perfection of the unified whole from the harmony of its various parts with one another, and with the whole. Alfred North Whitehead's aesthetics of experience results from the application of these antique accounts of beauty to the patterns formed by the feelings that grow together in our experience. For him, beauty qualifies our experience when it is the unification of a variety of contrasting feelings, that is, the concrescence of many feelings of which the individual emotional intensities promote the intensity of the whole emotional pattern, and vice

[1] This paper is closely related to Van Bendegem and Desmet (2016). We are particularly grateful to an anonymous referee for a detailed and extensive critique, inviting us to considerably reshape the paper, so much so that the end result of that process is a paper that is now much more of a distant relative. In other words, the original has undergone a major transformation.

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versa. Whitehead's aesthetics of experience can inform the aesthetics of mathematical theorems: a mathematical theorem is beautiful if it has the inherent capability for the production of beauty of experience in a person with the appropriate background.

But what if we turn our attention from theorems to proofs? And what if we turn our attention from mathematical products to the production of mathematics? To what extent are the antique accounts of beauty in terms of unity and harmony still relevant when we focus on mathematical proof and creation? And how informative are Whitehead's writings when we broaden the discussion of mathematical beauty to include the aesthetics of the mathematical practices of proving and creating? This paper aims at throwing some light on these issues by bringing together elements from the research of both authors in their respective fields of expertise: the philosophy of mathematical practices, and Whitehead's process philosophy.

In section 2 we start with a presentation of the philosophers and mathematicians who are important for their views on aesthetics in mathematics: Whitehead, Hardy, Poincaré, Birkhoff, and Rota. This will lead us via economic-aesthetic considerations to insight, enlightenment and, most importantly, the Whiteheadian notion of self-evidence. This in turn will lead us, we dare to think quite naturally, to an extension of the discussion on aesthetics in mathematics to include the creation process, which will be the core topic of section 3. Putting the two sections together, it is our belief that a more complete picture of aesthetics in mathematics can be presented, as inspired by Whitehead[2].

2. First Act: Economic Aesthetics

2.1. *Overture: Whitehead and Hardy on Beautiful Proofs*

"The feeling," Whitehead wrote, "widespread among mathematicians, that some proofs are more beautiful than others, should excite the attention of philosophers" (MT 60). So let us explore the notion of a beautiful proof by means of two examples that require no specialized mathematical knowledge.

The first example is Euclid's proof of the theorem that there are infinitely many primes. A prime number or prime is an integer greater than 1, which

[2] We deliberately use the verb "inspired" as the intention of this paper's authors is not to present a careful reconstruction of the aesthetic ideas of Whitehead. The views presented on mathematical aesthetics are therefore our own but we do present Whitehead's views rather extensively so that the reader can judge for him- or herself whether or not our inspiration was well-guided.

is only divisible by 1 and itself. Thus 2, 3, and 37 are examples of primes. Primes are the material out of which all integers greater than 1 are built. Indeed, they can all be expressed as a unique multiplication of primes, which is called their 'prime factorization.' Thus 37 = 37 (for each prime the prime factorization is trivial) and 660 = 2.3.3.37 (a prime factor can occur twice or more). Now consider any finite list of primes, say $p_1, p_2, \ldots, p_n$. Let P be the product of all the primes in this list, and let Q be P plus 1. Thus $Q = (p_1 . p_2 . \ldots p_n) + 1$. Also, let q be one of the factors of the prime factorization of Q (possibly $q = Q$, namely if Q is prime). Clearly, q is a prime other than $p_1, p_2, \ldots, p_n$ because Q is divisible by q (since q is a prime factor of Q), whereas Q is not divisible by $p_1, p_2, \ldots, p_n$ (since it leaves the remainder 1 when divided by any one of these primes). This proves that for any finite list of primes there is a prime not on the list, and therefore there must be infinitely many primes.

The second example is Pythagoras's *reductio ad adsurdum* proof of the theorem that the square root of 2 is irrational, i.e. that it cannot be expressed as a fraction. Suppose that the square root of 2 can be expressed as a fraction, in other words, that the square root of 2 equals p/q where p and q are integers and have no common factor (since if they had we could remove it). From this hypothesis (by squaring) follows that $2 = p^2/q^2$. Hence $p^2 = 2q^2$. It follows that p^2 is even (since $2q^2$ is divisible by 2), and therefore that p is even (since the square of an odd number is odd). If p is even then $p = 2s$ for some integer s, and therefore $2q^2 = p^2 = (2s)^2 = 4s^2$. Hence $q^2 = 2s^2$. It follows that q^2 is even, and therefore that q is even. However, if both p and q are even, they have a common factor, namely 2. This contradicts our hypothesis, and hence the hypothesis is false and the theorem true.

Versions of these two examples are also given in Godfrey Harold Hardy's famous 1940 essay, *A Mathematician's Apology*, in which – by the way – no author is quoted more often than Whitehead. According to Whitehead, "mathematics is concerned with the investigation of patterns of connectedness" (AI 153) or, shorter, "mathematics is the study of pattern" (ESP 111). Likewise, Hardy writes: "A mathematician, like a painter and a poet, is a maker of patterns" (84). And he adds: "The mathematician's patterns, like the painter's or the poet's must be *beautiful*; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the final test: there is no permanent place in the world for ugly mathematics" (85). Hardy also agrees with Whitehead that it is an error to represent the love and aesthetic appreciation of mathematics as "a monomania confined to a few eccentrics in each generation" (85 & SMW 20), and he gives his account of Euclid's and Pythagoras's proofs because he is quite sure that most educated readers will be sensitive to their aesthetic appeal, and will recognize their beauty.

2.2. *Developing the Theme: Generality, Depth and other Quality Markers*

According to Hardy, beauty and seriousness are the two criteria by which the mathematician's patterns should be judged (98), and as beauty depends on seriousness (90), he first looks at 'serious' theorems. Note that when Hardy speaks of theorems, he includes their proofs. Hardy writes: "A 'serious' theorem is a theorem which contains 'significant' ideas, and … the qualities which make a mathematical idea significant … are … a certain generality and a certain depth" (103).

According to Hardy, generality involves, among other things, being "capable of considerable extension," being "typical of a whole class of theorems," and being "a constituent in many mathematical constructs" (104). Take Pythagoras's proof of the irrationality of the square root of 2. It can be extended to the square root of 3, namely, by replacing 'even' (which means: 'being a multiple of 2') by 'being a multiple of 3,' and by noticing that if the square of an integer is a multiple of 3, so is the integer itself (since if the integer itself does not have 3 as a factor in its unique prime factorization, its square cannot have it as such a factor either). Pythagoras's reasoning is typical of a whole class of proofs, which includes the proofs of the irrationality of the square root of 2, of 3, of 5, of 7, and so on. In other words, it can be generalized to the proof of the irrationality of the square root of any prime. Moreover, the proof that the square root of each of the primes cannot be a fraction implies the construction of a whole class of non-fractions or irrational numbers to supplement the fractions or rational numbers. This is another element contributing to the generality (and, hence, the seriousness or significance or importance) of Pythagoras's irrationality proof, and it is also an element contributing to the depth of this proof. Indeed, Hardy holds that "the idea of an 'irrational' is deeper than that of an integer" (or a fraction of integers), and that "Pythagoras's theorem is, for that reason, deeper than Euclid's" (110). To Hardy, "it seems that mathematical ideas are arranged somehow in strata" (110), and he imagines the world of numbers as consisting of ever deeper strata, and the stratum of the irrational numbers as deeper than that of the integers. Moreover, Hardy has another reason to think that Euclid's proof, even though very important, is not very deep: depth also involves "using the most powerful weapons" of mathematics, but "we can prove that there are infinitely many primes without using any notion deeper than that of 'divisibility'" (111). However, as Hardy admits, "this notion of 'depth' is an elusive one even for a mathematician who can recognize it" (112).

After dealing with the seriousness of proofs, Hardy turns to their aesthetic qualities, and writes:

> What 'purely aesthetic qualities' can we distinguish in such theorems as Euclid's and Pythagoras's? I will not risk more than a few disjointed remarks.

> In both theorems (and in the theorems, of course, I include the proofs) there is a high degree of *unexpectedness*, combined with *inevitability* and *economy*. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared to the far-reaching results; but there is no escape from the conclusions. There are no complications of detail – one line of attack is enough in each case; and this is true too of the proofs of many much more difficult theorems, the full appreciation of which demands quite a high degree of technical proficiency. We do not want many 'variations' in the proof of a mathematical theorem: 'enumeration of cases,' indeed, is one of the duller forms of mathematical argument. A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way. (113)

2.3. *A Short Digression: Three Comments on the Discussion so Far*

The above quote invites us to make three important comments[3].

(I) One might argue against unexpectedness or surprise as an essential aesthetic quality by giving many cases of beautiful proofs, since for some of them the quality of surprise is absent, whereas for others it is present and adds to the experience of beauty. And indeed, the search for the essential characteristics of beauty seems to be as idle as Ludwig Wittgenstein's search for the essential characteristics of games. This observation can inspire to give up the search for the essence of mathematical beauty, and to introduce, in Wittgenstein's footsteps, the notion of family resemblance in order to give a non-essentialist, and yet coherent account of beautiful proofs. Here, however, we focus on Whitehead not Wittgenstein.

(II) One might argue that all proofs are inevitable and, hence, that inevitability, even though it contributes to the beauty of proofs, does not shed any light on why one proof is more beautiful than another. And indeed, to the extent that only logically ideal proofs are taken to be real proofs, they are all inevitable. This remark presupposes that agreement rules with respect to the logical ideal, whereas the definition of this ideal is the topic of heated debates. Here, however, we ignore these debates.

(III) Hardy does not think that a case-by-case proof has the kind of simplicity that contributes to a mathematician's aesthetic delight. A "'proof by enumeration of cases' (and of cases which do not, at bottom, differ at all profoundly)" is the kind of proof, according to Hardy, "a real mathematician tends to despise" (114). Also, Hardy does not equate economy with the

[3] These comments will not be developed further in this paper as the main objective here is to present the outline of a more complete theory of aesthetics in mathematics. The reason why we nevertheless present the comments is rather to show that, once this (outline of a) theory is further elaborated, they surely will have to be dealt with.

right kind of simplicity as such, but conceives of a proof as economic if it is simple *in comparison with* its seriousness (its significance, its importance, its being far-reaching). Here, we take Hardy's aesthetic quality of economy, which explains how beauty depends on seriousness, to be the most relevant quality for further discussion.

2.4. *Articulating the Theme: Enters Poincaré (with Plato and Kant in the Background)*

In Henri Poincaré's 1908 book, *Science and Method*, we can witness a similar interplay of beauty, importance, generality, surprise, economy, simplicity, and so on:

> What is it that gives us the feeling of elegance in a solution or a demonstration? It is the harmony of the different parts, their symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts. But that is also precisely what causes it to give a large return; and in fact the more we see this whole clearly and at a single glance, the better we shall perceive analogies with other neighboring objects, and consequently the better chance we shall have of guessing the possible generalizations. Elegance may result from the feeling of surprise caused by the unlooked-for occurrence together of objects not habitually associated. In this, again, it is fruitful, since it thus discloses relations till then unrecognized. It is also fruitful even when it only results from the contrast between the simplicity of the means and the complexity of the problem presented, for it then causes us to reflect on the reason for this contrast, and generally shows us that this reason is not chance, but is to be found in some unsuspected law. Briefly stated, the sentiment of mathematical elegance is nothing but the satisfaction due to some conformity between the solution we wish to discover and the necessities of our mind, and it is on account of this very conformity that the solution can be an instrument for us. This aesthetic satisfaction is consequently connected with the economy of thought. (375-6)

Like Hardy, Poincaré connects aesthetic satisfaction with economy. He considers "economy of thought" to be "a source of beauty" (369), and he links it to importance by saying that "the importance of a fact is measured by the return it gives – that is, by the amount of thought it enables us to economize" (374). Applied to "mathematical demonstration," Poincaré holds that the return of a proof involves the "feeling" or "intuition" of "the whole of the argument at a glance" (389-90). Moreover, an economic and important proof, in which we see "all parts in a single glance" (376), allows us to "summarize it in a few lines" for "those that come after us" (377), and to "perceive immediately what must be changed to adapt it to all the problems of a similar nature that may be presented" (376). Initially, finding a solution to a particular problem, and proving it, can involve "blind groping," but to this Poincaré adds:

> My time will not have been lost if this very groping has succeeded in revealing to me the profound analogy between the problem just dealt with and a much more extensive class of other problems; if it has shown me at once their resemblances and their differences; if, in a word, it has enabled me to perceive the possibility of a generalization. Then it will not be merely a new result that I have acquired, but a new force. (374)

Poincaré's account is *similar* to Hardy's account, not identical. So there *are* differences. The main one is that Poincaré, contrary to Hardy, explicitly links surprise, economy, simplicity, importance, and generality with the human mind, with the amount of thought, with perception and intuition of the whole at a single glance, with the return of groping and reasoning, and with the recognition of analogy. In other words, Poincaré links beauty to the *subject* that feels, perceives, intuits, gropes, reasons, and recognizes, whereas Hardy aims at a more *objective* aesthetic account in terms of patterns of *ideas*. Poincaré's account is Kantian, whereas Hardy's is Platonic.[4] For Poincaré, in the field of mathematical activity, a person's interior forms of intuition crystalize in mathematical patterns,[5] whereas for Hardy mathematical patterns are simply a person's notes of observations of the exterior reality of mathematics. This also helps to explain the quite amazing fact that Hardy hardly talks of the aesthetic role that is played by self-evidence, understanding, insight, and enlightenment. However, despite the difference of philosophical language, Hardy's and Poincaré's accounts of mathematical beauty both culminate in a discussion of a similar economic aesthetics. In what follows we present a specific proposal how such economic aesthetics can be understood. The source of inspiration in this case is George David Birkhoff's search for an "aesthetic measure" in his 1933 book, *Aesthetic Measure*.

2.5. *The Theme Finally spelled out in Birkhoff's M = O/C*

In *Aesthetic Measure*, Birkhoff proposes a formula to quantify the qualitative, namely, to measure the aesthetic value as experienced by art-lovers and mathematicians alike. Birkhoff's analysis of the aesthetic experience suggests that the aesthetic measure M equals the harmony, symmetry, or order O of the artistic or mathematical object, divided by the complexity C, which is proportional to the preliminary effort necessary for the act of

[4] Whitehead's aesthetics of experience emphasizes both the subjective *and* the objective character of beauty. Also, even though Whitehead adopts some philosophical ideas of Kant, and several of Plato, his philosophy is neither Kantian nor Platonic.

[5] Poincaré's forms of intuition, however, do not involve Kant's sensible intuition of space and time, but the intellectual intuition of mathematical induction, continuity, and groups.

perception (3-4). Birkhoff's general comment on the formula *M=O/C* reads: "The well known demand for 'unity in variety' is evidently closely connected with this formula" (4) since "it is the intuitive estimate of the amount of order *O* inherent in the aesthetic object, as compared with its complexity *C*, from which arises the derivative feeling of the aesthetic measure *M*" (11-12).

A first way of reading Birkhoff's formula, *M=O/C*, is the following: when the order in an object increases, it will be experienced as more beautiful; when the complexity increases, it will be experienced as more ugly. This way of reading and applying the formula leads to the confirmation of commonplaces such as the antique cliché that the circle is the acme of beauty in mathematics because of its infinite symmetry and utter simplicity; or the prejudice that the most beautiful music is Western music based on the classical scales and harmony; or the banality that in all arts beauty is increased by symmetry. But then what are we to do with Piet Mondrian's explicit rejection of symmetry for being a negative property? Does this imply that all his paintings are ugly? And do we have to conclude from Birkhoff's formula that Arnold Schönberg, Alban Berg, and Anton Webern have heralded the end of beautiful music? And what about the contemporary mathematician's experience of the complex beauty of fractals, which defies the simplistic beauty of the circle? Clearly, Birkhoff's formula cannot deliver on the promise of being an adequate measure of the value of our aesthetic experiences. Rather, it lays bare the problematic nature of all quantitative aesthetics. This is not, however, our final verdict on Birkhoff's approach.

Another and more interesting way of reading (or rather reinterpreting) Birkhoff's formula is suggested by two analogies that he offers to justify it. He writes: "The definition of the beautiful as that which gives us the greatest number of ideas in the shortest space of time (formulated by Heemsterhuis in the eighteenth century) is of an analogical nature" (4). And Birkhoff draws a second analogy "from the economic field":

> In each business there is involved a certain investment *i* and a certain annual profit *p*. The ratio *p/i*, which represents the percentage of interest on the investment, is regarded as the economic measure of success. Similarly in the perception of aesthetic objects … there is involved a feeling of effort of attention, measured by *C*, which is rewarded by a certain tone of feeling, measured by *O* … By analogy, then, it is the ratio *O/C* which best represents the aesthetic measure *M*. (12)

These two analogies suggest a novel interpretation of *M=O/C*, not developed by Birkhoff himself but which is in line with Hardy's and Poincaré's economic accounts of mathematical beauty, especially when applied to a mathematical proof *P*. Indeed: *O(P)* can be taken as directly proportional to Hardy's qualities of depth and generality, and *C(P)* as inversely proportional to his quality of simplicity. Alternatively, *O(P)* can be taken as directly proportional to Poincaré's intuition of the whole and the possibility of generalization, and *C(P)* as directly proportional to Poincaré's amount of thought needed. Synthesizing, we can take *O(P)* as a measure of the mathematician's intuition of the depth and generality of the whole proof, and *C(P)* as a measure of the mathematician's time and effort to overcome the difficulty of the proof, that is, to unify the variety of mathematical ideas and logical deductions constituting the proof. Consequently, and analogous to the return-on-investment formula in economy, the aesthetic measure formula *M(P)=O(P)/C(P)* then means that the aesthetic satisfaction produced by a proof can be defined as the return of intuition of mathematical depth and generality of the whole on investment of time and effort in proof.

We should however temper our (and hopefully the reader's) enthusiasm. We did assume throughout our presentation up to now that proof is central to mathematical practice and hence that aesthetics must therefore somehow be linked directly to proof. But is that necessarily so? What role does in fact proof play? Is it a goal that, once achieved, tells the whole story? Or is it, in a sense, a means to an end, a tool that provokes the aesthetic experience *à la* Birkhoff, yet finds its source elsewhere? In addition, is it a solely cognitive pleasure or does it also involve affective elements? In the "entr'acte"[6] that follows we return to Whitehead, and let Rota express serious concerns along the way, to show how self-evidence and enlightenment are possible candidates for such sources.

2.6. *Entr'acte: Whitehead on Self-Evidence and Rota on Enlightenment*

This second, economic or return-on-investment interpretation of Birkhoff's formula, inspired by Hardy's and Poincaré's economic aesthetics of mathematical proof, and by Birkhoff's two analogies, is not only the interpretation we favor, but also a remarkably good summary of Whitehead's aesthetics of mathematical proof. Indeed, for Whitehead too, the beauty of a proof is all about the return of holistic or large-scale intuition on investment of logical effort in this proof. In *Modes of Thought*, Whitehead utilizes 'self-evidence' and 'understanding' as synonyms for 'intuition,' and he writes:

> There is very little large-scale understanding, even among mathematicians. There are snippets of understanding, and snippets of connections between these snippets. These details of connection are also understood. But these fragments

[6] We call this part of the paper an "entr'acte" as it is meant to make a bridge from the first act (product aesthetics) to the second act (process aesthetics). In contrast to the two acts (and as it should be for an "entr'acte"), we do not use Whitehead-Rota as sources of inspiration here but let them speak for themselves in order to make clear to us (and the reader) that the first act is perhaps rather an *esquisse* or a *Skizze* and not yet a fully-detailed opera.

> of intelligence succeed each other. They do not stand together as one large self-evident coordination. At the best, there is a vague memory of details which have recently been attended to.
> This succession of details of self-evidence is termed *proof*. But the large self-evidence of mathematical science is denied to humans. (MT 46-47)

And then, after emphasizing "the great variety of characters that self-evidence can assume, both as to extent and as to the character of the compositions which are self-evident" (MT 47),[7] Whitehead continues:

> The thesis that I am developing conceives proof, in the strict sense of that term, as a feeble second-rate procedure. When the word *proof* has been uttered, the next notion to enter the mind is halfheartedness. Unless proof has produced self-evidence and thereby rendered itself unnecessary, it has issued in a second-rate state of mind, producing action devoid of understanding. Self-evidence is the basic fact on which all greatness supports itself. But proof is one of the routes by which self-evidence is obtained. (MT 48)

A consequence of this doctrine, Whitehead claims, is that "proof should be at a minimum. The whole effort should be to display … self-evidence" (MT 48). On the other hand, Whitehead adds:

> Our understanding is not primarily based on inference. Understanding is self-evidence. But our clarity of intuition is limited, and it flickers. Thus inference enters as a *means* [our emphasis] for the attainment of such understanding as we can achieve. Proofs are the *tools* [our emphasis] for the extension of our imperfect self-evidence. (MT 50)

Whitehead's next claim is that the return of intuition or self-evidence (or insight or understanding) on investment of logical reasoning in formal proof is not merely a cognitive return, but an affective return as well, namely, that it involves not only proof-induced enlightenment, but also aesthetic delight. Prior to giving a long Whitehead quote justifying this claim, we first highlight that it is a controversial one.

In "The Phenomenology of Mathematical Beauty," starting in conformity with Whitehead's view, but ending with a critique of the economic aesthetics of mathematical beauty, Gian-Carlo Rota writes:

> Every teacher of mathematics knows that students will not learn by merely grasping the formal truth of a statement. Students must be given some enlightenment … or they will quit.[8] …

[7] For example, Ramanujan, contrary to Whitehead, had an impressively extensive intuition with respect to numerical patterns; and Whitehead, contrary to Ramanujan, especially enjoyed "patterns of relationships in which numerical and quantitative relationships are wholly subordinate" (MT 47).

[8] In his educational writings, Whitehead conceives of education as a cyclic process in which each cycle consists of three stages: first the stage of romance, then the stage of precision, and finally, the stage of generalization. These stages, continually recurring in cycles,

> If the statements of mathematics were formally true but in no way enlightening, mathematics would be a curious game played by weird people. Enlightenment is what keeps the mathematical enterprise alive …
> Mathematicians seldom explicitly acknowledge the phenomenon of enlightenment for at least two reasons. First, … enlightenment is not easily formalized. Second, enlightenment admits degrees: some statements are more enlightening than others. Mathematicians dislike concepts admitting degrees … Mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding acknowledgment of the fuzziness of this phenomenon. They say that a theorem is beautiful when they mean to say that the theorem is enlightening. … The term "mathematical beauty" … is a trick mathematicians have devised to avoid facing up the messy phenomenon of enlightenment. (132)

Whitehead does not fail to face the phenomenon of enlightenment by degree, and to acknowledge the vagueness of the notion of enlightenment, and of the similar notions of intuition, self-evidence, understanding, and insight. But it is true that, like most mathematicians, he holds that a proof is beautiful when the return of enlightenment on investment in proof is high. So the question arises whether Whitehead's correlation of aesthetic delight with proof-induced self-evidence is indeed a trick that philosophers better avoid. The answer to this question is "No," because such a correlation is only a fallacious trick in a philosophy which separates the faculty of affection from that of cognition, and which posits the correlation without justification – Whitehead's philosophy does neither of these two. Whitehead rejects all faculty psychology, and, as we will highlight now, he justifies the intimate relation between the aesthetic delight in a proof and the logic-induced enlightenment it involves.

According to Whitehead, "the feeling, widespread among mathematicians, that some proofs are more beautiful than others" can be justified by the fact that "*aesthetic experience is another mode of the enjoyment of self-evidence*" (our italics), and he adds:

> I suggest to you that the analogy between aesthetics and logic is one of the undeveloped topics of philosophy.

determine what Whitehead calls "the rhythm of education" (AE 15). Roughly, and applied to mathematics, on might say that Whitehead's three stages are the stage of undisciplined intuition, the stage of logical reasoning, and the stage of logically guided intuition. By skipping stage one, and never arriving at stage three, bad math teachers deny students the only possible motivation to love mathematics: enlightenment! Unfortunately, as Paul Lockhart correctly claims in *A Mathematician's Lament*, at present most math teachers are forced by the standard school mathematics curriculum to be bad math teachers: "What is happening is the systematic undermining of the student's intuition," whereas the goal of each proof, that is, of each mathematical argument, is to be "a beacon of light – it should refresh the spirit and illuminate the mind" (68).

> In the first place, they are both concerned with the enjoyment of a composition, as derived from the interconnection of its factors. There is one whole, arising from the interplay of the many details. The importance arises from the vivid grasp of the interdependence of the one and the many. If either side of this antithesis sinks into the background, there is trivialization of experience, logical and aesthetical.
> The distinction between logic and aesthetics consists in the degree of abstraction involved. Logic concentrates upon high abstraction, and aesthetics keeps … close to the concrete …
> The characteristic attitude of logical understanding is to start with the details, and to pass to the construction achieved. Logical enjoyment passes from the many to the one. … The understanding of logic is the enjoyment of the abstracted details as permitting that abstract unity. As the enjoyment develops, the revelation is the unity of the construct. …
> The movement of aesthetic enjoyment is in the opposite direction. We are overwhelmed with the beauty of the building … The whole precedes the details. We then pass to discrimination. (MT 61-62)

In other words, according to Whitehead it is justified to correlate aesthetic enjoyment (aesthetic delight) with logical enjoyment (logic-induced enlightenment) because the enjoyment of mathematical beauty is a mode of the enjoyment of self-evidence. But whereas logical enjoyment develops from the variety of details to their unification in an abstract pattern, the aesthetic enjoyment is an emotional flow from the concrete whole to the discrimination of details.

We could end this entr'acte here and move on to the second act, but instead we choose to develop the theme a bit further so that it prepares the ground for the opening of the next act. First Whitehead addresses the doubt whether or not, when the topic of aesthetic enjoyment is sufficiently explored, there will be anything left over for discussion:

> This doubt is unjustified. For the essence of great experience is penetration into the unknown, the unexperienced. … Our lives are passed in the experience of disclosure. As we lose this sense of disclosure, we are … descending to mere conformity with … the past. Complete conformity means the loss of life. (MT 62)
> The essence of life is to be found in the frustrations of established order. The Universe refuses the deadening influence of complete conformity. And yet in its refusal, it passes towards novel order as a primary requisite for important experience. (MT 88)

Frustrating the established aesthetic canon to enable the emergence of a novel one is characteristic of great artists and mathematicians alike. Think, for example, of David Bowie, a musician and performer who, again and again, renewed his music and performance. Bowie died while we were writing the first part of this chapter, and in one of the many interviews rebroadcasted following his death, Bowie said that whenever he started to see a thread of stability in what he was making, he felt the impulse of destroying it, and he added: "When you can predict the outcome of what you are doing, that's incredibly unsatisfactory." Bowie did not stop being a non-conformist and feeling the urge to make new kinds of songs and acts. He will be remembered as forever young in accord with what Whitehead once wrote: "Youth is not defined in years but by the creative impulse to make something" (AE 119). And what holds for Bowie, holds for great mathematicians too, which is why Eric Temple Bell once wrote: "The essence of mathematics is its eternal youth" (quoted by Le Lionnais 144). The creative and youthful character of mathematics is the topic of the second part of this chapter. In contrast to the static description of the beauty of mathematical proofs we now want to insert some dynamics into our presentation (which should probably be very much to the liking of Whitehead).

3. Second Act: The Dynamics of Aesthetics in Mathematics

3.1. *Extending the Original Theme: From Product Aesthetics to Creation Aesthetics (Enters Le Lionnais)*

When we think of the beauty of theorems and proofs, our thinking is primarily product-focused. It is the end result of the mathematical activity that is being evaluated aesthetically. That is why it is appropriate to say that our account so far has dealt with product aesthetics. However, the exploration of new territory is as important a part of mathematics as the enjoyment of old territory. In fact, the history of mathematics is the story of the exploration of new territory. The explorations of analytic geometry, group theory, non-Euclidean geometry, complex analysis, and chaos theory, are but a few examples of important historical breakthroughs in mathematics.

The exploration of fractals, one of the relatively recent branches of the tree of mathematical research, is most appropriate as an illustration to introduce the 1948 essay of François Le Lionnais, "Beauty in Mathematics," which will guide our thinking beyond a mere product aesthetics towards a creation aesthetics. When asking mathematicians what is so beautiful about fractals, one is likely to receive answers such as: "because they are so intricate," or: "because an extremely simple reiteration formula can generate an extremely complex object," or: "because no one expected such complexity to be hidden in mathematics." These answers feel bizarre because they seem to flatly contradict Birkhoff's view that mathematical beauty is about minimizing complexity rather than generating it. With respect to the exploration of fractals, however, mathematicians seem to behave like "romantics" – this is Lionnais's term, and it stands for his fundamental insight with respect to mathematical exploration, research, and creation.

"Works of art can," according to Le Lionnais, "be ranged under two grand banners: *classicism*, all elegant sobriety, and *romanticism*, delighting in striking effects and aspiring to passion" (123), and he holds that it is also possible to distinguish classical beauty and romantic beauty in mathematics.

"We say," Le Lionnais writes, "that a mathematical proposition has classical beauty when we are impressed by its austerity or its mastery over diversity, and even more so when it combines these two characteristics in a harmoniously arranged structure" (124). Clearly, according to Le Lionnais, classical beauty involves unity amid diversity, and harmony of parts and whole; and what he writes next on classical beauty can be read as a repetition of our account of Hardy and Poincaré. Classical beauty depends on the "value" or importance of a mathematical theorem, which, in its turn, "depends on the depth of the mathematics required to prove it" (126) as well as on its generality, that is, the light it throws on the "sublime interdependence" of mathematical concepts, theorems, proofs, and theories (130): "Classic are the methods which cast a new light on previously known facts, bringing together and unifying discoveries formerly considered disparate" (137). Moreover, classical beauty involves unexpectedness, for it "intrigues us especially when we are expecting a certain disorder" (124), as well as economy: "It seems to us that a method earns the epithet of classic when it permits the attainment of powerful effects by moderate means" (136).

3.2. *Variations: Cantor's Infinities and Von Koch's Snowflake: Romanticism at Work*

"By contrast with classical mathematical beauty," Le Lionnais writes, "we are now going to examine another sort of beauty which can be described as romantic. Its underlying principle is the glorification of violent emotion, nonconformism and eccentricity" (130). Le Lionnais first states that when mathematicians behave like romantics (say, like Bowie), that is, when they are in exploration-and-creation mode, their violent rejection of conformism can produce "what seem to be completely illogical results repugnant to common sense" (132). And then Le Lionnais gives a spectrum of examples, from which we select the example of Georg Cantor's exploration of transfinite set theory, and the example of the exploration of continuous functions without derivatives. With respect to the first of these two examples, Le Lionnais writes:

> Does not the modern theory of sets take as its point of departure concepts which seemed an insolent defiance of common sense when Cantor defended them? This exuberant theory had to enjoy repeated successes in other disciplines already classic like arithmetic and analysis before we would accept the existence of quantities "greater than infinity" (Cantor's expression) plus the startling number situated *on the other side of infinity*. Theologians were not the last to protest certain ideas as unfair competition.

> After the paradoxes come the anomalies, the irregularities, indeed the monstrosities. They arouse some people's indignation and to others bring delight. (133)

For example, Cantor's romantic exploration of transfinite set theory aroused Leopold Kronecker's indignation, and Kronecker famously wrote: "I do not know what predominates in Cantor's theory – philosophy or theology, but I am sure that there is no mathematics there." On the other hand, it delighted David Hilbert, who famously wrote: "No one will drive us from the paradise which Cantor created for us."[9] Today, the romantic exploration of Cantor has been turned into a more classical enjoyment of transfinite set theory, and most mathematicians can share Hilbert's aesthetic delight. This is the way mathematics advances: once a new territory has been mapped out, and its problems have been identified, the classicist ideal takes over again – problems must be solved, proofs must be found, and the investment of effort in logical reasoning must give a significant return of self-evidence.

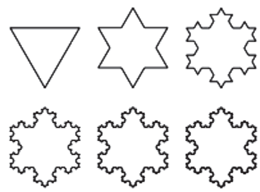
With respect to the second of the examples we selected, some preliminary remarks are due. A continuous function is one of which you can draw the graph without lifting your pen. And a continuous function has a derivative in a point of its graph, if it is possible to draw a tangent to this graph through this point. For a long time it was thought that all continuous functions have derivatives in all points of their graphs, except in a few, namely, those points in which these graphs are changing their direction not smoothly, but abruptly. This classical intuition, however, was falsified by the romantics who explored continuous functions *without* derivatives, that is, of continuous graphs (and curves in general) that are *nowhere* smooth and, hence, do not have a single tangent. With respect to these romantics, Le Lionnais writes:

> When Riemann and Weierstrass made known the existence of continuous functions without derivatives, what an outcry came from the mathematicians against these newcomers: "I turn with fright and horror from this lamentable plague of continuous functions having no derivatives," exclaimed Charles Hermite. If it is difficult to reason about such functions, it becomes impossible to visualize fully the infinite caprices of the curves representing them. (134)

Not only Hermite was horrified – Poincaré too! And in *Science and Method* Poincaré called these "weird" continuous functions without derivatives, "monsters," and the ensemble of curves without tangents, a "collection of

[9] Less known than Hilbert's remark in "Über das Unendliche," *Mathematische Annalen* 95(1926):367-94, is the fact that Whitehead, who wrote very few mathematical research papers, nonetheless published three papers in the *American Journal of Mathematics*, which dealt with Cantor's theory in the context of Peano's developments of mathematical logic, and Russell's symbolism for the logic of relations: "On Cardinal Numbers" *AJoM* 24 (1902):367-94; "The Logic of Relations, Logical Substitution Groups, and Cardinal Numbers" *AJoM* 25(1903):157-78; "Theorems on Cardinal Numbers" *AJoM* 26(1904):31-32.

monstrosities" (447). Consider, however, Helge von Koch's construction of the curve without tangents that is called the Koch snowflake: start with an equilateral triangle, and then recursively alter each line segment in three steps: (1) divide the line segment in three segments of equal length; (2) draw an equilateral triangle that has the middle segment of step 1 as its base and points outward; (3) remove the line segment that is the base of the triangle of step 2. In the figure below you can see the initial triangle and the first five iterations, but the Koch snowflake itself is the limit approached as the above steps are followed over and over again, and so it cannot be pictured.



When the Koch snowflake was discovered-cum-invented in 1904, it was indeed an example of a curve without tangents, that is, one of Poincaré's "monstrosities." However, today, the Koch snowflake is known as one of the many examples of beautiful fractals, and Le Lionnais writes:

> The romantic wildness of continuous functions without derivatives could evoke in the mystical Hermite the impression that he was battling demons escaped from some mathematical hell. Observe, however, the case of one of these functions, the celebrated Koch curve or *homunculus*. Every arc of this curve, no matter how short, is similar to the entire curve, whose exquisite arabesque it chisels into infinity with unfailing regularity. What could be more classical? (136)

Today, the Koch snowflake can be classified among other fractals by using a generalized notion of dimension. A straight line still has dimension 1, and the plane still has dimension 2, but curves like the Koch snowflake are given a dimension in between 1 and 2, log4/log3 to be exact. In other words, the Koch snowflake gradually became part of a larger whole, and the romantic exploration of the incoherent and wild territory of curves without tangents gradually turned into a classical enjoyment of the ever more coherent and structured territory of fractals. Today, we might say, Poincaré's monsters have been tamed. Clearly, once the more romantic mathematicians have explored and mapped out the new territory, they hand over their maps to more classical mathematicians for further refinement, and there is beauty to be found in both the romantic explorations and the classical refinements.

"We must not think," Le Lionnais concludes, "that mathematics progresses only via the royal road of classicism" (138). Mathematics is in need of both classicism and romanticism, of both classical and romantic mathematicians. In fact, mathematical progress consists essentially in “an ever-renewed antagonism” between the classical “desire for unity” and the romantic “rebellions” which erupt at every new attempt to get at the heart of mathematics (141). So if we want to arrive at a satisfactory aesthetics of mathematics, we cannot separate mathematics from other art forms, but will have to take into account at least the two basic forms here treated: the classical or economic aesthetics that is primarily oriented on the (end) products of mathematics, especially proofs; and the romantic or exploring aesthetics that is oriented on prospection of new mathematical territory[10].

3.3. *The Finale: Return to Whiteheadian Themes*

Le Lionnais’s essay makes clear the role of romantic exploration in the progress of mathematics, but it does not offer a clear definition of romantic beauty as opposed to classical beauty. We learn that romantic beauty involves striking effects and violent emotions, but do not get anywhere near a clear-cut definition such as the antique unity and harmony definitions of classical beauty, or their Whiteheadian offspring. In *Emblems of Mind*, however, Edward Rothstein does shed some additional light on romantic beauty in mathematics (180-189). Skipping all the details he gives, and immediately focusing on his conclusion, Rothstein holds that whereas classical beauty enlightens and delights, romantic beauty disturbs and overwhelms, and, following Immanuel Kant, Rothstein reserves the term ‘beautiful’ to qualify an instance of classical beauty, and the term ‘sublime’ to qualify an instance of romantic beauty. In fact, the beautiful and the sublime are so different, that one might question the use of the expression ‘romantic beauty.’

And what about Whitehead? The romantic spirit, which Le Lionnais claims to be vital to mathematical progress, is prominently present in Whitehead’s oeuvre. This youthful spirit of creativity, which sacrifices aesthetic harmony and delight to be rewarded with it at a higher level, and which Ferdinand Gonseth called “the spirit of adventure” (quoted by Le Lionnais 144), is a key ingredient of the aesthetics of experience that Whitehead develops, for example, in his *Adventure of Ideas*.

In *Adventures of Ideas* – a romantic title indeed – Whitehead highlights that there are different types of beauty (AI 252), and that it is important, when developing a theory of aesthetics, not only to distinguish lower and

[10] An anonymous referee suggested that there might be a nice parallel to be drawn here between the classicist-romantic mathematics conception and the normal-revolutionary science conception in the early Thomas Kuhn. We do indeed believe this to be the case, keeping in mind of course that it is not a good idea to simply equate mathematics and the sciences. As it turns out, even one of the core concepts in Kuhnian thinking, namely the notion of revolution, is not as easily transposable as one might expect. One of the authors of this paper has dealt with this problem, together with Karen François in François & Van Bendegem (2010).

higher types of beauty, but also to account for the role of discordant feelings in the transition from type to type. According to Whitehead:

> There are in fact higher and lower perfections, and an imperfection aiming at a higher type stands above lower perfections. … Progress is founded upon the experience of discordant feelings. The social value of liberty lies in its production of discords. There are perfections beyond perfections. All realization is finite, and there is no perfection which is the infinitude of all perfections. Perfections of diverse types are among themselves discordant. Thus the contribution to Beauty which can be supplied by Discord – in itself destructive and evil – is the positive feeling of a quick shift of aim from the tameness of outworn perfection to some other ideal with its freshness still upon it. …
> Ancient Greek civilization … attained its proper beauty in human lives to an extent not surpassed before or since. Its arts, its theoretic sciences, its modes of life, its literature, its philosophic schools, its religious rituals, all conspired to express every aspect of this wonderful ideal. Perfection was attained, [but] with the attainment inspiration withered. With repetition in successive generations, freshness gradually vanished. Learning and learned taste replaced the ardor of adventure. … To sustain a civilization with the intensity of its first ardor requires more than learning. Adventure is essential, namely, the search for new perfections. (AI 257-258)

According to Whitehead, the destructive feelings that are needed when a civilization falls prey to the tameness of outworn perfection are none other than the romantic feelings that are needed when a mathematical theory "which overwhelms us the first time we meet it, comes to appear trite in the end" (Le Lionnais 127); the discordant feelings that lie at the basis of human progress in general are none other than the sublime feelings that lie at the basis of mathematical progress in particular.

Whitehead claims that adventure – the search for new perfections – is essential to the advance of humanity in general, and mathematics in particular. The latter claim is not different from Le Lionnais' claim that romanticism – the search for novel aesthetic delights – is essential to the advance of mathematics.

In *Process and Reality*, Whitehead writes:

> There are various types of order, and some of them provide more trivial satisfaction than do others. Thus, if there is to be progress beyond limited ideals, the course of history by way of escape must venture along the borders of chaos in its substitution of higher for lower types of order. (PR 111)

Very few additions are needed to turn this quote into an appropriate ending of our discussion of the romantic aesthetics of mathematical creation:

> There are various types of order, and some of them provide more trivial aesthetic satisfaction than do others. Thus, if there is to be progress beyond limited mathematical ideals, the course of the history of mathematics must venture along the borders of chaos in its substitution of higher for lower types of order.

4. Conclusion

We have arrived at the end of our performance, the curtain is about to drop. Instead of a mere summary, we want to pay a final tribute to Whitehead in order to emphasize how important he has been as a source of inspiration to develop our own ideas.

Whitehead's aesthetics of experience includes a classical account of the aesthetic delight in proof, which emerges from the return of intuitive enlightenment on investment of logical effort in proof. It also includes a romantic account of mathematical creation, of the adventurous explorations of the mathematical landscape, which lead mathematicians from one mountain top to another, from one panoramic view to another, from one type of order to another, and from aesthetic delight to aesthetic delight, but not without descending in the valleys of discord, not without suffering the loneliness of non-conformism, not without venturing along the borders of chaos, and not without exercising the challenging freedom that can save us from our smallness and lack of vision.

Whitehead's aesthetics of experience, not surprisingly for an aesthetics developed *by* a mathematician, allows us to understand not only that it is "alluring aesthetic satisfactions which have motivated modern mathematicians to cultivate their cherished study with such ardor" (Le Lionnais 121), but also that it is "the spirit of adventure (which) animates the mathematician far more than his formulas" (Gonseth, quoted by Le Lionnais 144), or, in other words, that Cantor is right: "The essence of mathematics lies precisely in its freedom" (896).

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Abbreviations of Books by Alfred North Whitehead:

AE = *The Aims of Education and other essays*. (New York: The Free Press, 1929/1967.)

AI = *Adventures of Ideas.* (New York: The Free Press, 1933/1967.)

ESP = *Essays in Science and Philosophy.* (Westport, Connecticut: Greenwood Press, 1947/1968.)

MT = *Modes of Thought*. (New York: The Free Press, 1938/1968.)

PR = *Process and Reality.* Corrected Edition. Edited by D. R. Griffin and D. W. Sherburne. (New York: The Free Press, 1929/1985.)

SMW = *Science and the Modern World.* (New York: The Free Press, 1925/1967.)