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ON MEREOLOGICAL COUNTERPARTS OF SOME PRINCIPLE FOR SETS

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ABSTRACT

In this paper we deal with the following problem – *what are consequences of adopting the following property of Cantorian sets*:

$$\forall_z(z \text{ is element of the set of all } S\text{-es} \rightarrow z \text{ is an } S)$$

for mereological sets?

We show that the abovementioned principle does not hold for any notion of *mereological set* considered here. Further we prove that in case of some classical definitions of *mereological set*, enriching the theory of such sets with its counterpart leads to trivial, one-element theories. We also consider some less popular (but more natural) definition of mereological set in the form of the so called *aggregate of objects* and prove that in case of this adoption of the counterpart of the Cantorian principle reduces *proper part of* relation (i.e. one which is irreflexive) to set theoretical $\in$.

In the introduction we present an informal argument for some unwelcome consequences of adopting the principle for mereological sets as defined by means of mereological sums. In the sequel we are more formal and we turn to application of tools of mathematical logic to analyze the problem in its full scope.

1. Introduction

Among pre-theoretical intuitions connected with sets the following one seems to be fundamental and crucial for understanding the notion of *set*[1]:

$$\forall_z(z \text{ is element of the set of all } S\text{-ses} \longleftrightarrow z \text{ is an } S). \qquad (\star)$$

So, x is element of the set of natural numbers iff x is a natural number and $\sqrt{3}$ is not element of this set, since $\sqrt{3}$ is not a natural number.

Things are different when we consider mereological (collective) sets. What is *a mereological set*? According to one of the very first characterizations by the creator of mereology, Stanisław Leśniewski:

[1] For interesting remarks concerning the nature of sets see: [4, pp. 14–38], [6, pp. 119–123, 128], [7, pp. 22–26].

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> an object x is *a mereological set* of a group of S-es iff every S is part of x and every part of x has some common part with at least one of S-es.

In order to understand the definition, we may try to build some intuitions about mereological sets by interpreting *part of* in spatio-temporal manner. In this sense, the territory of the United States can be seen as a mereological set of the territories of its states, since every state is part of the U.S. and whatever part of the U.S. we take it must have some part in common with at least one of the 50 states. What we can see as well is that if a is a state, then it must be part of the U.S., but reverse situation is not the case, i.e. there are parts of the U.S., which are not states (the Yosemite National Park, for example).

In general we see that if a is an S, then it is part of a mereological set of S-es, but not every part of the set is an S. So the following is true:

$$\forall_a (a \text{ is an } S \rightarrow a \text{ is part of a mereological set of } S\text{-es}), \qquad (\alpha)$$

while the reverse implication is not.

Thus we can ask what are the consequences of adopting the following principle:

$$\forall_a (a \text{ is part of a mereological set of } S\text{-es} \rightarrow a \text{ is an } S) \qquad (\beta)$$

in theory of mereological sets.[2]

The following two theorems partially answer this question. By *a proper part* of an object x we understand y such that y is part of x and y is distinct from x.

Theorem 1. *If* (β) *holds and the relation of* parthood *is reflexive, then no object has any proper parts.*

Proof. First, let us notice that if we take a group of S-es to consist of just one object x:

$$z \text{ is an } S \overset{\text{df}}{\longleftrightarrow} z = x,$$

[2] (α) and (β) are closely related to the so called Collapse Principle (see [8]). We do not use the term, since the original formulation of Collapse Principle is in plural logic (see [1]), which is also the standard setting for its analysis. Our formulation and investigation are done with application of set theory. Moreover, while writing about *consequences* we have in mind these mathematical in nature, i.e., we answer the question how adoption of (α) and (β) (formulated in suitable way) influences relational structures satisfying basic mereological principles in presence of various definitions of mereological set. We do not consider philosophical importance of the problem. Collapse Principle, on the other hand, is deeply involved in metaphysical issues. The reader interested in the latter is asked to consult [8].

then x is a mereological sum of the group. Indeed, reflexivity of parthood entails that every S (namely x itself) is part of x, and whatever part y of x we take, y is a common part of y and x. Therefore if (β) holds and a is part of x, then a is just x. $\square$

Theorem 2. *If* (β) *holds, the relation of* parthood *is reflexive and for any pair of objects a and b there exists a mereological set of a and b, then for all objects a and b, either a is part of b or b is part of a.*[3] *In consequence there exists only one object.*

Proof. Fix any objects a and b and consider the following property:

$$z \text{ is an } S \overset{\text{df}}{\longleftrightarrow} z = a \vee z = b.$$

Let x be a mereological sum of a and b. Since x is its own part, by (β) it is the case that either $x = a$ or $x = b$. In the first case, since b is part of x, by (α) we have that b is part of a. In the second case, a is part of b. This together with Theorem 1 entail that there can only be one object. $\square$

2. Formalization of the problem

In mereology we deal with relational structures $\langle M, \sqsubseteq\rangle$, where $\sqsubseteq \subseteq M\times M$ is called *part of* relation. In case $x \sqsubseteq y$ we say that *x is part of y*. In terms of $\sqsubseteq$ we define the following auxiliary relations:

$$x \not\sqsubseteq y \overset{\text{df}}{\longleftrightarrow} \neg x \sqsubseteq y, \tag{df$\not\sqsubseteq$}$$

$$x \sqsubset y \overset{\text{df}}{\longleftrightarrow} x \sqsubseteq y \wedge x \neq y, \tag{df$\sqsubset$}$$

$$x \bigcirc y \overset{\text{df}}{\longleftrightarrow} \exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y), \tag{df$\bigcirc$}$$

$$x \wr y \overset{\text{df}}{\longleftrightarrow} \neg\exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y). \tag{df$\wr$}$$

In case $x \not\sqsubseteq y$ (*resp*. $x \sqsubset y$, $x \bigcirc y$, $x \wr y$) we say that x is *not part* of y (*resp. x is a proper part of y, x overlaps y, x is external to y*). We assume that $\langle M, \sqsubseteq\rangle$ is a reflexive structure, i.e. *part of* relation is reflexive:

$$\forall_{x\in M} x \sqsubseteq x. \tag{r$_\sqsubseteq$}$$

One of possible formalization of the notion of *mereological set* is the following hybrid relation between elements of M and its subsets[4]:

[3] Our attention was first drawn to this fact at Mathoverflow site at this address: http://mathoverflow.net/questions/58495/why-hasnt-mereology-suceeded-as-an-alternative-to-set-theory.

[4] See [3] for a more thorough analysis of the problem of defining mereological (collective) sets.

$$x \text{ Sum } X \overset{\text{df}}{\longleftrightarrow} \forall_{z\in X} z \sqsubseteq x \wedge \forall_{y\in M}(y \sqsubseteq x \to \exists_{x\in X} x \bigcirc y), \qquad (\mathsf{df}\,\text{Sum})$$

called *the relation of mereological sum*. In case x Sum X we say that x is *a mereological sum* of the set X. It is routine to verify that if $\sqsubseteq$ is reflexive, then:

$$\neg\exists_{x\in M} x \text{ Sum } \emptyset, \tag{2.1}$$

$$x \text{ Sum}\{x\}. \tag{2.2}$$

We may now express the formal counterparts of (α) and (β) as below[5]:

$$\varphi(a) \wedge x \text{ Sum } \{y \in M \mid \varphi(y)\} \to a \sqsubseteq x, \qquad (\alpha^*)$$

$$a \sqsubseteq x \wedge x \text{ Sum } \{y \in M \mid \varphi(y)\} \to \varphi(a). \qquad (\beta^*)$$

As it can be easily seen, (α^*) is true solely by (df Sum). As for (β^*) it is almost equally as easy to show that it fails even in a very strong class of structures (in the sense of the amount of sentences being satisfied within) examined in mereology, the so-called *mereological structures*, i.e. one that are reflexive and satisfy the following set of postulates[6]:

$$\forall_{x,y\in M}(x \sqsubseteq y \wedge y \sqsubseteq x \to x = y), \qquad (\mathsf{antis}_{\sqsubseteq})$$

$$\forall_{x,y,z\in M}(x \sqsubseteq y \wedge y \sqsubseteq z \to x \sqsubseteq z), \qquad (\mathsf{t}_{\sqsubseteq})$$

$$\forall_{x,y\in M}(x \not\sqsubseteq y \to \exists_{z\in M}(z \sqsubseteq x \wedge z \wr y)), \qquad (\mathsf{SSP})$$

$$\forall_{x\in \mathcal{P}_+(M)}\exists_{x\in M} x \text{ Sum } X. \qquad (\exists\mathsf{Sum})$$

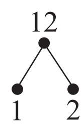

Figure 1: A mereological structure in which (β^*) fails.

We take $M := \{1, 2, 12\}$ and $\sqsubseteq := \mathrm{id}_M \cup \{\langle 1, 12\rangle, \langle 2, 12\rangle\}$ and put:

$$\varphi(x) \overset{\text{df}}{\longleftrightarrow} x = 12.$$

[5] Since we have not specified any formal language the notion of *condition* $\varphi(a)$ is imprecise. However, this imprecision is intended and does not influence correctness of the sequel. We simply assume that while writing about conditions we limit ourselves to these which could be expressed in the first-order part of a formal exposition of second-order mereology.

[6] For a set X, $\mathcal{P}(X)$ is its power set, $\mathcal{P}_+(X) := \mathcal{P}(X) \setminus \{\emptyset\}$. ($\mathsf{SSP}$) is the acronym for *strong supplementation principle*. Partially ordered sets which satisfy the principle are called *separative*.

By (2.2) we have that 12 Sum $\{x \in M \mid \varphi(x)\}$, yet $1 \sqsubseteq 12$ and $1 \neq 12$.

We are now in a position to formulate a formal version of Theorem 1.

Theorem 3. *If* $\langle M, \sqsubseteq\rangle$ *satisfies* ($\mathsf{r}_{\sqsubseteq}$), ($\mathsf{df}\bigcirc$), (df Sum) *and* (β^*), *then*:

$$\forall_{x,y\in M}(x \sqsubseteq y \to x = y).$$

Proof. Let $a \in M$ and, as in the above model, take:

$$\varphi(x) \overset{\text{df}}{\longleftrightarrow} x = a.$$

Assume $b \sqsubseteq a$. By (β^*) we have $b = a$. □

The theorem may be treated as a negative result about adopting (β^*) in the so-called *non-existential mereology*, i.e. these systems of mereology in which we do not assume anything about existence or non-existence of mereological sums. In light of the theorem the only structures satisfying the schema in question would be those consisted of isolated elements only.

Moreover, as it was already noticed in an informal way in Theorem 2, even modest existential assumption leads to even less interesting structures. To be more precise, if we adopt the axiom of existence of mereological sum for pairs of objects:

$$\forall_{y,z\in M}\exists_{x\in M}\, x \text{ Sum}\{y, z\}, \tag{$\exists$Sum$_2$}$$

then the only reflexive structures that satisfy this and (β^*) are degenerate one-element structures.

Theorem 4. *Let* $\langle M, \sqsubseteq\rangle$ *be a reflexive structure which satisfies* ($\mathsf{df}\bigcirc$), (df Sum), ($\exists$Sum$_2$) *and* (β^*). *Then* $\forall_{a,b\in M}(a \sqsubseteq b \vee b \sqsubseteq a)$. *In consequence* Card $M = 1$.[7]

Proof. Let $a, b \in M$ and consider:

$$\varphi(x) \overset{\text{df}}{\longleftrightarrow} x = a \vee x = b.$$

Let y Sum $\{x \in M \mid \varphi(x)\}$. Since by ($\mathsf{r}_{\sqsubseteq}$) we have $y \sqsubseteq y$, then by (β^*) $y = a \vee y = b$. Since $a \sqsubseteq y$ and $b \sqsubseteq y$ we have that $a \sqsubseteq b \vee b \sqsubseteq a$. We now apply Theorem 3 to conclude that $a = b$. □

[7] For a given set X, Card X is its cardinal number.

3. The principles and the fusion relation

Defined above the mereological sum relation is not the only mathematical interpretation of the notion of *collective set*. Another one is the so-called *fusion relation* which is characterized in the following way:

> an object *x* is a fusion of a group of *S*-es iff for every object *y*, it is the case that *y* overlaps *x* iff *y* overlaps one of the *S*-es.

In the language of relational structures $\langle M, \sqsubseteq\rangle$ augmented by suitable definitions the fusion relation may be couched in the following way:

$$x \text{ Fus } X \overset{\text{df}}{\longleftrightarrow} \forall_{y\in M}(y \bigcirc x \longleftrightarrow \exists_{a\in X} a \bigcirc y), \qquad (\mathsf{df}\ \text{Fus})$$

or, equivalently in light of (df $\bigcirc$) and (df $\wr$), as:

$$x \text{ Fus } X \overset{\text{df}}{\longleftrightarrow} \forall_{y\in M}(y \wr x \longleftrightarrow \forall_{a\in X} a \wr y). \qquad (\mathsf{df}'\ \text{Fus})$$

It is a well known fact that in the class of posets Fus is weaker than Sum, in the sense that every mereological sum is fusion but not vice versa (see for example [4, pp. 114–121]). The inclusion Sum $\subseteq$ Fus is provable in all structures $\langle M, \sqsubseteq\rangle$ that are transitive (see [5, p. 218]).

In the presence of the poset axioms[8] the fusion relation does not satisfy counterparts of (α^*) or (β^*):

$$\varphi(a) \wedge x \text{ Fus}\{y \in M \mid \varphi(y)\} \to a \sqsubseteq x, \qquad (\alpha^*_{\mathrm{F}})$$

$$a \sqsubseteq x \wedge x \text{ Fus}\{y \in M \mid \varphi(y)\} \to \varphi(a). \qquad (\beta^*_{\mathrm{F}})$$

Failure of (β^*_{F}) follows from the fact that every sum must be fusion, but (β^*) fails for Sum. This argument shows that (β^*_{F}) must be false in every stronger structure as well. To see that Fus does not satisfy (α^*_{F}) in the class of all posets consider the structure in Figure 3, in which $M := \{1, 2, 12, 21\}$ and

$$\sqsubseteq := \mathrm{id}_M \cup \{\langle 1, 12\rangle, \langle 1, 21\rangle, \langle 2, 12\rangle, \langle 2, 21\rangle\}$$

and put:

$$\varphi(x) \overset{\text{df}}{\longleftrightarrow} x = 12.$$

We see that 21 Fus $\{x \in M \mid \varphi(x)\}$, yet $12 \not\sqsubseteq 21$. This structure is also a classical model demonstrating that Fus $\not\subseteq$ Sum, as $\neg 21$ Sum $\{x \in M \mid \varphi(x)\}$.[9]

[8] That is $(\mathsf{r}_{\sqsubseteq})$, $(\mathsf{antis}_{\sqsubseteq})$ and $(\mathsf{t}_{\sqsubseteq})$.

[9] It is rather easy to notice that (α^*_{F}) is a weaker version of the principle saying that fusion is an upper bound. More on consequences of accepting this property of fusions as an axiom can be found in [2].

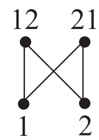

Figure 2: Fusion does not satisfy (α^*_{F}).

It is interesting to notice that if we adopt the following condition:

$$a \text{ Fus } \{x \in M \mid \varphi(x)\} \rightarrow (y \sqsubseteq a \longleftrightarrow \varphi(y)), \tag{$*$}$$

(which is equivalent to the conjunction of (α^*_{F}) and (β^*_{F})) then we may prove that for every subset of the domain expressible by some condition $\varphi(x)$, its fusion must be sum.

Before we prove the above, notice that (α^*_{F}) entails $(\mathsf{r}_{\sqsubseteq})$ (and the more so $(*)$ entails reflexivity of *parthood*). This follows from the fact that for every a, a Fus $\{x \in M \mid x = a\}$ solely by (df Fus) and logic. Since $a = a$, $a \sqsubseteq a$ by (α^*_{F}).

Theorem 5. *Let* $\langle M, \sqsubseteq\rangle$ *be a structure that satisfies* ($\mathsf{df}\bigcirc$), (df Fus), (df Sum) *and* $(*)$. *Then*:

$$a \text{ Fus } \{x \in M \mid \varphi(x)\} \rightarrow a \text{ Sum } \{x \in M \mid \varphi(x)\}. \tag{$\dagger$}$$

Proof. Let a Fus $\{x \in M \mid \varphi(x)\}$. We are to show that:

$$\forall_{y \in M}(\varphi(y) \rightarrow y \sqsubseteq a) \wedge \forall_{y \in M}(y \sqsubseteq x \rightarrow \exists_{z \in M}(\varphi(z) \wedge y \bigcirc z)).$$

The first conjunct follows immediately from the assumption and $(*)$. The second one is a consequence of $(*)$ and $(\mathsf{r}_{\sqsubseteq})$. □

In light of this and Theorems 3 and 4 we have the following corollary.

Corollary 1. *Let* $\langle M, \sqsubseteq\rangle$ *be a transitive structure satisfying* ($\mathsf{df}\bigcirc$), (df Fus), (df Sum) *and* $(*)$ *in which the following axiom of fusion existence holds*:

$$\forall_{y,z \in M}\exists_{x \in M}\, x \text{ Fus } \{y, z\}. \tag{$\exists\mathrm{Fus}_2$}$$

Then Card $M = 1$.

Proof. First notice that thanks to $(\dagger)$ we obtain $(\exists\mathrm{Sum}_2)$. Transitivity of $\sqsubseteq$ entails Sum $\subseteq$ Fus. So if a Sum $\{x \in M \mid \varphi(x)\}$ and $b \sqsubseteq a$, then a Fus $\{x \in M \mid \varphi(x)\}$ and by $(*)$ we obtain that $\varphi(b)$. Thus (β^*) holds. So now it is enough to refer to Theorems 3 and 4 to draw the conclusion. □

4. The principles and irreflexive *part of* relation

For an irreflexive relation $\sqsubset$ of *proper part*[10] even the counterpart of (α^*):

$$a \text{ Sum } \{x \in M \mid \varphi(x)\} \wedge \varphi(b) \to b \sqsubset a. \tag{$\alpha^*_{\sqsubset}$}$$

fails under (df Sum). Too see that it is enough to consider:

$$\varphi(x) \overset{\text{df}}{\longleftrightarrow} x = a$$

for a distinguished $a \in M$. We have:

$$a \text{ Sum } \{x \in M \mid x = a\} \wedge a = a \wedge a \not\sqsubset a.$$

In light of this we can see that taking $(\alpha^*_{\sqsubset})$ as an axiom schema leads to an inconsistent theory for irreflexive part of relation in the presence of (df Sum). On the other hand, if we accept:

$$a \text{ Sum } \{x \in M \mid \varphi(x)\} \wedge b \sqsubset a \to \varphi(b) \tag{$\beta^*_{\sqsubset}$}$$

then with $\sqsubset$ irreflexive and in the presence of (df Sum) we obtain that only atoms[11] may exist. Indeed, if $a \notin$ Atom and $a_0 \sqsubset a$, then since a Sum $\{x \in M \mid x = a\}$, we get that $a_0 = a$, a contradiction.[12]

The main reason for the negative results above (especially the failure of $(\alpha^*_{\sqsubset})$) seems to be a consequence of introducing mereological sets by means of $\sqsubseteq$ while formulating the principles for $\sqsubset$. In order to avoid this we may consider an alternative definition of mereological set, which is based on $\sqsubset$, not $\sqsubseteq$. To this end we define the notion of *an aggregate* of objects[13]:

$$\begin{aligned} z \text{ Agr } X \longleftrightarrow X \neq \emptyset \wedge \forall_{x \in X}\, x \sqsubset z \wedge \\ \forall_{y \in M}(y \sqsubset z \to \exists_{x \in X}\, y \bigcirc x). \end{aligned} \tag{df Agr}$$

In consequence, for a condition $\varphi(x)$, we have:

$$\begin{aligned} z \text{ Agr } \{x \in M \mid \varphi(x)\} \longleftrightarrow \exists_{x \in M}\, \varphi(x) \wedge \forall_{x \in M}(\varphi(x) \to x \sqsubset z) \\ \wedge \forall_{y \in M}(y \sqsubset z \to \exists_{x \in M}(\varphi(x) \wedge y \bigcirc x)). \end{aligned}$$

[10] If we chose $\sqsubset$ to be primitive, then we introduce $\sqsubseteq$ by means of the following definition:

$$x \sqsubseteq y \overset{\text{df}}{\longleftrightarrow} x \sqsubset y \vee x = y. \tag{df $\sqsubseteq$}$$

The relation $\sqsubseteq$ introduced as above is of course reflexive and in case we demand $\sqsubset$ is a strict partial order (irreflexive and transitive), $\sqsubseteq$ is a partial order and the properties of all relations introduced by means of $\sqsubseteq$ remain unchanged.

[11] See Definition 1 on page 543.

[12] The reader will easily convince herself that similar results hold for the fusion relation and the suitable counterparts of (α^*_{F}) and (β^*_{F}).

[13] The notion is patterned on the ideas presented in [3] and [4]. There are reasons for which aggregates are the most natural and intuitively the best mathematical embodiment of the notion of *mereological set*. Suitable arguments can be found in the aforementioned works.

We see that the counterpart of $(\alpha^*_\sqsubset)$ for aggregates is satisfied:

$$\varphi(b) \wedge a \text{ Agr } \{x \in M \mid \varphi(x)\} \rightarrow b \sqsubset a. \qquad (\alpha^*_A)$$

In order to present a structure $\langle M, \sqsubset\rangle$ which is a strict partial order but in which the following counterpart of $(\beta^*_\sqsubset)$ fails:

$$a \text{ Agr } \{x \in M \mid \varphi(x)\} \wedge b \sqsubset a \rightarrow \varphi(b) \qquad (\beta^*_A)$$

consider the set $M := \mathcal{P}_+(\{1, 2, 3\})$ with $\sqsubset := \subsetneq$. Let:

$$\varphi(x) \stackrel{\text{df}}{\longleftrightarrow} x = \{1, 2\} \vee x = \{2, 3\}.$$

We get $\{1, 2, 3\}$ Agr $\{x \in M \mid \varphi(x)\}$, $\{1\} \subsetneq \{1, 2, 3\}$ but $\neg\varphi(\{1\})$.

Our aim is to prove that if we adopt (β^*_A) as an axiom schema, then *proper part of* relation reduces to set theoretical $\in$.[14]

Definition 1. An object x is *an atom* iff it is minimal with respect to *proper part of relation*:

$$x \in \text{Atom} \stackrel{\text{df}}{\longleftrightarrow} \neg\exists_{y \in M}\, y \sqsubset x. \qquad (\text{df Atom})$$

Let for any x:

$$\text{Atom}(x) := \{a \in \text{Atom} \mid a \sqsubseteq x\}. \qquad (\text{df Atom}(x))$$

Lemma 1. *Assume $\langle M, \sqsubset\rangle$ is a strict partial order which satisfies* (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$), (df Agr) *and* (β^*_A). *Then*:

$$\forall_{a,b \in M}(a \sqsubset b \rightarrow a \in \text{Atom}).$$

In consequence, for any structure $\langle M, \sqsubset\rangle$ which satisfies the conditions listed above it must be the case that:

$$\sqsubset \subseteq \text{Atom} \times M \quad \textit{and} \quad \text{Agr} \subseteq M \times \mathcal{P}(\text{Atom}).$$

Proof. Let $a \sqsubset b$ and let $a \notin$ Atom. Define:

$$\varphi(x) \stackrel{\text{df}}{\longleftrightarrow} (x \sqsubset b \wedge x \wr \mathrm{a}) \vee x = a.$$

Notice that b Agr $\{x \in M \mid \varphi(x)\}$. Firstly, the set $\{x \in M \mid \varphi(x)\}$ is not empty by the fact that $\varphi(a)$. Secondly, $\varphi(x) \rightarrow x \sqsubset b$ follows from the construction of $\varphi(x)$. Thirdly, let $y \sqsubset b$. We have two possibilities:

[14] Of course, $\in$ is not relation so what we have in mind writing that $\sqsubset$ *reduces to* $\in$, is that a structure $\langle M, \sqsubset\rangle$ is isomorphic with some structure $\langle D, R\rangle$, where $R := \{\langle x, y\rangle \in D \times D \mid x \in y\}$. For simplicity we will write '$\langle D, \in\rangle$' instead of '$\langle D, R\rangle$'.

(a) $y \wr a$ or (b) $y \bigcirc a$. In (a) it is the case that $\varphi(y)$ and since $y \bigcirc y$ we have that $\exists_{z \in M}(\varphi(z) \wedge z \bigcirc y)$. In (b), a is such that $\varphi(a) \wedge y \bigcirc a$.

Since $a \notin$ Atom, then there must be a_0 such that $a_0 \sqsubset a$. So $a_0 \sqsubset b$ and $\varphi(a_0)$, i.e. either $a_0 \wr a$ or $a_0 = a$, a contradiction. □

Corollary 2. *If* $\langle M, \sqsubset \rangle$ *is a strict partial order which satisfies* (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$), (df Agr), (SSP) *and* (β^*_A) *then*:

$$\emptyset \neq \mathrm{Atom}(x) = \mathrm{Atom}(y) \rightarrow x = y.$$

Proof. Let $\langle M, \sqsubset \rangle$ satisfy the assumptions. By (SSP) we obtain the so called *proper parts principle*[15]:

$$\emptyset \neq \{a \in M \mid a \sqsubset x\} \subseteq \{a \in M \mid a \sqsubset y\} \rightarrow x \sqsubseteq y.$$

From this and ($\mathsf{antis}_{\sqsubseteq}$) we have that:

$$\emptyset \neq \{a \in M \mid a \sqsubset x\} = \{a \in M \mid a \sqsubset y\} \rightarrow x = y.$$

Now, if x and y are such that $\emptyset \neq \mathrm{Atom}(x) = \mathrm{Atom}(y)$, then by Lemma 1 we have that x and y have exactly the same proper parts, so we obtain that $x = y$. □

Theorem 6. *Every strict partial order* $\langle M, \sqsubset \rangle$ *satisfying* (df $\sqsubseteq$), (df $\bigcirc$), (df $\wr$), (df Agr), (SSP) *and* (β^*_A) *is isomorphic to some structure* $\langle D, \in \rangle$.

Proof. In the proof we use the von Neumann's construction of ordinal numbers and the definition of the cardinality of a given set A as the smallest ordinal equinumerous with A.[16] In particular we use the following property of ordinals:

if α is an ordinal, then for all $\beta, \gamma \in \alpha$: $\beta = \gamma \vee \beta \in \gamma \vee \gamma \in \beta$.

Let $\langle M, \sqsubset \rangle$ satisfy all hypotheses of the theorem. If $\sqsubset = \emptyset$, then (trivially) it is enough to take $\langle M, \in \rangle$ where $\in = \emptyset$.

Let then $\sqsubset \neq \emptyset$. By Lemma 1 we have that Atom $\neq \emptyset$. Let Card Atom $= \kappa$ and let $\kappa^+ = \kappa \cup \{\kappa\}$.[17] Take the set $\kappa^+ \setminus \{1\}$, where $1 = \{\emptyset\}$. $\kappa^+ \setminus \{1\}$ contains no singletons and therefore the elements of the set:

$$K := \{\{\alpha\} \mid \alpha \in \kappa^+ \setminus \{1\}\}$$

[15] For a proof of this fact see for example [4, p. 77]

[16] For details see for example [10, Chapter 3] or any of the classic textbooks on set theory.

[17] While writing *let* Card Atom $= \kappa$ we tacitly assume the Axiom of Choice (or rather its equivalent over Zermelo-Fraenkel set theory: Zermelo's Well-Ordering Theorem) which lets us conclude that there exists such κ.

are incomparable with respect to $\in$:

$$\forall_{X,Y\in K}(X \notin Y \wedge Y \notin X). \tag{‡}$$

Let i : Atom $\to K$ be a bijection. We define function $f : M \to K \cup \mathcal{P}(K)$ such that:

$$f(x) := \begin{cases} i(x) & \text{if } x \in \text{Atom}, \\ \{i(b) \mid b \sqsubset x\} & \text{otherwise.} \end{cases}$$

Notice that f is one-to-one. To see that assume $a \neq b$.

- If $a, b \in$ Atom, then $f(a) = i(a)$ and $f(b) = i(b)$, so $f(a) \neq f(b)$.
- If $a \in$ Atom, $b \notin$ Atom, then $f(a) = i(a)$ while $f(b) = \{i(x) \mid x \sqsubset b\}$. If $f(a) = f(b)$, then since $f(b) \neq \emptyset$ there would have to be an atom a_0 such that $i(a_0) \in i(a)$ which contradicts (‡). The other case while $a \notin$ Atom and $b \in$ Atom is proved analogously.
- If a, $b \notin$ Atom, then both a and b have proper parts which must be atoms by Lemma 1. Therefore by Corollary 2 we obtain that $\emptyset \neq \text{Atom}(a) \neq \text{Atom}(b)$ and thus $f(a) \neq f(b)$.

Now we show that:

$$a \sqsubset b \longleftrightarrow f(a) \in f(b).$$

($\to$) Assume that $a \sqsubset b$, that is $a \in$ Atom by Lemma 1. Since $b \notin$ Atom we have that $f(b) = \{i(y) \mid y \sqsubset b\}$, so $f(a) \in f(b)$.

($\leftarrow$) Let $f(a) \in f(b)$. First notice that $b \notin$ Atom. Suppose otherwise. Therefore $f(a) = i(b)$ and $i(b) = \{\alpha\}$ for some ordinal $\alpha \in \kappa^+ \setminus \{1\}$. Thus we have that $f(a) = \alpha$. In consequence $a \notin$ Atom, since otherwise $i(a) = \alpha$ and $i(a) \in i(b)$, a contradiction. So $f(a) = \{i(z) \mid z \sqsubset a\} = \alpha$. Since $a \notin$ Atom, $f(a) \neq \emptyset$. Moreover by (SSP) a must have at least two atomic parts, so $f(a)$ has at least two elements incomparable with respect to $\in$. But then α is an ordinal number which has elements incomparable with respect to $\in$, a contradiction.

Since $b \notin$ Atom and $f(b) = \{i(y) \mid y \sqsubset b\}$, for some $y \sqsubset b$: $f(a) = i(y)$, so $a = y$ and $a \sqsubset b$.

To conclude, $\langle M, \sqsubset\rangle$ is isomorphic to $\langle f[M], \in\rangle$. □

5. Summary

In the prequel we have demonstrated that, under very natural constraints put upon *parthood* relation, each notion of *mereological set* considered in this paper is different from the classical notion of *the Cantorian set*. This difference has been emphasized before from, so to say, *ontological* point of view. We have proved that this difference has also some strong *technical* background, which is embodied in right-to-left part (or rather counterpart) of (⋆). Whatever formalization of the notion of *mereological set* we choose to consider, acceptance of a suitable counterpart of the aforementioned principle leads to theories which are either trivial (in the sense that they are one element structures) or such in which *part of* relation reduces to set theoretical $\in$.

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