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FROM LINGUISTICS TO DEONTIC LOGIC VIA CATEGORY THEORY

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Abstract

The present paper aims to bridge the gap between deontic logic, categorial grammar and category theory. We propose to analyze Forrester's (1984) paradox through the framework of Lambek's (1958) syntactic calculus. We first recall the definition of the syntactic calculus and then explain how Lambek (1988) defines it within the framework of category theory. Then, we briefly present Forrester's paradox in conjunction with standard deontic logic, showing that this paradox contains some features that reflect many problems within the literature. Finally, we analyze Forrester's paradox within the framework of the syntactic calculus and we show how a typed syntax can provide conceptual insight regarding some of the problems that deontic logic faces.

Keywords: Categorial grammar, Joachim Lambek, Forrester's paradox, Modal logic, Deductive systems, Type theory.

1. Introduction

Categorial grammar, understood as a (well defined) formal syntax, was introduced in the work of Ajdukiewicz (1935) and Bar-Hillel (1953), although the expression 'categorial grammar' appeared later (see Bach 1988). In contrast with approaches that concentrate on the phrase's structure, categorial grammar analyzes sentences in terms of *syntactical categories* (Bach 1988, p. 1). In 1958, Lambek introduced a syntactical calculus based on Gentzen's (1934) sequent calculus as a tool to analyze these *syntactical categories*. Later, Lambek (1968; 1969) realized that his syntactic calculus behaved as a *closed category*, and so Lambek (1988) revisited categorial grammar within the framework of category theory. Although the term 'category' is used to refer to *syntactical categories* in categorial grammar, 'category' in the context of category theory refers to the concept introduced in the work of Eilenberg and Mac Lane (1945). Following Lambek (1988, p.298), we shall restrict the use of the term 'category' to category theory only. We will speak of *syntactic types* rather than *syntactic categories* to avoid confusion.

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Also introduced in the fifties but utterly unrelated to categorial grammar and category theory, von Wright (1951) proposed to construct a (monadic) deontic logic in analogy with alethic modal logic to formalize the behaviour of terms such as *obligatory*, *permitted* or *forbidden*.[1] It did not take much time for his work to become the target of multiple objections, now known as the *paradoxes* of deontic logic. The first objection was formulated by Prior (1954) and aimed to show that von Wright's notion of *commitment* fails in context of derived obligations. Von Wright (1956) took this paradox very seriously and answered it by introducing the building blocks of what is nowadays known as *dyadic deontic logic*. Even though dyadic deontic logic attracted the interest of some philosophers (for instance Rescher 1958), the initial system of von Wright stayed a choice target for the paradoxes. In 1958, Prior presented the paradox of the good Samaritan, precursor of Nozick's and Routley's (1962) robbery paradox, which aimed to show that the relation of deontic consequence in von Wright's system leads to paradoxical results. But the *coup de grâce* against his approach was given by Chisholm (1963), who showed that monadic deontic logic is simply not powerful enough to represent conditional obligations, a central notion for normative reasoning. Dyadic deontic logic was further developed by von Wright (1967) and has since then become a distinct field of study in philosophical logic (for an overview, see Tomberlin 1981). Chisholm's objection was however not literally a mortal blow to monadic deontic logic. After the introduction of possible world semantics by Kripke (1963), von Wright's system was analyzed in the framework of modal logics, which gave rise to the well-known *standard system of deontic logic* (SDL for short).[2] Despite many objections against SDL as a logic that represents deontic entailment, it remains a central point in the literature insofar as philosophers either build their systems as extensions of SDL or position themselves according to it, pointing out which principles they reject and why. Some times after the development of SDL, Forrester (1984) presented an objection against the standard system to show that SDL's notion of consequence simply does not represent the relation of deontic entailment within the natural language.

Although deontic logic was introduced roughly at the same time as categorial grammar and category theory, it has been developed and utilized independently from both these fields.[3] The present paper aims at making the bridge between deontic logic, on the one hand, and category theory and

[1] In a *monadic* deontic logic, there is only one formula in the scope of the deontic operators, while in a *dyadic* one there are two.

[2] Note that von Wright's initial system is not equivalent to the standard system.

[3] To the best of our knowledge, there has been no approach joining categorial grammar with deontic logic and scant few regarding deontic logic and category theory. Johanson (1996) used category theory to model normative systems but there was no follow up, and Lucas (2006; 2007; 2008) used category theory to model actions in deontic contexts.

categorial grammar, on the other. This will be done by analyzing Forrester's paradox through the categorical framework of Lambek's syntactic calculus. We will first present the syntactical calculus in the context of category theory, and then we will briefly present SDL and Forrester's paradox, explaining why we concentrate on this paradox rather than others. We will analyze the paradox through the framework developed so far and conclude in the last section with remarks for future research.

2. Lambek's syntactic calculus

The general idea of categorial grammar is to analyze sentences in terms of syntactical types rather than to concentrate on the phrase's structure (Bach 1988, p.1). Assuming a finite set of primitive types $\mathcal{T} = \{\tau_1, \ldots, \tau_n\}$, the set of complex types is defined recursively from the language $\mathcal{L} = \{(,), \otimes, /, \backslash, \mathcal{T}\}$ by:

$$\varphi := \tau_i \mid \varphi \otimes \psi \mid \varphi/\psi \mid \varphi\backslash\psi$$

The operator $\otimes$ expresses concatenation of types and $\varphi \otimes \psi$ is read 'φ times ψ'. The types φ/ψ and $\varphi\backslash\psi$ are respectively read 'φ over ψ' and 'φ under ψ'. Here, a syntactical type is understood as a linguistic functor (not to be confused with a functor between two categories), which takes a type and transforms it into another. For instance, if we assume a set $\mathcal{T} = \{s, n\}$ of two primitive types s (for declarative sentences) and n (for nouns), then the type $n\backslash s$ is understood as a linguistic functor which takes a noun and transforms it into a declarative sentence *from the right*. Similarly, the type s/n would be understood as a linguistic functor which takes a type n and transforms it into a declarative sentence *from the left*. Let us borrow two examples from Lambek (1958, p.156) to see how it works.

Paul	steals
n	$n\backslash s$

While 'Paul' is of type n, 'steals' is of the type that takes a noun and transforms it into something of type s from the right. Similarly, the word 'poor' in the following sentence is of the type that takes a noun and transforms it into a noun from the left, thus n/n.

poor	Paul	steals
n/n	n	$n\backslash s$

A word must be of *at least* one type, but it can have different types depending on the context. For instance, 'steals' in the latter example could also be understood as of the type $((n/n) \otimes n)\backslash s$, that is, of the type which takes something of type $(n/n) \otimes n$ and transforms it into a declarative sentence from the right. By concatenation, the sentence ‘poor Paul steals’ is of the following type.

$$((n/n) \otimes n) \otimes (n\backslash s)$$

Having this in mind, an obvious question would be to ask: how can one prove that something which is of type $((n/n) \otimes n) \otimes (n\backslash s)$ is of type s? Put differently, how can we prove that ‘poor Paul steals’ is a sentence? This is where Lambek’s syntactic calculs enters the picture.

The syntactic calculus is defined on the grounds of a *deductive system*, which in turn is defined within the framework of category theory. Let us consider first the axiomatic definition of a category (cf. Mac Lane 1971, pp.7-8).

Definition 1. A category $\mathcal{C}$ is composed of:

1. $\mathcal{C}$-objects;
2. $\mathcal{C}$-arrows;
3. an operation which assigns to each $\mathcal{C}$-arrow two $\mathcal{C}$-objects (the domain and the codomain);
4. an operation which assigns to each pairs of $\mathcal{C}$-arrows $f : a \to b$ and $g : b \to c$ the *composite arrow* $gf : a \to c$, which respects associativity, as represented by the following commutative diagram;

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h(gf)=(hg)f"'] \arrow[dr, "gf"] & b \arrow[d, "g"] \arrow[dl, "hg"'] \\ d & c \arrow[l, "h"] \end{tikzcd}$$

5. an operation assigning to each $\mathcal{C}$-object an identity arrow $1a : a \to a$ which respects the *identity law*, meaning that the following diagram commutes.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, "1_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"'] & c \end{tikzcd}$$

Now, consider Lambek’s (1988, pp.302,307) definition of a deductive system.

Definition 2. A *deductive system* is composed of:

1. types, considered as objects;
2. proofs, considered as arrows between types;
3. the identity arrow $1_A : A \to A$ for any type A respecting the identity law;
4. composition $gf : A \to C$ of arrows $f : A \to B$ and $g : B \to C$ respecting the associative law.

$$f\,1_A = f = 1_B\,f \qquad \text{(Identity law)}$$

$$h(gf) = (hg)f \qquad \text{(Associative law)}$$

As such, a deductive system can be defined as a category where the arrows are proofs (deductions) and the objects are formulas (types).[4] Now, consider the definition of a *monoidal category* (cf. Mac Lane 1971, p.161).

Definition 3. A *monoidal category* is a category $\mathcal{C}$ composed of:

1. a tensor product $\otimes$ (i.e., a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$);
2. a unit object 1 such that there are arrows (natural isomorphisms) f_x and g_x;

$$f_x : 1 \otimes x \to x$$
$$g_x : x \otimes 1 \to x$$

3. an arrow a_{xyz} for associativity (which is a natural isomorphism);

$$a_{xyz} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$

4. the following commutative diagrams, representing respectively the triangle and pentagon identities.

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr] & & x \otimes (1 \otimes y) \arrow[dl] \\
& x \otimes y &
\end{tikzcd}$$

$$\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[r] & w \otimes (x \otimes (y \otimes z)) \\
((w \otimes x) \otimes y) \otimes z \arrow[ur] \arrow[dr] & & \\
& (w \otimes (x \otimes y)) \otimes z \arrow[r] & w \otimes ((x \otimes y) \otimes z) \arrow[uu]
\end{tikzcd}$$

[4] It should be noted that Lambek (1988; 1999) proceeds backwards and defines a category on the grounds of a deductive system instead of defining a deductive system as a category, although he himself points out that this definition of a category is unorthodox (cf. Lambek 1988, p.298). Since Lambek's work, it has been shown that there is a strong connection between logical systems and different kinds of categories. As such, from a conceptual point of view, it is better to define a deductive system as a category that respects specific criteria, and this is what will be done within this paper.

In a nutshell, a monoidal category is a category $\mathcal{C}$ together with an associative tensor product and a unit. From the definition of a deductive system, Lambek (1988, p.302) proposes the following definition for the syntactic calculus. As we will see, the syntactic calculus is defined as a monoidal category given that concatenation of types is an associative tensor product that comes with a unit I.

Definition 4. The *syntactic calculus* is a deductive system with a special type I and which is closed under three binary operations $\otimes$, $/$ and $\backslash$ that satisfy the axiom schema (ρ_A)–(α_A^{-1}), which are natural transformations, and the rules of inference (f^*)–$({}^+g)$.

$$\rho_A : A \otimes I \to A \tag{ρ_A}$$

$$\rho_A^{-1} : A \to A \otimes I \tag{ρ_A^{-1}}$$

$$\lambda_A : I \otimes A \to A \tag{λ_A}$$

$$\lambda_A^{-1} : A \to I \otimes A \tag{λ_A^{-1}}$$

$$\alpha_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \tag{α_{ABC}}$$

$$\alpha_{A\,BC}^{-1} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \tag{$\alpha_{A\,BC}^{-1}$}$$

$$\frac{g : A \to C/B}{g+ : A \otimes B \to C}\,(g^+) \qquad \frac{g : B \to A\backslash C}{{}^+g : A \otimes B \to C}\,({}^+g)$$

$$\frac{f : A \otimes B \to C}{f^* : A \to C/B}\,(f^*) \qquad \frac{f : A \otimes B \to C}{{}^*f : B \to A\backslash C}\,({}^*f)$$

Taken together, the axiom schema (ρ), (ρ^{-1}), (λ) and (λ^{-1}) make I into a nullary type (we can think of I as a space), and the axiom schema (α) and (α^{-1}) state that $\otimes$ respects associativity. Note that Lambek (1988, p.309) assumes the following proofs, and as such the syntactic calculus satisfies the triangle and pentagon identities.

$$m_1 : A \otimes B \to (A \otimes I) \otimes B$$

$$m_2 : (A \otimes I) \otimes B \to A \otimes (I \otimes B)$$

$$m_3 : A \otimes (I \otimes B) \to A \otimes B$$

$$n_1 : ((A \otimes B) \otimes C) \otimes D \to (A \otimes B) \otimes (C \otimes D)$$

$$n_2 : (A \otimes B) \otimes (C \otimes D \to A \otimes (B \otimes (C \otimes D))$$

$$n_3 : A \otimes (B \otimes (C \otimes D)) \to A \otimes ((B \otimes C) \otimes D)$$

$$n_4 : A \otimes ((B \otimes C) \otimes D) \to (A \otimes (B \otimes C)) \otimes D$$

$$n_5 : (A \otimes (B \otimes C)) \otimes D \to ((A \otimes B) \otimes C) \otimes D$$

To understand properly the categorical structure induced on the syntactic calculus via the operations / and \, we must introduce the notion of a *closed category* (cf. Mac Lane 1971, p.184).[5]

Definition 5. A *closed category* is a monoidal category $\mathcal{C}$ where the tensor product has a right adjoint.

To properly understand this definition, we must introduce some terminology. A *functor* is a mapping (an arrow) between two categories that preserve compositions and identities. Following Mac Lane (1971, p.13):

Definition 6. A *functor* $\mathcal{C} \xrightarrow{\Phi} \mathcal{B}$ is a morphism between two categories such that:

1. there is $F(c)$ in $\mathcal{B}$ for each c in $\mathcal{C}$;
2. there is $F(c_1) \xrightarrow{F(f)} F(c_2)$ in B for each $c_1 \xrightarrow{f} c_2$ in $\mathcal{C}$;
3. $F(1_c) = 1_{F(c)}$;
4. $F(gf) = F(g)F(f)$.

As it happens, a tensor product of some arbitrary formula and a formula A can be seen as a functor $- \otimes A : \mathcal{C} \rightarrow \mathcal{D}$. Note that since we are in a monoidal category, and not a symmetric monoidal category (where the tensor product would be commutative), $A \otimes -$ and $- \otimes A$ are different functors. Now, consider the following definition (cf. Mac Lane 1971, p.16).

Definition 7. Let $f : a \rightarrow b$ be a $\mathcal{C}$-arrow and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A *natural transformation* $\eta : F \rightarrow G$ is a family of arrows such that for every a of $\mathcal{C}$ there is $\eta_a : F(a) \rightarrow G(a)$ in $\mathcal{D}$ making the following diagram commute (in $\mathcal{D}$).

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) & G(b) \arrow[l, "\eta_b"] \end{tikzcd}$$

A *natural isomorphism* is a natural transformation that possesses an inverse, that is, when there is also a natural transformation $n^{-1} : G \rightarrow F$ such that $n^{-1}n = 1_F$ and $nn^{-1} = 1_G$.

[5] Note that Mac Lane (1971) defines a closed category as a symmetric monoidal category where the tensor product has a right adjoint functor. Here, we modified slightly the definition and do not require that the category is symmetric, i.e., that the tensor product is commutative.

An *adjunction* between two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ arises when there is an exact correspondence between the arrows $f : a \to G(b)$ and $g : F(a) \to b$ (cf. Goldblatt 2006, p.439). This notion was introduced in the work of Kan (1958). Formally, an adjunction is defined as follows (cf. Mac Lane 1971, p.80).[6]

Definition 8. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with two natural transformations $\eta : 1_C \to GF$ and $\zeta : FG \to 1_D$, called respectively the *unit* and the *counit* of the adjunction, such that $\zeta_F F(\eta) = 1_F$ and $G(\zeta)\eta_G = 1_G$.

This definition implies that the two following diagrams are commutative.

$$\begin{tikzcd} F(a) \arrow[r, "F(\eta_a)"] \arrow[dr, "1_{F(a)}"'] & FGF(a) \arrow[d, "\zeta_{F(a)}"] \\ & F(a) \end{tikzcd} \qquad \begin{tikzcd} G(b) \arrow[r, "\eta_{G(b)}"] \arrow[dr, "1_{G(b)}"'] & GFG(b) \arrow[d, "G(\zeta_{(b)})"] \\ & G(b) \end{tikzcd}$$

A closed category is thus a monoidal category where there is an isomorphism between the class of morphisms (of $\mathcal{C}$) from $A \otimes B$ to C and the class of morphisms from A to $[B, C]$, with $[B, C]$ a right adjoint functor of $\otimes$. In other words, we have:

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{C}}(A, [B, C])$$

Given non-commutativity of the tensor product, there can be another right adjoint functor such that there is an isomorphism between the class of morphisms (of $\mathcal{C}$) from $A \otimes B$ to C and the class of morphisms from B to $\langle A, C\rangle$.

Lambek's syntactic calculus is defined as a *biclosed* monoidal category, that is, a monoidal category where the tensor product possesses two right adjoints. Indeed, $/ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a right adjoint of $\otimes$ inasmuch as the isomorphism can be proven from (f^*) and (g^+). Moreover, the tensor product possesses another right adjoint, namely $\backslash : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. The isomorphism between the Hom-sets can be proven from $({}^*f)$ and $({}^+g)$.

It is noteworthy that it is not assumed that there are proofs from $A \otimes B$ to $B \otimes A$, nor from $B \otimes A$ to $A \otimes B$. Hence, the syntactic calculus is not a *symmetric* biclosed monoidal category, nor a *braided* biclosed monoidal category. Actually, the fact that the syntactic calculus is *not* a symmetric monoidal closed category implies that the two right adjoints are different.

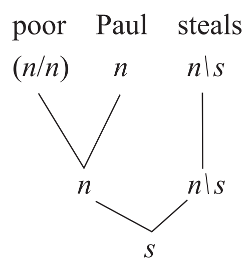
[6] There are various equivalent definitions of an adjunction. Note that here we use Mac Lane's (1971, p.83) theorem 2 and provide an alternative definition.

Returning to the aforementioned example, we can now prove within the syntactic calculus that a sentence of type $((n/n) \otimes n) \otimes (n\backslash s)$ is of type s. To do so, it suffices to show that there is a proof of s from $((n/n) \otimes n) \otimes (n\backslash s)$.

Example 1. There is an arrow $h : ((n/n) \otimes n) \otimes (n\backslash s) \to s$, namely the arrow $[(({}^+(1_{n\backslash s}))^*)(1_{n\backslash n})^+]^+$.

$$\dfrac{\dfrac{\dfrac{1_{n/n} : n/n \to n/n}{(1_{n/n})^+ : n/n \otimes n \to n} \qquad \dfrac{\dfrac{1_{n\backslash s} : n\backslash s \to n\backslash s}{{}^+(1_{n\backslash s}) : n \otimes (n\backslash s) \to s}}{({}^+(1_{n\backslash s}))^* : n \to s/(n\backslash s)}}{(({}^+(1_{n\backslash s}))^*)\,(1_{n\backslash n})^+ : n/n \otimes n \to s/(n\backslash s)}}{[(({}^+(1_{n\backslash s}))^*)(1_{n\backslash n})^+]^+ : n/n \otimes n \otimes n\backslash s \to s}$$

Note that one could have found an easier way to obtain that arrow. Intuitively, one could be tempted to assume that if it is possible to prove that there is an arrow $f : (n/n) \otimes n \to n$, and then another arrow $g : n \otimes (n\backslash s) \to s$, then one would obtain a proof from $((n/n) \otimes n) \otimes (n\backslash s)$ to s. The intuition would be to go from $((n/n) \otimes n)$ to n, and then from $n \otimes (n\backslash s)$ to s, as represented in the following tree.



But this is not how it works. Indeed, the composite arrow gf would not be defined in that case since g's domain would be different from f's codomain. In order to apply transitivity, one would first need to prove that there is h: $((n/n) \otimes n) \otimes (n\backslash s) \to n \otimes (n\backslash s)$ and then show $i : n \otimes (n\backslash s) \to s$, which is quite different from our starting intuition. Nonetheless, it is still possible to show that there is a derived rule such that if one assumes that there are arrows $f : A \otimes B \to C$ and $g : C \otimes D \to E$, then one can conclude that there is an arrow $h : (A \otimes B) \otimes D \to E$.[7]

$$\dfrac{\dfrac{(\mathrm{H})\, f : A \otimes B \to C \qquad \dfrac{(\mathrm{H})\, g : C \otimes D \to E}{g^* : C \to E/D}}{(g^*)f : A \otimes B \to E/D}}{((g^*)f)^+ : (A \otimes B) \otimes D \to E} \qquad (\beta)$$

[7] I am indebted to Marc-Kevin Daoust for that idea.

This makes the proof more intuitive insofar as it is easier to show the two following arrows.

$$\frac{1_{n/n} : n/n \to n/n}{(1_{n/\mathrm{n}})^{+} : (n/n) \otimes n \to n}$$

$$\frac{1n\backslash s : n\backslash s \to n\backslash s}{{}^{+}(1_{n\backslash s}) : n \otimes n\backslash s \to s}$$

Hence, the aforementioned derived rule can be applied to obtain the following proof.

$$\frac{(1_{n/n})^{+} : (n/n) \otimes n \to n \quad +(1_{n\backslash s}) : n \otimes n\backslash s \to s}{[(({}^{+}(1_{n\backslash s}))^{*})1_{n/n}]^{+} : ((n/n) \otimes n) \otimes (n\backslash s) \to s}$$

Summing up, Lambek's syntactic calculus can be defined through category theory as a biclosed monoidal category since the operations respect the minimal properties of a tensor product with its adjoint(s). From a semantical point of view, Lambek interpreted the syntactic calculus within a topos semantics via a structure preserving functor (see Lambek 1988, p.313 for details).[8] Note, however, that the syntactic calculus can be interpreted in some semantics that are different than toposes. The advances that have been made in category theory showed that monoidal categories, and not just toposes, play an important role in the development of logical frameworks. There are actually many semantical interpretations available in structures *weaker* than toposes. For instance, various substructural logics are extension of the syntactic calculus and can be interpreted within the framework of residuated lattices. Already in 1968, Lambek (1968, p.292) defined his syntactic calculus as a *residuated category*, which was defined a year later as a biclosed monoidal category (Lambek 1969, p.98). As such, one can see why / and \ are sometimes named the left and right *residuals* of $\otimes$.

While linear logic was introduced by Girard in 1987, Abrusci (1990a, 1990b) showed how Lambek's syntactic calculus is actually a fragment of the noncommutative intuitionnistic linear logic (i.e., NILL without intuitionnistic negation).[9] The non-commutativity of the syntactic calculus comes from the fact that it is defined as a biclosed monoidal category instead of a symmetric (or braided) monoidal closed category. Casadio and Lambek (2002) then showed how different categorial grammars can be obtained

[8] This requires the introduction of the usual lattice operations of a Heyting algebra (cf. Lambek 1999, p.281). A topos is a Cartesian closed category with a subobject classifier (cf. Mac Lane 1971, p.106).

[9] See also Casadio et al. (2004) for the comparison between the syntactic calculus and NILL.

through the extension of the calculus via some additional properties, depending on whether one wants commutativity, compact duality or involutive duality. In these cases, one would respectively obtain classical bilinear logic, compact bilinear logic or Curry's semantic calculus. The reader may consult Casadio and Lambek (2002) for details. See also Moortgat (2009) for symmetric categorial grammar, which would be an extension of classical bilinear logic (where the tensor is non-commutative).

3. SDL and Forrester's paradox

Leaving aside the categorical framework of the syntactic calculus, we now turn our attention to modal logic. It is worth mentioning that the term 'standard system' in the deontic logic literature is sometimes used ambiguously. A distinction must be made between *the* standard system and *a* standard system (or *the* standard *systems*). *The* standard system, SDL, is equivalent to the modal logic *KD*.[10] Following Åqvist (2002, p.205), SDL is the smallest set constructed from the language $\mathcal{L} = \{(,), Prop, \neg, \supset, O\}$ (with the usual definitions for the other logical connectives, interdiction $FA =_{def} O\neg A$ and weak permission $PA =_{def} \neg O \neg A$, and *Prop* a denumerable set of atomic propositions) which is closed under the rules *modus ponens*, O-necessitation and contains the axiom schema (A1)–(A4).

$$\frac{\vdash A \supset B \quad \vdash A}{\vdash B} \quad (\textit{modus ponens})$$

$$\frac{\vdash A}{\vdash OA} \quad (\text{O-necessitation})$$

$$\text{every propositional tautology of } \mathcal{L} \tag{A1}$$

$$PA \equiv \neg O \neg A \tag{A2}$$

$$O(A \supset B) \supset (OA \supset OB) \tag{A3}$$

$$OA \supset PA \tag{A4}$$

While (A1) implies that SDL is an extension of the (classical) propositional calculus, (A2) assumes that weak permission is the dual operator of O, (A3) is the K axiom of distribution and (A4) is the well-known axiom schema (D). The set of well-formed formulas is defined recursively by:

$$A := p_i \mid \neg A \mid A \supset B \mid OA$$

[10] See Chellas (1980) for an introduction to modal logics.

The modal logic KD is the smallest set which satisfies these conditions. On the other hand, *a* standard system is a conservative extension of KD. Following Åqvist (2002, p.155), Δ is a normal system of deontic logic if $K \subseteq \Delta$, and a strongly normal deontic logic if $KD \subseteq \Delta$.[11] Thus, *a* standard system is a strongly normal deontic logic or, put differently, the class of standard systems contains extensions of KD.

Forrester's paradox aims to show that the rule (ROM), a derived rule in KD, fails to represent the intuitive notion of deontic entailment in *the* standard system.

$$\frac{\vdash A \supset B}{\vdash OA \supset OB} \qquad \text{(ROM)}$$

The paradox can be formulated as follows (see Peterson and Marquis 2012):

(1) Jones murders Smith (p)

(2) Jones ought to not murder Smith ($O\neg p$)

(3) If Jones murders Smith, then Jones ought to murder Smith gently ($p \supset Oq$)

(4) If Jones murders Smith gently, then Jones murders Smith ($q \supset p$)

By modus ponens between (1) and (3) we get (5).

(5) Jones ought to murder Smith gently (Oq)

By (ROM) and *modus ponens*, (6) follows from (4) and (5).

(6) Jones ought to murder Smith (Op)

But from (A4) and (6) we can derive (7), which contradicts (2).

(7) it is false that Jones ought to not murder Smith ($\neg O\neg p$)

We follow Åqvist's (2002, pp.161-173) presentation of the paradoxes. Let $\tau : NDL \rightarrow KD$ be a translation function from the *Natural Deontic Language* to KD's language, and τ^{-1} be the inverse map of τ from KD to NDL. NDL can be viewed as a set containing the (intuitive) semantical consequences of the normative English language. The translation function assigns equivalence classes of formulas.

Paradoxical results arise in two situations. Either there is a formula $A \in KD$ such that $\tau^{-1}(A) \notin NDL$ because intuitively $\tau^{-1}(A)$ seems invalid, or there is a formula $A \notin KD$ such that $\tau^{-1}(A) \in NDL$ because $\tau^{-1}(A)$ seems intuitively valid. These are respectively a *right-to-left inadequacy* and a

[11] Note that a formal system can be understood as a set of theorems, containing specific axioms, that is closed under some rules of inference.

left-to-right inadequacy. They imply that the logic is not an adequate representation of the natural language. Let:

$$\phi = p \wedge [(p \supset Oq) \wedge (q \supset p)]$$
$$\pi = O\neg p \wedge \phi$$

Forrester's paradox states two right-to-left inadequacies, that is:

$$\phi \supset Op \in KD \qquad \tau^{-1}(\phi \supset Op) \notin NDL$$
$$\pi \supset \bot \in KD \qquad \tau^{-1}(\pi \supset \bot) \notin NDL$$

It is shown in Peterson and Marquis (2012) that Forrester's paradox is actually *not* a paradox for *the* standard system, although it *is* a paradox for *a* standard system, which either admits $q \supset p$ as an axiom or includes a necessity operator of type $\Box$ and can translate (4) by $\Box\,(q \supset p)$. Forrester's paradox is not a paradox for SDL per se since:

$$\phi \supset Op \notin KD$$
$$\pi \supset \bot \notin KD$$

Put differently, neither $\phi \supset Op$ nor $\pi \supset \bot$ are derivable in *KD*. Nonetheless, Forrester's paradox can be objected to some (conservative) extensions of *KD*.

We chose to analyze Forrester's paradox insofar as it incorporates three different problems for deontic logic. First, given that the paradox uses an obligation that arises in a context of violation, it incorporates Chisholm's (1963) paradox and shows that the standard systems cannot deal appropriately with conditional obligations. Secondly, the step from (4), (5) and (ROM) to (6) is an instance of Prior's (1958) good Samaritan paradox, or of Nozick's and Routley's (1962) robbery paradox.[12] In addition to these two features, Castañeda (1986) pointed out that Forrester's paradox brought to light a more subtle, linguistic problem for deontic logic. Indeed, Castañeda (1986) argued that the paradox appears when one does not distinguish between *propositions* and *practitions*. In other words, if one wants to avoid the paradox, then one must distinguish between the *context* in which an obligation arises (described by a proposition) and the *action* which is in the scope of the deontic operator. While a *proposition* can be true or false (i.e., it is declarative), a *practition* cannot (Castañeda 1986, p.37).[13] We chose

[12] This step cannot be done in *KD*, but assume a standard system in which it can.

[13] Note that his formal framework was not proposed as a tool that was meant to resolve Forrester's paradox but rather that it was a happy consequence that the framework provided the material to deal with the paradox. Indeed, Forrerster's paradox was published in 1984 (although Castañeda became aware of the paradox in 1982), and the solution proposed by Castañeda (1986) is based upon the formal framework presented in Castañeda (1981), which is the result of his work done in Castañeda (1959; 1968; 1970; 1977).

to analyze Forrester's paradox through the framework of categorial grammar via category theory insofar as it involves:

1. contrary-to-duties (cf. Chisholm's paradox);
2. the notion of deontic entailment (cf. the good Samaritan paradox);
3. a syntactical distinction between propositions and practitions (cf. Castañeda's analysis of Forrester's paradox).

As we will see, these three aspects can be dealt with by using a typed syntax.

4. The paradox revisited

4.1. Deontic logic within a typed syntax

Deontic logic is usually interpreted as a modal logic. It should be noted, however, that our approach is neither oblivious nor incompatible with the modal tradition. Indeed, there have been many approaches that developed modal logics within categorical settings.[14] Even though these approaches mostly dealt with $S4$ modalities, it remains that modal logic can be incorporated within a categorical framework. We wish to show how this could be beneficial for deontic logic. In what follows, we illustrate how category theory provides an interesting framework to analyze many problems that deontic logic faces. We show that most of these problems can be solved using a typed theory. Since categorical logic, once the quantifiers are introduced, is mainly a typed theory, our analysis suggests that category theory is a likely candidate for the formalization of deontic logic.

Obviously, the application of categorial grammar to deontic logic requires more primitive types than only s and n. In his 1958 paper, Lambek introduced n_c, n_s and n_p for count nouns, substance nouns and plural nouns. A year after, Lambek (1959) introduced more syntactic types to analyze the English verb phrase. For instance, he augmented $\mathcal{T}$ with i, p and q, standing respectively for the types of infinitive of intransitive verb, present participle of intransitive verb and past of intransitive verb. Of course, more syntactical types can be introduced, as it is done in Casadio and Lambek (2002), where one can find types for past and present questions, past and present declarative sentences and objects. Although our analysis does not require that we burden ourselves with types for questions, the analysis of Forrester's paradox requires that we introduce types for tensed declarative sentences. As it was noted by Lambek (1988, p.315), thus constructed, the syntactic calculus does not account for *imperative* sentences, which we often distinguish from factual propositions since Hume's naturalistic fallacy.

[14] See for instance Reyes and Zolfaghari (1991) and Makkai and Reyes (1995). See also Steve Awodey and Kohei Kishida. Topology and Modality: the Topological Interpretation of First-Order Modal Logic. *The Review of Symbolic Logic*, 1 (2): 146-166, 2008.

It is worth pausing here and consider carefully the type we wish to introduce for imperative sentences (or, more accurately, normative propositions).[15] Hume's semantical dichotomy between *facts* and *norms* states that one cannot infer a descriptive sentence from a set constructed only of normative premises, and, vice versa, that one cannot infer a normative sentence from a set of descriptive premises. This semantical dichotomy causes some issues when one considers Jørgensen's (1937) dilemma. In a nutshell, the dilemma arises if one assumes that the truth value of a declarative sentence depends upon the world, that is, if one adopts a (naive) correspondence theory of truth. Since there is a semantical dichotomy between facts and norms, one can safely assume that if normative propositions can be true or false, then it is not in the same conditions that descriptive propositions do. But if one assumes that the truth value of a sentence depends upon its correspondence with reality, then, assuming the semantical dichotomy, normative propositions are not declarative given that normative propositions cannot be verified empirically. Thus the dilemma: normative inferences seem to follow some logical rules, but since the object of logic (when applied to inferences) is *declarative* sentences, it seems that logic cannot deal with normative inferences seeing that normative propositions are not declarative (assuming the correspondence theory of truth).[16]

Our solution to the dilemma is to reject the correspondence theory of truth. Normative propositions *are* declarative. For example, it is true that one should not steal since there are legal norms that forbid stealing. However, we nonetheless assume Hume's semantical dichotomy since normative propositions are not true in the same conditions that descriptive propositions are.[17] But still, both are declarative. This leads us to distinguish between two syntactic types. While s_d is the syntactic type of descriptive (declarative) sentences, s_n is the syntactic type of normative (declarative) sentences.

Thus, we assume a language $\mathcal{L}$ as defined earlier but constructed from the set of primitive types $\mathcal{T} = \{s_d, s_n, n, i, p, q\}$. As it was mentioned in Lambek (1999, p.281), one must assign to each word at least one syntactical type by means of a dictionary. This begs the question of the syntactical types of words such as *ought*, *should*, *can* or *must* in a normative context, but also of words such as *obligatory*, *permitted* or *forbidden*. Since von Wright, the tradition has been to interpret these words as modalities that influence declarative sentences. For instance, the following sentence would be translated in SDL by Op, with p being 'John eats'.

John ought to eat

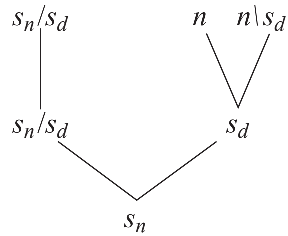
[15] We distinguish between a norm (or an imperative) and a normative proposition. While the latter is declarative, the former is not.

[16] See Peterson (2011) for an analysis of the dilemma.

[17] Following Alchourrón and Bulygin (1981), the truth value of a normative proposition depends upon a norm.

From the point of view of syntactical types, this would require that the sentence be written as:

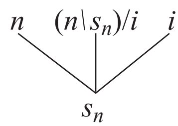
ought (John eats) (8)



In this case, ‘ought’ would be of type s_n/s_d, that is, of the type that takes a descriptive sentence and transforms it into a normative one from the left.

Another option would be to consider the term ‘ought’ as it was written in the first sentence.

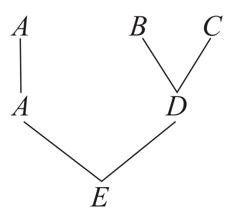
John ought (to eat) (9)



In both cases, it can easily be proven that the sentences are of type s_n. The proof of (8) requires (but not necessarily) a derived rule similar to β as presented previously. The proof (left to the reader) becomes obvious when one can prove (δ).

$$\dfrac{\dfrac{\text{(H) } g : \mathrm{B} \otimes \mathrm{C} \to \mathrm{D} \qquad \dfrac{\text{(H) } f : A \otimes D \to E}{{}^{*}f : D \to A\backslash E}}{({}^{*}f)g : B \otimes C \to A\backslash E}}{{}^{+}[({}^{*}f)g] : A \otimes (B \otimes C) \to E} \qquad (\delta)$$

The proof of (δ) can be visualized by the following tree.



The proof of (9) is straightforward.

$$\dfrac{\dfrac{1_{n\backslash s_n/i} : (n\backslash s_n)/i \rightarrow (n\backslash s_n)/i}{(1_{n\backslash s_n/i})^{+} : ((n\backslash s_n)/i) \otimes i \rightarrow n\backslash s_n}}{^{+}[(1_{n}s_{n/i})^{+}] : n \otimes (((n\backslash s_n)/i) \otimes i) \rightarrow s_n}$$

Now, an obvious question would be to determine whether these two types are equivalent, that is, if there are arrows:

$$\gamma : s_n/s_d \rightarrow (n\backslash s_n)/i$$
$$\gamma^{-1} : (n\backslash s_n)/i \rightarrow s_n/s_d$$

Without pretending that such arrows do not exist, we did not find any (although this may be unsatisfactory for the reader). For now, we assume that the distinction between the two different syntactical types is sound.

With that in mind, we can revisit Forrester's paradox and see which type each sentence is. Let us consider the first type for 'ought' (hereafter the a-translations).

(1)	Jones	murders	Smith
	n	$(n\backslash s_d)/n$	n

(2a)	ought	(not	(Jones	murders	Smith))
	s_n/s_d	s_d/s_d	n	$(n\backslash s_d)/n$	n

(3a)	Jones	murders	Smith	implies	ought	(Jones
	n	$(n\backslash s_d)/n$	n	$(\mathbf{s_d}\backslash\mathbf{s_n})/\mathbf{s_n}$	s_n/s_d	n
	murders	Smith	gently)			
	$(n\backslash s_d)/n$	n	$s_d\backslash s_d$			

(4)	Jones	murders	Smith	gently	implies	Jones	murders	Smith
	n	$(n\backslash s_d)/n$	n	$(s_d\backslash s_d)$	$(s_d\backslash s_d)/s_d$	n	$(n\backslash s_d)/n$	n

(5a)	ought	(Jones	murders	Smith	gently)
	s_n/s_d	n	$(n\backslash s_d)/n$	n	$s_d\backslash s_d$

(6a)	ought	(Jones	murders	Smith)
	s_n/s_d	n	$(n\backslash s_d)/n$	n

(7a)	not	(ought	(not	(Jones	murders	Smith)))
	$\mathbf{s_n}/\mathbf{s_n}$	s_n/s_d	$\mathbf{s_d}/\mathbf{s_d}$	n	$(n\backslash s_d)/n$	n

The proofs of the types of these sentences are relatively easy and so are left to the reader.[18] Let us now look at the translations for the second interpretation of 'ought' (hereafter the b-translations).

(2b)	Jones	ought	to not	murder	Smith	
	n	$(n\backslash s_n)/i$	i/i	i/n	n	

(3b)	Jones	murders	Smith	implies	Jones	ought
	n	$(n\backslash s_d)/n$	n	$(\mathbf{s_d}\backslash \mathbf{s_n})/\mathbf{s_n}$	n	$(n\backslash s_n)/i$
	to murder	Smith	gently			
	i/n	n	$i\backslash i$			

(5b)	Jones	ought	to murder	Smith	gently
	n	$(n\backslash s_n)/i$	i/n	n	$i\backslash i$

(6b)	Jones	ought	to murder	Smith
	n	$(n\backslash s_n)/i$	i/n	n

(7b)	not	(Jones	ought	to not	murder	Smith)
	$\mathbf{s_n}/\mathbf{s_n}$	n	$(n\backslash s_n)/i$	$\mathbf{i/i}$	i/n	n

4.2. Ought-to-be/Ought-to-do

The reader familiar with the deontic logic literature will in all likelihood have noticed that the first interpretation of 'ought' is related to the *Ought-to-be* interpretation of normative sentences, while the second interpretation refers to the *Ought-to-do* (see von Wright 1999). While the *Ought-to-be* interpretation takes 'ought' as a modality that changes the truth value of a descriptive proposition (e.g., modal deontic logic within possible worlds semantics), the *Ought-to-do* interpretation considers that there are actions (rather than propositions) in the scope of the deontic operator (as in dynamic deontic logic and deontic logic with algebras for actions).

It is noteworthy that from the perspective of categorial grammar, the *Ought-to-do* interpretation of the operator O allows us to construct normative sentences that are more closely related to the English language. Indeed, the compositionality of $n\backslash sn/i$ seems to be naturally related to the natural language. When interpreted in terms of *Ought-to-be*, 'ought' is taken as a modality that modifies the truth value of a descriptive statement, and thus the formal translation requires that we rearrange the phrase's structure. This results in an asymmetry between the formal language and the English language. Moreover, interpreting 'ought' in terms of *Ought-to-do* not only

[18] Hint: construct a tree and then use (β) and (δ).

allows us to preserve the phrase's structure within the formal language of the syntactic calculus, but it also preserves the meaning of term 'ought' in the natural language. Indeed, actions, and not declarative sentences, are obligatory, permitted or forbidden. In this respect, interpreting 'ought' in terms of *Ought-to-do* allows for a more faithful representation of the sentence's structure and meaning into the formal language. Hence, the analysis of normative sentences from the point of view of syntactical types suggests that normative (declarative) sentences should be interpreted as being of the *Ought-to-do* type rather than of the *Ought-to-be*, meaning that the deontic operator O is of type $(n\backslash s_n)/i$ rather than s_n/s_d.

That being said, if one wants an *Ought-to-do* interpretation of O, then one might want to replace the primitive type i by a primitive type a for action verbs. Understood this way, the deontic operator O would be of type $(n\backslash s_n)/a$ rather than $(n\backslash s_n)/i$. This would mean that O is of the type that takes a noun and an action verb and transforms it into a normative sentence. This view is consistent with Hume's semantical dichotomy given that it implies that the truth value of a normative proposition does not rely upon the descriptive proposition in the scope of O. Thus, the *Ought-to-do* interpretation should be preferred to the *Ought-to-be* insofar as it preserves the syntactical structure of normative sentences within the English language and that it is consistent, contra the modal interpretation, with Hume's naturalistic fallacy.

4.3. Conditional obligations and detachment

In addition to revealing the *Ought-to-do* structure of normative sentences, the analysis of Forrester's paradox through the framework of the syntactic calculus allows us to provide a formal explanation to the problem that is implicit to Chisholm's paradox, i.e., that a material conditional is inappropriate to deal with conditional obligations. Indeed, one can see in both the a-translations and the b-translations that the material conditional in (3) is of the type that takes a normative statement and a declarative statement and turns it into a normative statement (a conditional obligation). Our analysis provides a formal explanation as to why the material conditional is not appropriate to translate conditional obligations: the conditional $\supset$ should not be interpreted as a connective of type $(s_d\backslash s_n)/s_n$. As such, Chisholm's paradox arises when a deontic conditional is interpreted as a connective of the type that takes a descriptive sentence (a context) and a normative sentence and transforms it into a normative sentence.

Many deontic logicians think that material conditionals are not sufficient to represent the formal properties of conditional obligations. We agree.[19] In a nut-shell, to model conditional obligations, one requires a primitive dyadic

[19] As it is shown in Peterson (2014b), it is nonetheless possible to model conditional obligations with a monadic O and a connective similar to the linear implication.

operator that can represent situations where the initial conditions for the obligation are encountered.[20] In a possible worlds semantics, this is usually done in minimal models, where one can define the accessibility relation for the primitive operator in terms of a subset of a more general accessibility relation (see for example Chellas 1974). Another way of answering this problem is to introduce temporal deontic logics. Decew (1981) for instance argued that the solution to the conditional obligations problem requires temporal modalities. The main objection is that dyadic systems fail to represent the temporal character implicit to conditional obligations. An interesting solution was proposed by van Eck (1982a, 1982b), who introduced a quantified temporal deontic logic.

With that being said, one can see that there are (at least) two things going on with conditional obligations. First, it is clear that a material conditional such as $\supset$ is not of type $(s_d \backslash s_n)/s_n$, and thus neither (3a) nor (3b) are appropriate translations. This, which begs the question of the syntactical type of a conditional obligation, brings us to the next point: some might argue that there is a temporal parameter implicit to conditional obligations. Hence, using $\triangleright$ to represent conditional obligations, one could try to define the type of $\triangleright$ with the help of primitive types that are indexed to some states, allowing the representation of the temporal parameter. A possible solution would be to consider the conditional $\triangleright$ as of the syntactical type that takes a descriptive proposition in the past (or the present) and makes it into a normative proposition in the present (or in some possible future related to that present). Put differently, if one considers that the world is in a specific state at a moment i, then $\triangleright$ is of the type that takes a descriptive sentence at i and a normative sentence in a state $i+1$ (accessible from i) and makes it into a normative conditional in state i. Hence, a solution is to consider that $\triangleright$ is of the type $s_d^{(i)} \backslash s_n^{(i)} / s_n^{(i+1)}$, where the set of primitive types is augmented with types that are indexed to states.

Issues concerning conditional obligations can also be seen through the problem of detachment. Following van Eck (1982a, p.263), the problem of detachment comes from the fact that although sometimes one might want to conclude the normative consequent from the conjunction of the descriptive antecedent and the normative conditional, as in the following reasoning, it remains that sometimes the addition of other conditions results in situations where one does not want to detach the normative consequent.

P1 If Jones drinks alcohol, then he must not drive.
P2 Jones drinks alcohol.
———————————————
$\therefore$ Jones must not drive.

[20] There are however problems regarding the detachment of conditional obligations. See Vorobej (1986) for an overview.

From a categorical perspective, the problem of detachment can be reduced to the fact that there is not always a proof of the normative consequent from the conjunction of the conditional obligation and the descriptive antecedent.

Put differently, we do not have the following arrow for every A and B.[21]

$$f : A \wedge (A \rhd B) \to B$$

From a categorical point of view, the classical connectives $\wedge$, $\neg$ and $\supset$ can be defined on the basis of a deductive system (i.e., a category) where arrows are proofs and objects are propositions. In doing so, one obtains a Cartesian closed category with conjunction commutative and idempotent, respecting the triangle and pentagon identities and the associative law (with $\neg$ defined as $\neg A =_{def} A \supset \bot$ and $\bot$ an initial object). From this perspective, $\wedge$ and $\supset$ are adjoint. This can easily be seen from the fact that:

$$\frac{g : A \wedge B \to C}{g' : A \to B \supset C}$$

$$\frac{h : A \to B \supset C}{h' : A \wedge B \to C}$$

Using these rules, one can easily show that:

$$\frac{1_{A \supset B} : A \supset B \to A \supset B}{g'\,(1_{A \supset B}) : (A \supset B) \wedge A \to B}$$

In other words, there is a proof from A implies B and A to B. The problem of detachment follows from the fact that it might happen that one does *not* want to conclude the normative consequent from the descriptive antecedent in a normative conditional. When modelling a conditional obligation through a material conditional, the problem of detachment arises from the fact that $\supset$ is the adjoint of $\wedge$. As such, to solve this problem, one would need to define a connective $\rhd$ for conditional obligations which is not an adjoint to classical conjunction.[22]

All things considered, our analysis provides two arguments to support that $\supset$ is not an adequate connective to model conditional obligations. On the one hand, a material conditional is not sufficient to represent conditional obligations insofar as a connective for conditional obligations is not of the same syntactical type as a material conditional. On the other hand, the detachment problem shows that the connective for conditional obligations

[21] Note that we are looking at the logical connectives from a categorical perspective rather than from the point of view of the syntactic calculus.

[22] On this subject, see Peterson (2014b)

is not an adjoint to conjunction ($\wedge$), and hence that it cannot be represented by a material conditional.

4.4. Action negation

Returning to the syntactic calculus, the analysis of the paradox enables us to specifically formulate one of the problem that dynamic deontic logic (and more generally action logics) faces. Consider the logical connective 'not', as it is used in (7a) and (7b). In the a-translations, 'not' can be considered as of two types: either it takes a (declarative) descriptive sentence and transforms it into a descriptive sentence or it takes a (declarative) normative sentence and turns it into a normative one. This is perfectly consistent with our assumption that normative inferences are governed by logical rules and that normative sentences are declarative. In this case, negation can be applied to both descriptive and normative statements. An interesting fact is that the b-translation admits a third type for 'not'. Indeed, 'not' can be seen as something which takes an action verb (or an intransitive verb) and turns it into another action verb (or intransitive verb).

It seems fairly uncontroversial to assume that the 'not' of types s_d/s_d and s_n/s_n behave according to the same logical rule inasmuch as they both apply to declarative statements. In other words, this type of 'not' is a propositional negation that can be applied to declarative statements. But is this also the case for the 'not' of type a/a (or i/i)?

The work of Kanger (1957) and Pörn (1970) lead, with the help of Belnap and Perloff (1988) and Xu (1995), to a broad class of approaches that can be regrouped under the name *action logics*. Action logics in philosophy can be divided in two main groups, namely the *stit* logics (where an agent *sees to it that*...) and dynamic logic, introduced in the deontic logic literature by Meyer (1988). The main difference between these approaches is that dynamic logic requires an algebra for actions and another for propositions, while *stit* logics only need truth functional connectives (dynamic deontic logics are of the *Ought-to-do* type while *stit* logics are of the *Ought-to-be*). However, there are still problems regarding the behaviour of logical connectives when applied to actions (or descriptions of actions in the case of *stit* logics). While the tradition in deontic logic is to use Boolean algebras for actions (see for example Segerberg 1982), it is noteworthy that there are still problems regarding the formalization of *action negation*.[23]

In short, the problem revolves around the interpretation of the negation of an action: must the negation of an action be considered as something which is simply not done or as something which is deliberately not done?

[23] See Broersen (2004) for a definition of action negation within the framework of dynamic deontic logic.

This problem becomes clear when one introduces a deontic operator. There is a distinction between not doing an action and doing the negation of an action, and so $O\neg p$ will mean two completely different things depending on how one interprets the action negation $\neg p$. Does $O\neg p$ mean that one must do $\neg p$ or does it mean that one must not do p? Although it is not the aim of the present paper to argue in favour of that point (this is the subject of another paper, see Peterson 2014a), we think that a monoidal deductive system weaker than intuitionistic logic would be better suited than a Boolean algebra to represent the formal behaviour of complex actions. The main reason for this assumption is that the equivalence $a \equiv \neg\neg a$ is dubious when a is considered as an action proposition. Is doing a equivalent to refraining from refraining from doing a? If it is false that one did not do a, then can we conclude that one actually did a? In a nutshell, the negation of an action proposition does not seem to have a classical behaviour. As a result, the negation of a declarative sentence and the negation of an action proposition should not be considered as of the same syntactical type. While the 'not' of types s_d/s_d or s_n/s_n are truth functional and classical, it is unclear whether the 'not' of type a/a is truth functional or classical.[24]

4.5. Deontic entailment

So far, we have seen how our analysis sheds light upon Chisholm's paradox and the *Ought-to-be*/*Ought-to-do* distinction, and moreover it has shown us how to formulate precisely the problems of action negation and detachment of conditional obligations. In addition to these points, our analysis also enables us to see the good Samaritan paradox from a different perspective. The paradox of the good Samaritan in Forrester's argument can be seen through the step from (4) to (★).

(★)	Jones	ought	to murder	Smith	gently	implies
	n	$(n\backslash s_n)/i$	i/n	n	$i\backslash i$	$(s_n\backslash s_n)/s_n$
	Jones	ought	to murder	Smith		
	n	$(n\backslash s_n)/i$	i/n	n		

Although this is not mentioned in Forrester (1984), the step from (4) to (★) is an instance of the principle of deontic consequences, as stated in Castañeda (1968, p.13):

> If an act A entails an act B, then (1) the obligatoriness of A entails the obligatoriness of B and (2) the forbiddenness of B entails the forbiddenness of A.

[24] See Peterson (2014a) for a discussion.

Clearly, if Jones ought to murder Smith in a gentle fashion, then obviously he also ought to murder Smith. We argued elsewhere (cf. Peterson and Marquis 2012) that, contra Forrester, this paradox is not problematic at all since despite its validity, the argument is not sound. But still, one could wonder whether there is something that can be done via the syntactical types to solve that problem. La Palme Reyes et al. (1994) used category theory to analyze the relations between types, nouns and properties. One of their example shows how some properties cannot be passed between two nouns, although these two nouns are related somehow. For instance, although a baby is a person, a big baby is not a big person. So perhaps the same kind of phenomenon is at work here. Although a gentle murder is a murder, it does not necessarily follows that a gentle murder which is obligatory is also a murder which is obligatory. The question would thus be to determine how one can represent this phenomenon through the syntactic calculus. One way of answering this problem is to fix a primitive *class* of types $\mathcal{A} = \{a_1, \ldots, a_n\}$ (replacing i) which would be composed of a finite number of types of action verbs. It is thus possible to obtain (⋆⋆).[25]

(⋆⋆)	Jones	ought	to murder	Smith	gently
	n	$(n\backslash s_n)/a_2$	a_1/n	n	$a_1\backslash a_2$

On these grounds, one could then argue that the step from (4) to (★) requires that both actions in the scope of 'ought' be of the same type, which is not the case in Forrester's paradox. Indeed, the action of 'murdering Smith' is of type a_1 while 'murdering Smith gently' is of type a_2. Therefore, the step from (4) to (★) is illegitimate insofar as the property 'obligatory' cannot be passed between action verbs that are not of the same type.

The same strategy can be applied to solve the good Samaritan paradox, which goes from (10) to (11).[26]

(10) The good Samaritan binds the traveler's wounds implies the traveler is wounded

(11) The good Samaritan ought to bind the traveler's wounds implies the traveler ought to be wounded

In this case, it is easy to argue that the action 'binding the traveler's wound' is not of the same type as 'the traveler is wounded'. Indeed, the action of being wounded only concerns the traveler while the action of binding the traveler's wound is performed by the good Samaritan. Hence, this use of

[25] Here, we are not following the solution proposed by La Palme Reyes et al. (1994).

[26] This version is taken from Garson (2006, p.46). Note, however, that thus formulated it is not derivable neither in *K* or *KD*. That said, the formulation of Åqvist (1967) can be derived.

(ROM) is illegitimate: in order to be correct, its use must be restricted to action verbs that are of the same type in both the antecedent and the consequent of the conditional. As such, the application of (ROM) requires that the action verbs in the scope of the deontic operator are of the same syntactic type, otherwise the application is illegitimate.

4.6. Propositions and actions

Finally, it is worth mentioning that our approach implicitly incorporates Castañeda's (1986) distinction between propositions and practitions. In showing that the syntactical type of the 'ought' operator in the English language behaves according to the *Ought-to-do* interpretation, our approach implies a distinction between declarative statements and action statements. Since the syntactical type of 'ought' is $(n\backslash s_n)/a_i$ rather than s_n/s_d, it follows that Castañeda's distinction is implicit to our framework. However, there is more to our approach than that. Indeed, in addition to distinguishing between declarative statements and action statements, our approach also allows us to distinguish between different types of declarative statements, namely descriptive propositions of type s_d and normative propositions of type s_n. To the best of our knowledge, this is something that has not been previously done within the literature. Thus, our framework incorporates Castañeda's distinction and goes beyond: it also incorporates Hume's semantical dichotomy between facts and norms, which is important if one wants to apply logic to normative inferences. Furthermore, although Castañeda distinguishes between propositions and practitions, he still uses the same logical rules to govern both, and as we saw this is problematic if we consider action negation and conditional obligations.

5. Conclusion

Summing up, the contribution of the present paper was to open the dialog between three disciplines that were otherwise blind to each other. Although the bridge between category theory and linguistics was made by Lambek and further developed by Moot and Retoré (2012)[27], category theory and categorial grammar remained unrelated to deontic logic. The present paper contributes to the literature given that it provides new conceptual insights on many important features of deontic logic. First, we showed how one can formally explain the problems regarding the *Ought-to-be*/*Ought-to-do* distinction, the connective for conditional obligations and action negation.

[27] The use of deductive systems in categorial grammar is actually an active field of study and developments were made by many authors. See Moortgat (2012) for an overview.

In a nutshell, these issues arise since they involve different syntactical types. We showed that the *Ought-to-do* interpretation of normative sentences is a more faithful representation of the English language, and thus that it is the one that should be adopted. Moreover, we showed how our approach implicitly involves Castañeda's distinction between propositions and practitions, but goes further insofar as it distinguishes between different syntactical types for declarative statements. Incidentally, it enables us to give a formal representation of Hume's semantical dichotomy. Our framework also provided a different perspective on Forrester's and the good Samaritan paradoxes, showing that they only arise when (ROM) is applied to a conditional that contains different types of actions. Hence, the analysis of deontic logic via Lambek's syntactic calculus shed light upon many issues that it faces.

As a result of our analysis, we can see that the problems concerning conditional obligations and action negation could benefit from category theory. For instance, we saw that the detachment problem happens when the connective for conditional obligations is considered as an adjoint to conjunction. Although the main aim of this paper was to expose that deontic logic can benefit from Lambek's type theory, we also saw a glimpse of how deontic logic could benefit from a categorical analysis. As such, it opens the door to the application of category theory to deontic logic. Since categorical logic is in the first place a theory of types, our analysis suggests that deontic logic could benefit from the formal framework of category theory. For future research, we intend to provide a categorical analysis of deontic and action logics in order to precisely define the categories within which their connectives behave (cf. Peterson 2014a). Furthermore, we intend to analyze the problems regarding conditional normative inferences through the framework of categorical logic (cf. Peterson 2014b;d). Now that the dialog between deontic logic, category theory and linguistics is open, we face a broad, rich and interesting research avenue for deontic logic.

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