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THROUGH FULL BLOODED PLATONISM, AND WHAT PARACONSISTENTISTS COULD FIND THERE

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Abstract

Really full blooded platonism (RFBP) is aimed to achieve a complete picture of the mathematical landscape by accepting inconsistent mathematics. I distinguish various senses of completeness relevant to the debate and argue that the move to inconsistency-tolerance might not be enough to achieve a complete picture of the mathematical landscape in any of those senses, for there is positive reason to endorse a richer Platonism with place for trivial parts in the mathematical landscape.

Keywords and phrases. Platonism; inconsistency; completeness, triviality, degenerate toposes.

1. Introduction: The promises of Really Full Blooded Platonism

Benacerraf's [7] epistemic challenge to mathematical Platonism is the challenge of explaining how we, spatiotemporal beings, could gain knowledge of mathematical Platonic entities which are causally inert, unlike many other entities about which we do gain knowledge.

Balaguer [4] argues that *full blooded Platonism* — the idea that any mathematical entity that can exist does exist[1] — is the only Platonist theory that can solve Benacerraf's epistemic challenge. Balaguer says that full blooded Platonism implies that any mathematical theory that is internally

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[1] Full blooded Platonism should be distinguished from views like Linsky and Zalta's [13], which extend full bloodedness to all abstract objects, not only the mathematical ones. It cannot be said that Linsky and Zalta's view implies full blooded Platonism, though, for their conception of abstract objects is different from the more traditional one discussed by Balaguer and other full blooded Platonists.

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consistent, and only them, picks out a system of entities. Let me use 'FBP' to denote Balaguer's version of full blooded Platonism thus tied to consistency. The FBP strategy to solve Benacerraf's challenge can be summarized as follows. Since FBP says that all the mathematical objects that could exist actually do exist, it follows that if FBP is correct, then all consistent purely mathematical theories truly describe some collection of Platonic mathematical entities. Thus, to acquire knowledge of mathematical entities, all we need to do is acquire knowledge that some purely mathematical theory is consistent, since knowledge of the consistency of a theory — in particular a mathematical one — does not require any sort of interaction with the objects that the theory is about. Balaguer's strategy, then, is to enlarge ontology to the limit of consistent mathematics since knowledge of consistency provides the required access.

There is another brand of full blooded Platonism for which some of the mathematical entities that can exist are inconsistent. It is called "paraconsistent plenitudinous Platonism" by Priest in [22] and *really full blooded Platonism* by Beall in [6] to distinguish it from the full blooded Platonism which would restrict the possible to the consistent. For convenience, I will stick to Beall's label and abbreviate it to 'RFBP'. Priest considers RFBP when discussing the implications for the philosophy of mathematics of the thoroughgoing noneist claim that every characterization characterizes an object — even inconsistent characterizations —, but he opts for a plenitudinous noneist way (where mathematical objects do not exist) rather than for a Platonist one. For his part, Beall [6] argues that Balaguer's version of FBP is neither the only Platonist theory that can meet Benacerraf's epistemic challenge nor the one with best credentials to do so, but RFBP could do that as well. Similarly to the case of FBP, Beall's RFBP would imply that any mathematical theory, consistent or inconsistent alike, picks out a system of entities. Then, since RFBP simply expands the Platonic heaven even further than FBP, then RFBP solves the problem if FBP does, so FBP would not be the only Platonist view that meets Benacerraf's challenge.[2] Moreover, Beall argues in a dilemma that RFBP is preferable to FBP as follows. Either FBP is informal or it is not. If FBP is informal, it is inconsistent — Beall appeals to Priest's argument in [24, ch. 17] for the inconsistency of the informal notion of proof — and then already is RFBP; if it is not informal, then RFBP is a better option since inconsistency can afford completeness, as proved by results reported for example in [16, ch. 2] and [24, ch. 17].

[2] Beall does not mention what the analogue of knowledge of consistency for accessing objects of inconsistent mathematics is, but one can reckon that knowledge of non-triviality may do the trick.

Independently of whether Beall's remarks about Balaguer's Platonism are right, and even granting that the strategies suggested by any of these full blooded Platonisms provide a solution to Benacerraf's dilemma, my concern here is whether RFBP really provides us with a complete picture of the mathematical landscape. I think it does not. First, the proof-theoretical completeness of some inconsistent mathematical theories is extrapolated to other kinds of completeness a philosophical theory about the whole mathematical landscape might have, and that is not an easy step. Second, the "expansion of the Platonic heaven" is made only up to its non-trivial limits, but it is never given positive reason as to why leave trivial theories out nor is a warrant provided that in avoiding triviality we are still in the business of completeness. I think a more comprehensive picture of the mathematical landscape in full blooded terms would require an extension to include triviality. A point on dialectics is in order. My argument is aimed at a particular audience, namely those who already are prepared to give full blooded Platonism a chance. I am attempting to push the members of that audience into seeing that, given their commitments, they should give trivial mathematics a chance, too, and all the more so when they already are ready to give RFBP a chance. In particular, if other audiences consider that the arguments presented here amount to a *reductio*-like refutation of full blooded Platonism, that is also good for my present purposes, since my aim here is to show certain consequences of full blooded Platonism, and such a *reductio*-like refutation of full blooded Platonism would proceed precisely by granting that it has the consequences I am presenting.

The plan of the paper is as follows. In section 2 I discuss some kinds of (in)completeness relevant to the discussion on the prospects of providing a complete picture of the mathematical realm and argue that Beall's RFBP fails to be complete in some important senses which imply the inclusion of trivial mathematics. In section 3 I expound three reasons to be afraid of triviality which would recommend against a full blooded Platonism more ambititious than Beall's RFBP. One is a logico-ontological worry, according to which admitting triviality even in the smallest parcel of the mathematical realm would spread. Another is a practical worry, according to which triviality is boring and theoretically useless. A last one is an epistemological worry, according to which triviality is epistemically inaccessible given its structurelessness. In section 4 I give a very quick presentation of certain bits of topos logic which will be helpful to address in section 5 those worries about triviality. I find these worries largely unfounded and show that the discussion of triviality is important because it connects to other general topics in logic and metaphysics, being thus not only of interest as pure mathematics or for elucidating the proper scope of a full blooded Platonism.

2. The incompleteness of inconsistency

Let me distinguish some kinds of completeness relevant to Beall's discussion, with the understanding that I will not provide a full taxonomy of kinds of completeness nor delve into the exact connections between them, but that constitute nonetheless working characterizations enough for my present purposes.[3] A theory is *domain-complete* if it contains all the subtheories it might contain according to a previously agreed picture of what such a theory must be. For example, physics would not be considered domain-complete if it were described in a way such that it ought to leave thermodynamics out of it. Similarly, a description of mathematics as a whole would not be considered domain-complete if it failed to incorporate, say, number theory. A theory is *expressively complete* when it has the resources to describe itself completely. A special case of this is when a theory is *proof-theoretically complete*, i.e. if it can prove all the statements that it regards as true. Finally, a theory is *ontologically complete* if it has in its "official ontology" all the objects that do exist and that ought to be included in such official ontology — to make sense of the truths to be captured by proof-theoretical completeness, for example.

Beall's work can be interpreted as the claim that if a full blooded Platonism aims at completeness, it must go really full blooded and go inconsistent. However, he does not make sufficiently clear what the kinds of completeness involved in his discussion are. In some parts of the paper, Beall treats full blooded Platonism as a mathematical theory or, at least, a theory with a provability predicate (cf. [6, p. 324]). Thus he runs his argument for the preferability of RFBP over FBP mentioned in the previous section. He says that if full blooded Platonism is an informal theory, then we should go with RFBP, since in effect we are there already because of the inconsistency of the informal notion of proof. But if full blooded Platonism was a formal theory, then since completeness is an "obvious virtue" that "can be regained only by endorsing inconsistency", one should go for RFBP. This is problematic, first, because Beall's discussion is entirely in terms of "why *informal* mathematics is inconsistent", but it is not clear that a philosophical theory of mathematics need be itself mathematics, or at least Beall did not argue enough for the transition from the inconsistency of informal mathematics to the inconsistency of a philosophical theory. Second, a philosophical theory of inconsistent mathematics need not be itself inconsistent. True, RFBP might be regarded as the claim that the mathematical realm is an (inconsistent) set, parts of which are truly described by some mathematical theories. In this sense, RFBP could be taken as a kind of inconsistent union of all those theories, and this might

[3] Other kinds of completeness are discussed in [10] and [3].

be thus a formal theory too, plus the claim that they collectively describe completely the mathematical landscape. In any case his concern seems to be some sort of expressive or proof-theoretical completeness.

Nonetheless, in other parts of the paper he seems to suggest that recognizing these inconsistent mathematical theories as parts of mathematics is needed to secure the domain-completeness of mathematics because, to borrow an expression of Mortensen [16], "inconsistent mathematics is out there": "The difference between FBP and RFBP is that the latter but not the former admits inconsistent theories" ([6, p. 323]); "Either way [leaving things informal or formalizing but aiming at expressive or proof-theoretical completeness], then, we seem to be led to inconsistent mathematics" ([6, p. 324]). Finally, the additional objects required by these inconsistent theories, if any, should be included in the Platonist heaven, so it becomes really full blooded to be ontologically complete.

However, going inconsistent — or really full blooded — is not enough to secure that a full blooded Platonism is complete in all the senses mentioned above. First, not all incomplete mathematical theories can gain expressive completeness by mere inconsistency, i.e. by accepting only some contradictions; some of them are either incomplete or have to include all inconsistencies, so they would be trivial. Since it is important but not widely discussed in philosophical circles, at least a quick explanation of the idea that some theories are either incomplete or trivial through a generalized version of Cantor's theorem due to William Lawvere in [12] would be worth. An excellent source for a non-categorial presentation of Lawvere's work is Noson Yanofsky's [28], and here I follow his approach.

Cantor proved that there is no surjection from the set of natural numbers, $\mathbb{N}$, to its power, $\mathbf{2}^{\mathbb{N}}$, where $\mathbf{2}$ is any set with two elements. Cantor's theorem can be generalized to every set T, not only $\mathbb{N}$. Moreover, it can in its turn be generalized to any base Y, not only $\mathbf{2}$. In analogy with $\mathbf{2}$, Y can be thought of as the set of possible "values" or "properties" of elements of T. The only exception for this generalization is when the base is a singleton. In that case, the base is "degenerate" or "trivial" in the sense that the base does not have enough structure to support even usual exchanges of elements without fixed points, as (classical) negation does.[4] So the theorem holds for non-degenerate bases, i.e. sets with at least one endofunction without fixed points.[5]

[4] Recall that an *endofunction* is a function such that its domain and codomain are the same, and a *fixed point* for an endofunction f with domain (and codomain) X is an element $a \in X$ such that $f(a) = a$.

[5] Yet another generalization, Lawvere's generalization properly speaking, is done no longer in terms of sets and functions, but of objects and morphisms in a category. However, the usual membership-based set-theoretical jargon can be maintained at this point without losing the generality required for my purposes.

Then, a generalized Cantor's theorem roughly says that if Y is "non-degenerate" then there is no surjection from T to Y^T. Now, every function f from T to Y^T can be converted to a function f^* from $T \times T$ to Y, where $f^*(t, t') = f(t')(t) \in Y$. Saying that f is not surjective is equivalent to saying that there is a $g(-) \in Y^T$ such that for all $t' \in T$ the function $f(t') = f^*(-, t')$ is not the same as the function $g(-)$. Said otherwise, there is a $t \in T$ such that $g(t) \neq f^*(t, t')$. Then, a function g from T to Y is *representable by* t_x if $g(-) = f^*(-, t_x)$. So if f is not a surjection, then there is a $g(-) \in Y^T$ not representable by any $t \in T$.

This generalized Cantor's theorem says, in a philosophical reading, that as long as the properties of T are non-trivial, there is no way that T itself can "talk about" its own properties.[6] When the Ys are truth values, Cantor's theorem says that the only case in which a theory T can talk completely about itself is when there is only one truth value. For reasons that will become clear in section 4, that truth value is usually taken to be the value "true", so the only way in which T can talk completely about itself is just when it is *trivial* in the logical sense, i.e. when everything is true in it.

Thus, the strategy of adding structure to Y — for example, augmenting the number of elements of Y (truth values) — only makes things worse, because that produces more functions without fixed points that only fail to exist in the degenerate case. Another, more promising attempt of blocking the outcome is pointing out that the proof of Cantor's theorems usually relies on a *reductio* and that it might not be valid in a paraconsistent context. However, Zach Weber has recently showed that there is a proof of the theorem using a *reductio* in perfectly paraconsistent terms.[7] Thus, some mathematical theories remain incomplete in spite of the inconsistencies tolerated, unless these amount to triviality.

This result gives reason to at least doubt the expressive completeness of RFBP. Beall says that (proof-theoretical or expressive) completeness is an "obvious virtue". But then incompleteness must be shown as obviously preferable over triviality in case one refrains from achieving completeness

[6] Lawvere's merit is to have noticed that most of the usual "paradoxes" and limitative results concerning "self-reference", from Russell's paradox to Rice's theorem, through Gödel's incompleteness results, Tarski's indefinability theorem and more, are but instances and applications of this general version of Cantor's theorem. Needless to say, this connects with several current hot topics, like the diverse attempts to find a "revenge-free" solution to the paradoxes as well as discussions on Russell's and Priest's schemas of paradoxes. Unfortunately, I just can mention this and leave a more detailed discussion for further work.

[7] See [27]. Admissible forms of *reductio* have been a lasting interest for paraconsistent logicians; see for example the footnote 3 in Beall's paper. Note also that the contrapositive form of Cantor's theorem has a direct proof with with no paraconsistently questionable step. I have preferred the other version because it makes explicit the non-trivial structure of "truth values" and because it makes my case stronger by allowing me to mention Weber's result on the paraconsistent admissibility of the *reductio* employed.

at the cost of triviality. Let us suppose that that is not a big deal. Even in that case, the domain- and ontological completeness of RFBP could be called into question, since those trivial mathematical theories which are expressibly complete perhaps should also be taken into account. But before arguing that trivial mathematical theories with their trivial mathematical objects could have been wrongly left out, in the next section some reasons against the idea of including them are probed.

3. Why not triviality?

The expressive completeness of a trivial mathematical theory might not be enough reason to include them in a comprehensive picture of the mathematical landscape like full blooded Platonism because the cost of such completeness might be too high. There are several arguments against trivialism in general, the idea that everything is true[8], but I focus on three that have explicit significance for mathematics. Surely they are not the only ones that can be raised against trivial mathematics, but I think they are a good start for a greatly neglected topic.

Let me start with a *logico-ontological objection*. One reason to step out of triviality could be that, even if admitted in the slightest piece of the Platonic heaven, it might spread and infect all of it. Less colorfully, the worry is that, if trivial theories are admitted, they trivialize every other theory: "the worst sort of expansion", as Beall [6, p. 323] calls it. Prima facie this is not a grounded worry. Just as the inclusion of inconsistent theories (or objects or pairs of propositions) does not make inconsistent ipso facto every other theory (or object or pair of propositions), the inclusion of trivial theories or trivial objects does not make trivial ipso facto every other theory or object. Still, it might be maintained that triviality is so sui generis that it could be infectious in a way that inconsistency is not. For example, if a universe of worlds contains a trivial world and the accessibility relation between them is reflexive, transitive and symmetric (as in S5), then all worlds are trivial. Something similar could happen if trivial parcels are admitted in the Platonic heaven.

Another reason, a *practical objection*, could appeal to a different yet related sense of the term 'trivial': It can be said that a trivial mathematical theory is uninteresting and neither including it nor leaving it out makes a difference regarding the domain — and ontological completeness of a comprehensive theory of the mathematical landscape. What could be the mathematical significance of mathematical theories in which everything is true? They are

[8] Starting from the fourth book of *Metaphysics*. For more recent discussion see [23, chapter 3] whereas a defense of trivialism can be found in [11].

so simple (trivial) that we would know in advance what its theorems and the properties of its objects are: Everything. Add it for domain- and ontological completeness, but they fill no interesting lacuna nor play any substantive role in a complete description of the mathematical landscape. There is research related to triviality which is indeed interesting for a wider spectre of mathematicians or logicians, namely the conditions under which a theory becomes trivial. However, the interest here is in the extent to which a theory can be expanded with a view to avoid triviality rather than in triviality itself and including it, not excluding it.

Finally, in an *epistemic objection* it might be argued that knowledge of consistency gives epistemic access to mathematical objects of a consistent theory, whereas knowledge of non-triviality gives epistemic access to mathematical objects of inconsistent theories, but triviality leads to structurelessness and no knowledge can be obtained of that, so those putative objects of trivial mathematical theories could not be really accessible.

Thus, RFBP is incomplete in an important way that could be circumvented by accepting triviality in the mathematical landscape. However, according to the objections presented in this section, such a move would be untenable; the costs of triviality would be much higher than the benefits of the completeness that it would provide. In section 5 I will show that these worries are unfounded but, for that, I first need the notion of degenerate topos.

4. Degenerate toposes

Categories are a kind of mathematical universe of objects and connections between them — their morphisms — satisfying very general conditions, like composability and associativity of connections as well as identity connections for every object with themselves.[9] An *isomorphism* is a connection i between two objects X and Y with a connection i^{-1} from Y to X such that the composition of i and i^{-1} is the identity connection for X. Two objects are said to be isomorphic if there is a an isomorphism between them. A central feature of standard category theory is that it is structural in the sense that each object in a category provably has all the same properties as any object isomorphic to it. For example, the defining property of a singleton is *having only one element*. Clearly, in usual set theories there are many singletons, but in a categorial set theory each singleton has only the

[9] Clear introductions to category theory in general, and topos theory in particular, can be found in [14]. I advisely use the 'connections' jargon rather than the usual of 'morphisms' simply to emphasize certain philosophical concerns. This leaves essentially unaffected the description of categorial terms that follows.

properties that all have in common, so anyone of them can be denoted by the same sign, say '**1**', and speak as if there were only one of them.[10]

Toposes are categories with extra structure which allow for the interpretation of set-theoretical notions and hence of significant parts of mathematics, some of them even as much as ZFC. I do not need all the details of topos theory here, but only some aspects presented in a rather informal way that convey the main logical ideas. One of the crucial features of toposes is that there is a truth value v which satisfies the

Comprehension axiom: The proposition $f(x)$ about an element x of a domain O is v if and only if x belongs to the part M of the Os which are fs.

The usual reading of this is that v is *true* and so M would be the *extension* of the predicate f.

This allows the definition of logical notions like *false* and zero- and higher-order connectives in a way that a topos comes with an *internal logic*. This internal logic is internal in the sense that it is defined using only the resources of the topos or mathematical universe in question and that it is the right logic to reason about the topos in question since it is determined by the definition of its objects and connections: attempting to use a different logic to reason about them would alter their defining properties and thus it would not be a logic at all for the intended objects and connections.

In short, truth values in such an internal logic have the following features implied by the **Comprehension axiom** and the characteristics of any topos:

(IL1) Truth values form a partial order, i.e. for every values p, q and r: (IL1a) $p \leq p$

(IL1b) If $p \leq q$ and $q \leq p$ then $p = q$

(IL1c) If $p \leq q$ and $q \leq r$ then $p \leq r$

(IL2) There is a truth value called *true* with the following property:

$$\text{For every } p,\ p \leq \mathit{true}$$

(IL3) One can define a truth value called *false* that has the following property:

$$\mathit{false} \leq \mathit{true}$$

and

$$\text{for every } p,\ \mathit{false} \leq p$$

[10] For those who might wonder of a definition in terms of objects and connections: a *terminal object* in a category **C**, denoted '$1_{\mathbf{C}}$', is an object such that for any object X there is exactly one connection from X to $1_{\mathbf{C}}$. The dual notion, *initial object*, denoted '$0_{\mathbf{C}}$', the categorial version of an empty set, is an object such that for any object X there is exactly one connection from $0_{\mathbf{C}}$ to X.

(IL4) Rather than implied by the categorial data, the traditional, "Tarskian", notion of logical consequence is assumed:

Let '$p \vDash_{\mathcal{E}} q$' denote that q is a logical consequence of p in a topos $\mathcal{E}$, i.e. that whenever p is *true* in $\mathcal{E}$, so is q. Equivalently: if q is not *true*, neither is p. $\vDash_{\mathcal{E}} p$ means that p is *true* in $\mathcal{E}$.

Nothing in the above rules out a mathematical universe where the following hold:

(T1) For every p, $p = true$

(T2) $true = false$

(T3) For every p, $\vDash_{\mathcal{E}} p$

(T4) By (T1) and (T2), $p = true$ and $p = false$, for every p

These conditions are satisfied in a mathematical universe where all the objects are isomorphic, so for all practical purposes it can be said that there is only one object, D, and only one connection, d, that must be with D itself. No element a and no part of D can make the propositional function "x belongs to the part M of D" distinct from *true*, because D is the only object, it has no proper parts, and all of them is included in itself, so to speak. Thus, every propositional function (the only one expressible given the characteristics of this universe) is satisfied by every element of D — which is just D itself — and every proposition — for practical purposes, only one, since all propositions turn out to be equivalent given the characteristics of this universe — is *true* — the only truth value given the characteristics of this universe.

This goes further. As I have mentioned, a topos is a category which allows for the interpretation of set-theoretical notions. Thus, one has in it general categorial versions of, say, Cartesian binary products, disjoint unions or power sets. D and d are enough for a degenerate topos to satisfy the definitions of all these notions, so in a degenerate topos, a singleton is an empty set[11]; a Cartesian product is a disjoint union and a power set and so on.

An FBP-ist, and a fortiori an RFBP-ist as well, accepts the plurality of mathematical universes and, moreover, that standard or intended models have no privileged metaphysical status (cf. [4, pp. 58–62]; [22, p. 153]). But there are these degenerate categories, and they can be taken as mathematical universes (models of mathematical theories) just as non-degenerate toposes are. Given this and the fact that the basic truth value is interpreted as *true*, there are degenerate mathematical universes where propositions can be obtained, but that are so simple that the distinction between true and

[11] In fact, the usual non-degeneracy axiom states that terminal and initial objects are not isomorphic.

false cannot even be formulated. In these simple mathematical universes everything is true, even if only because there is at most one proposition in them. They are not made-up universes, though, even if usually only non-degenerate categories are taken into account. There might be good reasons to do so, but none of them could amount to the claim that degenerate categories fall short of mathematicality, and this is so to the point that leaving them out requires extra axioms on toposes.

5. Degeneracy as a way to really real completeness

Cantor's theorem discussed in section 2 implies that the only case in which T can talk completely about itself is when there is only one truth value. Trivial mathematical theories, as modelled in a degenerate topos, are expressively complete, as they meet this requirement. The case of toposes also shows how to cope with the worries raised in section 3. Regarding the *logico-ontological objection*, triviality does not necessarily spread; other toposes do not become degenerate or trivial only because they belong to the same family as degenerate toposes. As long as a mathematical universe has at least two non-isomorphic objects, this fact together with the conditions (IL1)-(IL6) imply that the internal logic of such universe is not trivial.[12]

As I anticipated in section 3, a related worry concerns the connections between these toposes qua mathematical universes, that is, if they in turn form a category, thus a universe of mathematical universes or a "mathematical multiverse". What if the connections are so strong that they amount to an equivalence relation and the presence of a degenerate topos trivializes the multiverse? The study of the properties of the relations between (models of) set theories is relatively recent, but there is reason to think that this worry is unfounded. The usual structure-preserving connections between toposes, which have been studied in entire independence of any logical or philosophical concern, do not amount to equivalence relations. In fact, one needs to assume for connections in this multiverse only the properties of connections in a category in general: reflexivity, since every object is connected to itself, and transitivity via the composition of connections.

[12] A sketch of the proof is as follows. Suppose that there is at least one connection f between two non-isomorphic objects X and Y. If X is the domain, f determines a part M of Y. But since f is not an isomorphism, there must be another connection h from X to Y such that for some connection g from Y to X the composition g and f is not the same connection as the composition of g and h. But then h determines another part M' of Y. So Y has at least two parts, one of them which can serve as the extension of a predicate not satisfied by those in the other part. For the former, $f(x)$ is *true*; for the latter, *false*, but *true* and *false* are referring then to different parts, unlike the case of a degenerate topos where, even if *false* can be defined, it refers to the same part as *true*, the only part existing in those universes.

The connections between mathematical objects in category theory are not imposed "externally", so to speak; they have to respect the definitions giving identity to the objects. In contexts when the properties of connections are largely independent of the objects connected — as in relational semantics, where the properties of accessibility relations determine what kind of worlds we have, not vice versa — and those properties are rather strong, maybe a trivial entity should be left out to avoid generalized triviality, but when connections and entities are on equal footing and the former preserve the structure of the latter as in category theory, then there might be a trivial entity, as in the case of toposes. Moreover, it has recently been proved [9] that the modal logic of forcing models of ZFC is (classical) S4.2, that is, the accessibility relation between them is reflexive, transitive and convergent.[13] A topos is in general a (model of a) theory weaker than ZFC and the logic available is not always classical, so this is another reason to think that the connections between toposes need not have the full power of classical S5 and its equivalence relation.

The questions about the strength of the connections between the parts of the mathematical realm raise in its turn questions about different kinds of mathematical trivialism. Between "the worst sort of expansion" in which every mathematical statement is true in every part of the mathematical realm and the innocuous situation in which in some parts of the mathematical realm every mathematical statement is true — for example, in a degenerate universe —, the order of the quantifiers allows another possibility: every mathematical statement is true in some part of the mathematical realm. Of course this is satisfied in the presence of degenerate mathematical universes, but a more interesting version would be one without those universes, for example, in which mathematical statements are true but not necessarily all of them in the same part of the mathematical realm. Consider philosophical positions according to which "anything is possible" (cf. [15], [19]). That would mean that every statement is true in some world. This may hold either collectively or distributively. In the first case, it would obtain if there was a trivial world where everything is true; in the latter case, it would obtain if all statements were true although maybe each of them at different worlds (see for example [8] for further details).

The mathematical trivialisms described above exploit the relativization of truth-evaluations to get different ways in which every mathematical statement is true, but there are other ways to obtain different varieties of trivialism. Building again upon some of Mortensen's ideas (cf. [17], [20]), we can coin the term *C-trivial theory* for a theory in which every statement of a class C is true. But then this serves to raise a new argument for the

[13] A relation R is *convergent* if for every w, x, y, if wRx and wRy then there is a z such that xRz and yRz.

infectiousness of triviality. Mortensen has suggested that *atomic triviality*, the truth of all logically atomic statements of a theory, although does not imply triviality *simpliciter* and does not rest on *ex contradictione quodlibet* to be produced, is "catastrophic for mathematics" [20, 635].[14] He exemplifies this with a simple proof by Dunn in the context of the theory of real numbers: Suppose $a \neq b$, that is, that a and b are distinct real numbers. Now add a classically false equation to this, for example $a = b$. Subtract a from both sides; since $(a - a) = 0$, one gets $0 = (b - a)$. Then one could multiply each side by any real number x one pleases to get $x(0) = x(b - a)$. Given that $x(0) = 0$ and if 'r' denotes the result of $x(b-a)$, one gets $0 = r$. Whether by the transitivity of = or the substitutivity of the identicals, every real number equals any other. Only one classically false equation was enough to produce atomic triviality in real number theory. Besides saying that this is catastrophic for mathematics, Mortensen falls within the tradition of people (from Aristotle to Putnam through McTaggart and many others, see [23, ch. 1]) who say that at least for some C, C-triviality implies C-meaningless: Mortensen says that calculations in such real number theory "would mean anything", would not be "possible" or would be "useless" [20, 635].

However, degenerate toposes shed some light on this, too. First, viewed from the outside or "externally", the claim that $a \neq b$ cannot be obtained in a degenerate topos since it consists of only one object and one morphism. There are no false equations in a degenerate topos, so Dunn's argument is unsound. But even internally, were $a \neq b$ expressible for an inhabitant of that universe, it would mean something, the same thing that any other proposition — including $a = b$ — means there, namely "x belongs to the part M of D". Whether this is useful or not, I do not want to say, but calculations are possible, even if extremely simple, and mean something; probably they would mean anything in the case that every equation were true and we demanded certain complexity that cannot be obtained nonetheless in this kind of universes. It is worth noting that Priest, working independently of Dunn's result and from any category theoretic inspiration, had put forward some ideas which serve to block Dunn's argument. For example, in [21] he considers models of arithmetic with (atomically) trivial objects in which, among other principles, neither the transitivity of = nor the substitutivity of identicals hold, so this is another way to see that no mathematical catastrophe needs to follow from a trivial arithmetical object.

[14] Mortensen calls the atomic triviality of a mathematical theory "mathematical triviality", since for him mathematicality is closely tied to *functionality* — the validity of substitutivity of identicals in atomic statements — because that would be "what ensures that calculations can proceed" [20, 636]. Anything of what follows depends on the connections between mathematicality and functionality, so I stick to 'atomic triviality' because it is a more general case, neutral on mathematicality theses.

Trying to figure out in detail more in-between options for expanding the mathematical realm into trivial limits would go beyond the scope of this paper, so I am content with merely pointing some of them out and leaving that for further work. I hope this makes clear that the logical and ontological difficulties typically associated with trivialism are not insurmountable, at least in certain mathematical contexts.

As to the *practical objection*, the simplicity of trivial mathematical theories and objects is a very interesting and instructive one. For a start, triviality has been important in logical practice since trivializing conditions are indeed non-trivial. For example, different non-obvious logical maneuvers can be used to trivialize various instances of a comprehension axiom with a Curry-like paradox.[15] But that "negative" importance need not be the only one in logical practice. Trivial universes should be added to cover limit cases that are not excluded in a non ad hoc way in a very competitive foundational framework, that of topos theory. However, they are not only limit cases but interesting limit cases with some possible interesting applications. For a start, they show how to go trivial without necessarily "changing the subject". The ingredients of the internal logic, parts of which were described in section 4, are the same for every topos. Degenerate toposes constitute the extreme case where there is only one truth value given the properties of that mathematical universe. Also, the discussion above of Dunn's theorem shows that we must be careful on reading a theorem as a catastrophe for mathematics when it could be enunciating instead the conditions under which the "catastrophic" result could be obtained with no catastrophe; a paradox can be read also as a limitative result.

But trivial mathematical universes might have a bear upon topics beyond pure logic and mathematics. Recall Priest on full blooded Platonism and the characterization principle:

> A thoroughgoing noneist holds that every characterization characterizes an object. And here, 'every' means *every*. Even inconsistent characterizations do this. This diet is probably too rich for even a plenitudinous platonist. Platonists are characteristically very much attached to consistency. (…) Of course, there is still another position out there. This belongs to what we might call the paraconsistent plenitudinous platonist. (…) Such a platonist can hold, quite generally, that every characterization characterizes an existent object. [22, pp. 153f]

What if '*every*' means in fact *every*, even trivializing characterizations? This diet is probably too rich even for a RFBP-ist, because they are characteristically very much attached to non-triviality. But the discussion thus far suggests that there could be a brand of Platonism that could hold, more generally than a RFBP-ist, that every characterization characterizes an

[15] Thanks to a referee for reminding me of the non-triviality of trivializing conditions.

existent object. As I mentioned in the preceding section, a degenerate topos is a mathematical universe in which objects and connections satisfy all the usual properties of objects and their connections in a mathematical universe.

I think this can be extrapolated to non-mathematical contexts. In a broader metaphysical setting, degenerate categories can shed light on, for example, Aristotle's anti-trivialist arguments — and that might help in turn to make sense of that extreme case of only one truth value as something more than gibberish. I will just give a brief description of the Aristotelian discussion, because it is not my purpose here to reconstruct in full detail Aristotle's argument nor to evaluate it, but merely pointing out a possible application of degenerate toposes outside their place as limit cases of a mathematical universe. In *Physics*, Aristotle argues that it is impossible that there is only one thing, whether substance or accident; in *Metaphysics* he argues that if everything is true then everything is the same, and so there is only one thing (primary substances in his examples). This, together with some logic moves, implies the impossibility of everything being true.[16] The case of degenerate toposes seems to support partially the argument in *Metaphysics*: if everything is true then the universe must be very simple, and in a degenerate topos everything is true because every object has to be D and every connection must be d, there is nothing distinct from that. But this trivial situation is in no way committed to the extreme simplicity perhaps rightly deemed as impossible in *Physics*, which could be translated into toposes jargon as the impossible simplicity that there was an object without a connection or a connection without an object, so the conclusion about the impossibility of trivialism could be resisted. More generally, degenerate toposes bear a resemblance to some metaphysical monisms, which now are being discussed in a lively fashion again. In degenerate toposes there is only one object and one connection, so they can model the only one existent of *existence monisms*, like the "blobject" of Horgan and Potrč's monism, the only one basic existent of *priority monisms* like Bradley's, or might suggest even other forms of monism not explored yet.[17] Again, this is material to be discussed fully on another occasion; pointing out some of triviality's philosophical potential is enough for my purposes of refuting the practical objection.

Lastly, the *epistemic objection* seems to have no force, for knowledge of the triviality of a mathematical theory gives us direct epistemic access to

[16] For Aristotle, a *primary substance* is an individual object, like me, that person, that horse, this chair, as opposed to a *secondary substance* which is a kind of individual objects: humanhood, horsehood, chairhood, etc. For the argument in *Physics*, see Book I, chapter 3; I have consulted the version in [1]; for the argument in *Metaphysics*, see [2, 1007^b18 to 1008^a2, approximately].

[17] For an overview of metaphysical monisms, see [26].

degenerate toposes in a similar way that knowledge of consistency or non-triviality would give us access to objects of, respectively, consistent and inconsistent mathematical theories, and perhaps in a simpler way given the characteristics of the objects of trivial theories.

Thus, I have showed that the logico-ontological objections are not founded, for the admission of trivial parts in the mathematical realm does not necessarily spread. The practical objections are also ungrounded, for triviality has important conceptual implications in both mathematics and philosophy. Finally, the epistemic objection leads us to consider that the epistemic access to trivial objects and theories is not really different from the epistemic access to consistent and non-trivial theories and objects admitted by FBP-ists and RFBP-ists. The assessment of all this is subject to discussion, of course; I am not claiming that I have dealt with all the objections that can be raised against trivial mathematical theories and their inclusion in a full blooded Platonism. I have merely tried to show that they can be coherently addressed. Degenerate toposes do not spread triviality into other mathematical universes, and I showed that it is not implausible that they come with both mathematical and philosophical import, so their inclusion would not only serve for formal purposes of including boring limit cases for gaining domain and ontological completeness due to a mere technicality. This also needs more elaboration, but for my present purposes and given the overpowering neglect of trivialism, it is enough to show that there is something there to be studied more carefully.

A case for the inclusion of what could be called "nihilist" theories (those where everything is false) similar to that made for trivial theories can be made using Mortensen and Lavers's complement-toposes, in which, via a different interpretation of some of the basic categorial ingredients of toposes, the basic truth value is *false*; see for example [16, chapter 11] and [18]. Then metaphysical nihilisms also appear, and again Aristotle and others can be engaged as to their arguments about the impossibility of everything being false.

Then, after all this, a formulation of *real really full blooded Platonism* (RRFBP) could be along the following lines:

(RRFBP) All mathematical objects exist.

where 'all' means just that, without adding the surnames 'consistent', 'non-trivial', 'possible' or the like. In terms of mathematical theories, as Beall discusses FBP[18], RRFBP would say that
Every mathematical theory truly describes a part of the mathematical realm.
again, without surnames like 'consistent', 'non-trivial' or 'possible' for 'mathematical theory'. Since mathematicality implies neither consistency

[18] And also Restall [25], but not Balaguer: see [5, pp. 70–72].

nor non-triviality, this leaves the interesting task of characterizing, without the notions of consistency or non-triviality (and non-nihility), what a mathematical object or a mathematical theory is if one wants to avoid the degeneracy of saying that everything is a mathematical object or every theory is mathematical.

6. Concluding remarks

Beall has argued for RFBP for the sake of completeness. I have suggested that a complete description of the mathematical realm should include its trivial limits, where everything is true (and by duality, also its nihilist limits, where everything is false). Beall's strategy is ingenious in many ways. But if we really are going to expand the Platonic heaven in an effort to ensure completeness, especially of the domain and ontological kinds, then we need to explore the option of expanding heaven as to encompass triviality, in a way such that beyond it there is truly only the non-mathematical. If this option is to be rejected, then we need good reason for rejecting it. For now, no such reason has been explicitly provided although one can figure out several of them, but I hope to have actually put forward instead good reasons for the idea that it is not impossible whether to accept it or to deal more or less successfully with probable objections when made explicit.

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