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A LOGIC OF CHANGE WITH MODALITIES

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Abstract

In the frame of classical sentential logic we introduce an operator C to be read as *it changes that* ... and $\Box$ as its counterpart to be understood as *unchangeability*. The considered notion of change is motivated by Aristotelian and Leibnizian philosophy of time and change. Typical axioms of our system are e.g. "CA implies Cnot-A", "$\Box A$ implies not-$C\Box A$", as basic rules we have e.g. "from A derive not-CA" (theorems don't change) and a version of an ω-rule connecting C with $\Box$. We prove the completeness of this calculus in respect to a semantics where we introduce "stages". We compare it with some other systems of temporal logic and show certain advantages of our calculus.

Keywords: logic of change, modalities, temporal logic

1. Introduction

The occurrence of changes of different kinds is usually linked with a flow of time and is therefore often described using just temporal modalities. This common way of speaking about change in terms of temporal logics has been realized in the frame of known temporal calculi like that of A. Prior, H. von Wright and J. Clifford (a detailed discussion of these logics extended by a notion of change has been presented by J. Wajszczyk, 1995). In our present considerations we follow the converse idea and treat the notion of change as a primitive one, which in particular may also be used to define some temporal operators. This intention is already expressed in the logic of change *LC* originally presented by K. Świętorzecka (2008) and extracted by Świętorzecka and J. Czermak (2012). This sets the frame of the logical basis of our work now.[1] *LC* was actually inspired by the metaphysics of Aristotle and was considered as a logic of the Aristotelian theory of *substantial changes*. But there is also another philosophical background for

[1] The idea of considering change as a primitive notion was also taken up by Kiczuk (1991) and formally worked out by Trypuz (2010). However Kiczuk's formalization is designed to describe so called *empirical changes* and this approach does not correspond to our motivations, it assumes different characteristics of the considered notion of change.

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LC – some fragments of Leibnizian philosophy of time and change. Our attention now is focused on enriching LC by considering next to change also some kind of necessity which may be regarded as *constancy*, as a counterpart of change. These two modalities, the first one expressed by a one place operator C read as *it changes that* ... and the second one by $\Box$, are not defined explicitly by each other but are connected via axioms and rules, especially by some kind of ω-rule. Within the framework of our formalization we show that $\Box$ has S4 properties and therefore we name our calculus $LCS4$. To interpret $LCS4$ we extend the original LC semantics and show that $LCS4$ is complete in relation to it. By this occasion we analyze the relation of $LCS4$ semantics to appropriate fragments of Aristotelian and Leibnizian methaphysics. At the end we notice a close connection of $LCS4$ with linear temporal logic and show some advantages of using our axiomatization.

2. The system *LC*

Logic LC was originally intended to be a formal basis of the theory of substantial changes formulated by Aristotle. Another philosophical background is found in the Leibnizian conception of time and change. These inspirations are sketched in (2.1). The formal system and its semantics is described in (2.2).

2.1. *Philosophical background*

According to Aristotle substantial changes consist in transitions from existence to non-existence (or conversely) of *individual substances* and these are

> […] first in every sense: (1) in definition, (2) in order of knowledge, (3) in time. For none of other categories none can exist independently, but only substance. (Met, VII, 1, 1028a 30-40)

The considered changes are dichotomic i.e. they occur between contradictory states. As Aristotle explains:

> One kind of change, then, being change in a relation of contradiction, where a thing has changed from not-being to being it has left not-being. Therefore it will be in being: for everything must either be or not be (Phys, VI, 5, 235b 13-16)

And so they are also immediate in a sense that there is no *middle term between contradictories*:

> Since all change is between opposites, and these are either contraries or contradictories, and there is no middle term for contradictories, clearly that which is *between* is between contraries. (Met, XI, 12, 1069a 2-5).

One of the crucial properties of substantial changes is that they are ontologically prior to time. Aristotle claims that time is only *an aspect of a movement* and is *not independent on change* (Phys, IV, 219a). The other thing is that substantial changes are of a relational nature, they should be considered in reference to some more basic ontological relation. Following suggestions from Metaphysics a candidate for this may be the relation of *the active to the passive* which becomes to be a change when it holds between substances (Met V, 15, 1020b28-30, 1021a15-19). To explain our intuition let us accept that substances are meant as existing *individual essences* (cf. Met, VII, 4, 1030 b 5-6). The mentioned relation holds between two individual essences a_1 and a_2 just when the existence of essence a_1 *enables* the existence of essence a_2 and a_2 comes to be a substance thanks to the fact that a_1 loses its existence so a_1 is not a substance any more:

> […] in substances the coming-to-be of one thing is always a passing of another, and the passing-away of one thing is always another's coming to be (GC, I, 3, 319a 20-22)

To describe such a transition which takes place in a chain of individual essences a_1, a_2, a_3, … which successively enable their existence, let us use a sentential $\neg\wedge$-language with constants $\alpha_1, \alpha_2, \alpha_3, \ldots.$ which describe situational counterparts of the mentioned existent essences (so called *elementary situations*). The history of substantial transformations consisting in passing-away a_1 to enable existence of a_2 and next passing-away a_2 to enable existence of essence a_3 (and so on) could be expressed by the sequence of the following conjunctions:

$$\sigma: \quad \alpha_1, \quad \neg\alpha_1 \wedge \alpha_2, \quad \neg\alpha_1 \wedge \neg\alpha_2 \wedge \alpha_3, \quad \ldots.$$

This sequence σ is one of many possible sequences that code possible changes consisting in occurring new elementary situations or their negations and transitions from A to not-A or from not-A to A. The set of all such sequences called *histories of changes* will be considered as a semantics of LC and then extended to $LCS4$ semantics.

The second philosophically interesting inspiration for LC comes from the Leibnizian conception of time and change. Although Aristotelian and Leibnizian ontologies are different in many respects, in the case of philosophy of time and change Leibniz adopted some essential Aristotelian ideas.

Thanks to the extensive discussion presented by M. Futch (2008) we may notice several similarities between the Aristotelian concept of substantial change and the idea of change by Leibniz. In (NE) Teofil quotes several times the Aristotelian description of change as transition from *potential* to *actual existence* (NE II, 169; III 297). Leibniz follows Aristotle in temporal reductionism, according to which, time is only *a measure of change* (NE II, 152 ff) and change is not dependent on time, since:

> It is obvious that [temporal] priority and posteriority do not enter into the definition of change (VE, 168)

Leibnizian changes are also dichotomic and immediate in the same sense as it was taken by Aristotle. Let us quote following Futch[2]:

> A change is made if ... two contradictory propositions are true (AK 6.4.167)
>
> If A is B, and A is not B, A is said to have changes (AK 6.4.629)
>
> Change (Mutatio) is an aggregate of two contradictory states. It is necessary, however, that these states be understood as immediate, since between contradictory things a third is not given (Grua, 323)

Leibniz follows Aristotle also in that he chooses as carriers of changes individual substances. Although primary he understood individual substances in Aristotelian style, finally they came out to be *monads* which were *eternal*, *imperishable* and *independent* one from another in their existence (cf. Mates, 1986, 191). In this way they cannot be considered as becoming and disappearing. However Leibniz links the phenomenon of change rather with *states* of monads (Mates, 1986, 228). Every state of a monad is determined by a collection of attributes possessed by this monad and so every change is actually the transition from possessing to non-possessing (or conversely) of these attributes. Although any two different monad states of the same monad are incompatible (because the difference between two states consists in the fact that some attribute possessed by a monad in the first state is lost in the second one or conversely), some states of different monads are *simultaneous*. Just the simultaneousness together with the relation of *being before/after* occurring between different states of the same monad, bring out the flow of time. This relational structure of time is a starting point of Futch's attempt to justify the Leibnizian temporal reductionism. Futch links one of two mentioned relations: the relation of being before/after with notion of *causality*. Changes (transitions) of states of any monad from possessing to non possessing (or conversely) its attributes are of course caused by something but they themselves do not form causal chains. For this reason we would propose to model the analyzed change by considering the relation of simultaneousness and to link it with a specific Leibnizian concept of *compossibility*. Leibniz does not describe criteria of being simultaneous but it seems to be unquestionable that compossibility is at least a necessary condition for this because:

> Compossible is what, with another thing, implies no contradiction (Grua, 325)
>
> Compossible things [are] those, one of which being given, it does not follow that the other is negated; or those of which one is possible, the other being assumed (AK 6.2.498)

[2] We use Futch's English translations of the original Latin texts.

Although every two simultaneous states are compossible, it happens that two compossible states are not simultaneous, even they can be considered to be consistent.

To realize our proposed idealization we start with any state of monad m_1 and successively consider compossible states of monads m_2, m_3, ... By this way we pass from one to the next more complex collection of states which also may change in dichotomic way. So we simulate a flow of occuring changes.

To describe these intentions let us use again the $\neg\wedge$-language. This time, constants α_1, α_2, α_3, … are interpreted as descriptions of some compossible states of monads m_1, m_2, m_3, …. (called *elementary states*). Depending on the fact that the state described by α_1 is actual or not it holds α_1 or $\neg\alpha_1$. This gives us the first *stage* of some history. In the next step we are referring to a compossible state of monad m_2 described by α_2 that also may be actual or not. In this way we can reconsider the value of α_1 but this is a new stage in which α_1 or $\neg\alpha_1$ is compossible with α_2 or $\neg\alpha_2$. If the value of α_1 is now different from the initial stage we speak about change. In general if each of the propositional constants α_1, …, α_n is true or false on stage n, we consider them in relation to α_{n+1} to come to the next stage $n + 1$ on which now each of α_1, …, α_{n+1} is true or false. In effect we obtain all possible histories of changes and one of them is just described by the mentioned sequence σ. In this way our interpretation of the Leibnizian concept of change leads us to LC semantics.

Let us now make the above intuitions precise.

2.2. *Formal system*

In our approach we use the notion of a language *of level* n to be the set of formulae of sentential logic built up from propositional constants out of the set $\{\alpha_1, ..., \alpha_n\}$ by classical connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and a one place operator C to be read as *it changes, that ....*[3]

To obtain the system LC we take as axioms all tautologies of classical propositional logic and all formulae of the following schemata:

(Ax1) $CA \rightarrow C\neg A$

(Ax2) $C(A \wedge B) \rightarrow CA \vee CB$

(Ax3) $A \wedge \neg CA \wedge CB \rightarrow C(A \rightarrow B)$

(Ax4) $\neg A \wedge \neg B \wedge CA \wedge CB \rightarrow C(A \wedge B)$

[3] The language of level n is the result of passing all previous stages up to stage n. To describe an existential transformation between essence a_n and a_{n+1} respectively to consider the next elementary state of monad m_{n+1} compossible with state of m_n we extend the language of level n to level $n+1$. In effect we get successively growing languages.

and as primitive rules:

(MP)	$A, A \to B \vdash B$
($\neg C$-rule)	$A \vdash \neg CA$
(Rep)	$A[B], B \leftrightarrow B' \vdash A[B']$

We replaced the original axiom A3 by the stronger Ax3, but this does not strengthen the system (cf. Świetorzecka, Czermak 2012).

The semantics for LC we describe as follows:

Let φ be a function from the set of natural numbers to subsets of the set of propositional constants. Let the expression $\varphi \models^n A$ be read as *the formula A is true at stage n in some history φ.*

The minimal level of a formula A ($lv(A)$) is the smallest number n such that A belongs to a language of level n. It is the highest index of the propositional constants occurring in A.

We give a standard definition of the relation $\varphi \models^n A$ for $n \geqslant lv(A)$:

Definition ($\models$)

For any constant α_k (where $1 \leq k \leq n$):

(i) $\varphi \models^n \alpha_k$ iff $\alpha_k \in \varphi(n)$

Let A, B be formulae of level n, then:

(ii)	$\varphi \models^n \neg A$ iff $\varphi \not\models^n A$
(iii)	$\varphi \models^n A \wedge B$ iff $\varphi \models^n A$ and $\varphi \models^n B$
(iv)-(vi)	for $A \wedge B$, $A \to B$, $A \leftrightarrow B$ as usual
(vii)	$\varphi \models^n CA$ iff ($\varphi \models^n A$ and $\varphi \not\models^{n+1} A$) or ($\varphi \not\models^n A$ and $\varphi \models^{n+1} A$)

Please note: If $n < lv(A)$ then $\varphi \models^n A$ is not defined.

An example of such a history φ is the function φ_a corresponding to our sequence σ: $\varphi_a(n) = \{\alpha_n\}$.[4]

Definition (Validity) *A is valid* iff $\forall_\varphi \forall_{k \geqslant lv(A)} \varphi \models^k A$

Completeness Theorem. A is valid iff A is derivable in LC

This is shown by Świetorzecka (2008).

[4] There may be considered histories with different special *rhythms* of changes of truth values. Some of them may correspond to the Parmenidenian theory of impossibility of change, others to e.g. theory of permanent changes of elementary situations (let us note that there are no histories which code global changes i.e. there are no histories that every contingent formula changes at every stage its truth value (as shown by Świetorzecka, 2009)).

3. The system *LCS*4

To introduce the usual symbol $\Box$ we define one place operators $(uC)^n$:

Def. (uC^n) $\quad (uC)^0 A \leftrightarrow A$

$(uC)^{n+1} A \leftrightarrow (uC)^n A \wedge \neg C(uC)^n A$

$(uC)^n A$ expresses that formula A will not change during the next n stages; to see this consider the first two steps:

$(uC)^1 A \leftrightarrow A \wedge \neg CA$

$(uC)^2 A \leftrightarrow A \wedge \neg CA \wedge \neg C(A \wedge \neg CA)$

$(uC)^k$ fulfills the corresponding semantic condition:

(viii) $\varphi \vDash^n (uC)^k A$ iff $\forall_m$ (if $n \le m \le n+k$ then $\varphi \vDash^m A$)

(Proof by straightforward induction on k.)

We define the meaning of the symbol $\Box$ as follows:

(ix) $\varphi \vDash^n \Box A$ iff $\forall_{k \geqslant n}\, \varphi \vDash^k A$[5]

We will also use $\Diamond$ as an abbreviation for $\neg\Box\neg$.
By the above conditions it may be seen that:

Fact 1. The following formulae are valid:

(Ax5) $\Box A \rightarrow (uC)^n A$ *for all* $n \geqslant 0$

(Ax6) $\Box A \rightarrow \neg C\Box$A

Fact 2. The validity of formulae is preserved by the following ω-rule:

$$(C\omega r)\ \frac{B \rightarrow (uC)^n A \quad \forall n \geqslant 0}{B \rightarrow \Box A}$$

(*Proof indirect*.)

When we add formulae Ax5 and Ax6 to the set of LC axioms and $C\omega$r to the set of primitive rules of LC we obtain a calculus which we call LCS4:

Axioms:

(Ax0) all tautologies of classical sentential logic

(Ax1) $CA \rightarrow C\neg A$

(Ax2) $C(A \wedge$ B$) \rightarrow CA \vee CB$

(Ax3) $A \wedge \neg CA \wedge CB \rightarrow C(A \rightarrow B)$

[5] Condition (ix) expresses constancy and not Aristotelian or Leibnizian *necessities* analyzed by e.g. Bocheński (1956, 94ff) and Mates (1986, 105ff).

(Ax4) $\neg A \wedge \neg B \wedge CA \wedge CB \to C(A \wedge B)$
(Ax5) $\Box A \to (uC)^n A$ *for all* $n \geqslant 0$
(Ax6) $\Box A \to \neg C \Box A$

Rules:

(MP) $A, A \to B \vdash B$
($\neg C$-rule) $A \vdash \neg CA$
(Rep) $A[B], B \leftrightarrow B' \vdash A$[B′]
($C\omega r$) $\dfrac{B \to (uC)^n A \quad \forall n \geqslant 0}{B \to \Box A}$

It is rather obvious that

$A \to B \vdash CA \to CB$ and $A \to B \vdash CB \to CA$

are not admissible rules in $LCS4$. However:

Fact 3. The following rules are admissible in $LCS4$:

(a) $A \to B \vdash A \wedge \neg CA \to B \wedge \neg CB$
(b) $A \to B \vdash (uC)^n A \to (uC)^n B$
(c) $A \to B \vdash \Box A \to \Box B$
(d) $A \vdash (uC)^n A$ and
(e) $A \vdash \Box A$

Proof:

Ad (a):

$$\dfrac{\dfrac{A \to B}{A \wedge \neg CA \to B} \qquad \dfrac{\dfrac{\dfrac{A \to B}{\neg C(A \to B)}\ \text{(by } \neg C\text{-rule)}}{\neg A \vee CA \vee \neg CB}\ \text{(by Ax3)}}{A \wedge \neg CA \to \neg CB}}{A \wedge \neg CA \to B \wedge \neg CB}$$

Ad (b): *Induction on n, using (a).*

Ad (c):

$$\dfrac{\dfrac{\dfrac{A \to B}{(uC)^n A \to (uC)^n B}\ \text{(by (b))}}{\Box A \to (uC)^n B}\ \text{(by Ax5)}}{\Box A \to \Box B}\ \text{(by } C\omega r\text{)}$$

Ad (d): *Induction on n.*

Ad (e): *Use (d) and $C\omega r$.*

Fact 4. $\Box$ has in $LCS4$ the standard $S4$ properties, i.e.

(T) $\Box A \to A$

(K) $\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$

(4) $\Box A \to \Box\Box A$

Proof:

(T) is a special case of Ax5.

For (K): $\Box(A \wedge B) \to \Box A \wedge \Box B$ follows from $A \wedge B \to A$ by Fact 3(c). To get the converse we prove $\Box A \wedge \Box B \to (uC)^n\, (A \wedge B)$ by induction on n.

If n = 0, it follows from (T).

1. $\Box A \wedge \Box B \to (uC)^n(A \wedge B)$ (ind. hyp.)
2. $\neg C(\Box A \wedge \Box B \to (uC)^n(A \wedge B))$ ($\neg$C-rule, 1)
3. $\neg(\Box A \wedge \Box B) \vee C(\Box A \wedge \Box B) \vee \neg C((uC)^n(A \wedge B))$ (Ax3, 2)
4. $\Box A \wedge \Box B \to C\Box A \vee C\Box B \vee \neg C((uC)^n(A \wedge B))$ (Ax2, 3)
5. $\Box A \to \neg C\Box A$ (Ax6)
6. $\Box B \to \neg C\Box B$ (Ax6)
7. $\Box A \wedge \Box B \to \neg C((uC)^n(A \wedge B))$ (prop. logic, 4, 5, 6)
8. $\Box A \wedge \Box B \to (uC)^n(A \wedge B) \wedge \neg C((uC)^n(A \wedge B))$ (1, 7)
9. $\Box A \wedge \Box B \to (uC)^{n+1}(A \wedge B)$ (Def.(uC^n))

To get $\Box A \wedge \Box B \to \Box(A \wedge B)$ we use $C\omega r$.

As usual we get also: (K) $\Box(A \to B) \to (\Box A \to \Box B)$

For (4): We prove $\Box A \to (uC)^n\Box A$ again by induction on n and use the $C\omega r$.

4. Completeness of *LCS*4

The soundness of $LCS4$ follows from the soundness of LC together with Fact 1 and Fact 2.

We give a Henkin-proof for the completeness of $LCS4$.

Theorem (*LCS*4-Completeness)

$$\text{If} \quad LCS4 \nvdash \neg A \quad \text{then} \quad \exists_\varphi \exists_{k \geq lv(A)}\, \varphi \vDash^k A$$

As usual we define: A set M of formulae is $LCS4$-*consistent* (or simply *consistent*) iff there is no finite set $\{B_1,\ldots, B_m\} \subset M$ such that $LCS4 \vdash \neg(B_1 \wedge \cdots \wedge B_m)$.

A consistent set M of formulae is *maximal* $LCS4$*-consistent* iff there is no formula $B \notin M$ such that $M \cup \{B\}$ is $LCS4$-consistent.

It is known that maximal consistent sets X have the following properties:

If $LCS4 \vdash B$ then $B \in X$
(ii)* $\neg B \in X$ iff $B \notin X$
(iii)* $B \wedge D \in X$ iff $B \in X$ and $D \in X$
(iv)*-(vi)* for $B \vee D$, $B \to D$, $B \leftrightarrow D$ as usual

We are looking for a sequence $\{L^k\}$ of maximal consistent sets such that the following additional conditions corresponding to (vii), (viii), (ix) hold:

(vii)* $CB \in L^k$ iff $(B \in L^k$ and $B \notin L^{k+1})$ or $(B \notin L^k$ and $B \in L^{k+1})$
(viii)* $(uC)^n\, B \in L^k$ iff $\forall_m$(if $n \leqslant m \leqslant k + n$ then $B \in L^m$)
(ix)* $\Box B \in L^k$ iff $\forall_{m \geqslant k}\, B \in L^m$

Now let us assume that $LCS4 \nvdash \neg A$. Then $\{$A$\}$ is $LCS4$-consistent. We consider an enumeration F_0, F_1, F_2, … of all formulae of the $lv(A)$-language. Take the sequence

$$F_0, \Box F_0, F_1, \Box F_1, F_2, \Box F_2, \ldots$$

We extend the set $\{A\}$ in accordance with the following definition:

Definition (L_k)

$$L_0 = \{A\}$$

$$L_{2k+1} = \begin{cases} L_{2k} \cup \{F_k\} & \text{if this is consistent} \\ L_{2k} \cup \{\neg F_k\} & \text{else} \end{cases}$$

$$L_{2k+2} = \begin{cases} L_{2k+1} \cup \{\Box F_k\} & \text{if this is consistent} \\ L_{2k+1} \cup \{\neg\Box F_k\} \cup \{\neg(uC)^m F_k\} & \text{for a certain } m \end{cases}$$

The last line here is well defined in view of the following

Fact 5. If M is finite, consistent, and $\neg\Box F \in M$, then there is $m \in \mathbb{N}$ such that $M \cup \{\neg(uC)^m\, F\}$is consistent.

Proof indirect. Let $\forall m \in \mathbb{N} : M \cup \{\neg(uC)^m F\}$ be inconsistent.
It means that for every $m \in \mathbb{N}$ there is a finite set $\{B_{m_1}, \ldots, B_{m_n}\} \subset M$ such that

$$LCS4 \vdash \neg(B_{m_1} \wedge \ldots \wedge B_{m_n} \wedge \neg(uC)^m F)$$

Since M is finite, let B be the conjunction of all formulae of M. Then

$$LCS4 \vdash B \to (uC)^m F \quad \forall m \in \mathbb{N}$$

and now

$$LCS4 \vdash B \to \Box F \text{ by } C\omega r.$$

Since $\neg\Box F \in M$, $LCS4 \vdash B \to \neg\Box F$, therefore $LCS4 \vdash \neg B$ which means that M is inconsistent.

From Definition (L_k) it follows that for every $k \geqslant 0$ the set L_k is consistent.
Corresponding to Lindenbaum's Lemma we conclude that $\mathbf{L} = \bigcup_{k=0}^{\infty} L_k$ is maximal consistent.
Now we take the following sequence $\{L^k\}$ of sets:

Definition (L^k)

For $k \geqslant lv(A)$: $L^{lv(A)} = \mathbf{L}$

$$L^{k+1} = \{B : (B \wedge \neg CB) \in L^k\} \cup \{B : (\neg B \wedge CB) \in L^k\}$$

We observe

Fact 6. For every $k \geqslant lv(A)$: L^k is maximal consistent.

Proof by induction on k. Let $k = lv(A)$, $L^k = \mathbf{L}$ and $\mathbf{L}$ (which is maximal consistent) and let us assume that L^k is maximal consistent.

(a) L^{k+1} is consistent.

If it is not, then there is a finite set

$$\{B_1, \ldots, B_m, D_1, \ldots, D_n\} \subset L^{k+1}$$

such that

$$\{B_1, \ldots, B_m, \neg CB_1, \ldots, \neg CB_m, \neg D_1, \ldots, \neg D_n, CD_1, \ldots, CD_n\} \subset L^k$$

and

$$LCS4 \vdash \neg(B_1 \wedge \cdots \wedge B_m \wedge D_1 \wedge \cdots \wedge D_n)$$

Let B be $B_1 \wedge \cdots \wedge B_m$ and D be $D_1 \wedge \cdots \wedge D_n$.
From $LCS4 \vdash B \to \neg D$ it follows by $\neg C$-rule

$$LCS4 \vdash \neg C(B \to \neg D)$$

and from that by Ax3, Ax2 and Ax1

$$LCS4 \vdash \neg B \vee CB_1 \vee \cdots \vee CB_m \vee \neg C \neg D$$

Since $\{B, \neg CB_1, \ldots, \neg CB_m\} \subset L^k$ we get $\neg CD \in L^k$ using also Ax1.
Now we have the following trivial generalization of Ax4:

$$LCS4 \vdash \neg D_1 \wedge \cdots \wedge \neg D_n \wedge CD_1 \wedge \cdots \wedge CD_n \to CD$$

and since $\{\neg D_1, \ldots, \neg D_n, CD_1, \ldots, CD_n\} \subset L^k$ we get $CD \in L^k$, which contradicts $\neg CD \in L^k$ (we assumed that L^k is consistent).

(b) Let L^{k+1} be not maximal consistent. Then there is a formula B such that $B \notin L^{k+1}$ and $\neg B \notin L^{k+1}$. It means that

$$B \wedge \neg CB \notin L^k,\ \neg B \wedge CB \notin L^k,$$
$$\neg B \wedge \neg C\neg B \notin L^k,\ \neg\neg B \wedge C\neg B \notin L^k$$

But then, since L^k is maximal consistent,

$$\neg(B \wedge \neg CB) \in L^k,\ \neg(\neg B \wedge CB) \in L^k,$$
$$\neg(\neg B \wedge \neg C\neg B) \in L^k,\ \neg(B \wedge C\neg B) \in L^k,$$

which entails the inconsistency of L^k.

In view of Fact 6 the conditions (vii)*, (viii)* and (ix)* may be proved easily:

(vii)* by definition of L^{k+1}, (viii)* by induction on n and (ix)* with the help of Ax5 and referring to (viii)*.

Now we define:

$$\varphi^*(n) = \{\alpha_i : \alpha_i \in L^n\}$$

We get by induction on formulae:

$$\varphi^* \models^n B \quad \text{iff} \quad B \in L^n$$

and, since $A \in \mathbf{L} = L^{lv(A)}$, at the end we get

$$\varphi^* \models^{lv(A)} A$$

which proves

$$\exists_\varphi \exists_{k \geq lv(A)}\, \varphi \models^k A$$

5. *LCS*4 and systems of linear temporal logic

The semantics for $LCS4$ shows a direct connection of $LCS4$ with systems of linear temporal logic. Indeed, we can define the next-operator N as follows:

Def. (N) $\quad NA \leftrightarrow (A \leftrightarrow \neg CA)$

with the corresponding semantical condition

(x) $\varphi \models^n NA \quad \text{iff} \quad \varphi \models^{n+1} A$

Now it can be proved in $LCS4$ that N has the usual characteristic properties:

Fact 7. The following formulae are derivable in $LCS4$:

(N1) $\neg NA \leftrightarrow N\neg A$

(N2) $N(A \star B) \rightarrow (NA \star NB)$ for $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

and the (N-rule) $A \vdash NA$ is admissible in $LCS4$.
(Proofs are given by Świetorzecka, Czermak, 2012).
Furthermore we observe:

Fact 8. The following formulae are derivable in $LCS4$

(N3) $\Box A \rightarrow N\Box A$

(N4) $\Box A \rightarrow N^k A$ for every $k \geqslant 0$

and also an N-version of the ω-rule:

($N\omega r$) $\frac{B\rightarrow N^n A \quad \forall n\geqslant 0}{B\rightarrow \Box A}$ is admissible in $LCS4$.

Proofs. To show the admissibility of $N\omega r$ it is convenient to introduce the definition:

$$\overline{N}^0 A \leftrightarrow A, \quad \overline{N}^{n+1} A \leftrightarrow \overline{N}^n A \wedge N\overline{N}^n A$$

for proving

(*) $\overline{N}^n A \leftrightarrow (uC)^n A$.

N3 follows directly from Ax6 and N4 from Ax5 and (*).
If we now conversely define C by N

Def. (C) $CA \leftrightarrow (A \leftrightarrow \neg NA)$

and take N1 through N4 as axioms and as basic rules the N-rule and $N\omega r$ we get a system definitionally equivalent to $LCS4$ and also equivalent to the $\Box\bigcirc$-fragment of propositional linear temporal logic called *PTL* (cf. Gabbay et al., 2006, 46ff).

Let us take the axiomatization of this fragment – we call it Θ^* – given by L. Goldblatt (1992, 87) (we use instead of $\bigcirc$ our symbol N):

Axioms:

all tautologies of classical sentential logic

(K) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(K_N) $N(A \rightarrow B) \rightarrow (NA \rightarrow NB)$

(Fun) $N\neg A \leftrightarrow \neg NA$

(Mix) $\Box A \rightarrow A \wedge N\Box$A

(Ind) $\Box(A \rightarrow NA) \rightarrow (A \rightarrow \Box A)$

Rules: (MP), (Rep), N-rule, Necessitation rule: $A \vdash \Box A$

Now we observe

Fact 9. Θ^* is deductively equivalent to $LCS4$.

Proof. K we noticed already in connection with Fact 4.
K_N is a special case of N2, Fun is N1, Mix follows immediately from N3 and N4. To derive Ind show at first by induction that

$$\Box(A \to NA) \to (A \to N^k A)\ \forall_{k \geqslant 0} \tag{5.1}$$

and apply $N\omega r$.

To prove the converse direction notice that N4 follows by induction from Mix, K_N and the N-rule. To see the admissibility of the $N\omega r$ in Θ^*, we have to use the finite model property of Θ^* (as shown by Goldblatt, 1992, 98).

In this way the philosophical motivations of our $LCS4$ stemming from the classical metaphysics of Aristotle and Leibniz also encounters epistemic issues modeled in particular by modern dynamic logics (described by e.g. Goldblatt, 1992; Harel et. al. 2000).

Besides philosophical reasons to base our logic on the primitive concept of change, $LCS4$ is a system that may be of interest in view of some formal properties.

The proposed calculus is, like the basic system LC, not regular in the sense that deleting C in derivable formulae sometimes leads to contradictions (in contrast to Θ^* where deleting $\Box$ and N in derivable formulae will yield tautologies).

Some advantages of using its basic ω-rule may also be noticed. The usefulness of $N\omega r$ can be seen e.g. in the derivations of Dummets formula

$$(Dum) \quad \Box(\Box(A \to \Box A) \to A) \to (\Diamond\Box A \to \Box A)$$

and of the characteristic axiom of $S4.3$

$$(L_1) \quad \Box(\Box A \to B) \vee \Box(\Box B \to A).$$

To derive Dum we prove at first

$$N\Box A \to \Box N\Box A \tag{5.2}$$

starting with N4, applying the N-rule to get

$$N\Box A \to N^k NA \quad \forall_k \geqslant 0$$

in connection with N3 and $N\omega r$ we have

$$\Box A \to \Box NA$$

Put now $\Box A$ for A and use $S4$ to get the (5.2).

Next we derive

$$N\Box A \to \Box(A \to \Box A) \tag{5.3}$$

We start with proving $N\Box A \to (A \to N^k A)$ by induction on k, using N3, N4 and the N-rule. Now use $N\omega r$ to get $N\Box A \to (A \to \Box A)$, the necessitation rule yields $\Box N\Box A \to \Box(A \to \Box A)$. By (5.2) we get (5.3).

To derive *Dum* we start like Goldblatt (1992, 88) with an instance of K

$$\Box(\Box(A \to \Box A) \to A) \to (\Box\Box(A \to \Box A) \to \Box A)$$

and use S4 with (5.3) to get

$$\Box(\Box(A \to \Box A) \to A) \to (N\Box A \to \Box A),$$

By propositional logic and the necessitation rule we have

$$\Box(\Box(A \to \Box A) \to A) \to \Box(\neg\Box A \to N\neg\Box A)$$

With help of (5.1) and $N\omega r$ to obtain *Dum*.

To derive (L_1) we start with the following instance of $\Diamond A \wedge \Box B \to \Diamond(A \wedge B)$ (which is an easily derivable formula of K):

$$\Diamond(\Box A \wedge \neg B) \wedge \Box B \to \Diamond(\Box A \wedge \neg B \wedge B)$$

to get by classical logic

$$\Diamond(\Box A \wedge \neg B) \wedge \Box B \to A \tag{5.4}$$

Then we show by induction

$$\Diamond(\Box A \wedge \neg B) \to N^k(\Box B \to A) \quad \forall k \geqslant 0 \tag{5.5}$$

For $k = 0$ it is (5.4). For the induction step use the N-rule to get

$$N\Diamond(\Box A \wedge \neg B) \to N^{k+1}(\Box B \to A)$$

In view of

$$\Diamond(\Box A \wedge \neg B) \leftrightarrow (\Box A \wedge \neg B) \vee N\Diamond(\Box A \wedge \neg B)$$

which is an instance of $\Diamond A \leftrightarrow A \vee N\Diamond A$ and can be derived from T, N3 and (5.3), we have only to show that

$$\Box A \wedge \neg B \to N^{k+1}(\Box B \to A) \quad \forall k \geqslant 0$$

which follows from N4.

Now apply $N\omega r$ to (5.5) to obtain L_1.[6]

The operator C can be also used to code, as already mentioned, rhythms of changes of truth values in a transparent and lucid manner. Take e.g. the formula $\Box CA$ which expresses that the truth value of A alternates; $\Box CCA$ means that we have rhythms like t, t, f, f, t, t, f, f.

The formula

$$C^3 A \leftrightarrow ((A \leftrightarrow \neg NNA) \leftrightarrow (NA \leftrightarrow N^3 A))$$

[6] Compare these derivations with these given by Goldblatt (1992, 87-89).

shows that C may be sometimes used to simplify formulae with N. Let us take as an example the formula (vii) considered by Harel (et al., 2000, 405):

$$A \wedge \Box(A \rightarrow N(\neg A \wedge N(\neg A \wedge N(\neg A \wedge NA))))$$

which is actually equivalent to:

$$A \wedge \Box(A \rightarrow \neg NA \wedge \neg NNA \wedge \neg N^3 A \wedge N^4 A)$$

and which is true at each multiple of 4 but false elsewhere. This can now be shortened to:

$$A \wedge \Box(A \rightarrow CA \wedge C^2 A \wedge C^3 \mathrm{A}).$$

Conclusion. The starting point of our considerations was a modal sentential logic LC originally motivated by Aristotelian and Leibnizian philosophy of change and time. The special feature of this formalism is that it's basic operator C expresses changeability considered as prior to time. We enriched LC by a $\Box$ operator identifying necessity with unchangeability (and relating C with $\Box$ by some kind of ω-rule). We proved its soundness and completeness relative to some semantics in a temporal framework. It turned out that $\Box$ has $S4.3$ properties. We showed some formal advantages of our calculus and that it can be translated in well-known systems of linear temporal logic. By this way our approach brings together concepts of classical ontology with modern developments in formal logic and computer science.

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