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IS LEWIS'S TRIVIALITY RESULT ACTUALLY A TRIVIALITY RESULT?

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1. Introduction

David Lewis (1976) proved that only a trivial language can have an indicative conditional "→" such that $P(A \rightarrow C) = P(C / A)$ if $P(A)$ is positive. A trivial language is such that if $P(A \wedge C) \neq 0$ and $P(A \wedge \neg C) \neq 0$, then $P(C / A) = P(C)$. As such, when both A and C and A and ¬C are cotenable, it follows that A and C are independent. Using few hypotheses, Lewis showed that when the probability of the conditional statement is the conditional probability, it trivializes the language. This result, which is known as the *first triviality result*, was presented in 1972 and published in 1976. It was almost unchallenged for forty years. It is quite curious that even after the publication of Morgan and Mares' paper (1995) the entire community still continues to consider Lewis' triviality result as a definitive one for any indicative conditional. Morgan and Mares showed that the positive part of intuitionistic logic (i.e. without negation) is the weakest logic for which the probability of the conditional can be interpreted as the conditional probability. They also showed that the definition of $\neg A$ as $A \rightarrow \perp$ does not break the consistency of the system.

The aim of the present paper is to show that Lewis' result does not hold for intuitionistic logic, whose conditional is an indicative conditional. First, I will summarize the relevant steps of Lewis' proof, and then I will show that one of Lewis' metalogical hypotheses (the law of expansion by cases) does not hold in intuitionistic logic. I will then show that the triviality result is back when we add the expansion by cases as an axiom to intuitionist logic. Indeed, adding the expansion by cases to intuitionistic logic yields classical logic and the triviality result holds for that system. Finally, I will show that the triviality result cannot be derived even if we add a weaker form of the law of expansion by cases to intuitionistic logic.

2. The First Triviality Result

Lewis' starting point is Adams's idea that the probability of the indicative conditional is the conditional probability. The assertability of "if A, then B"

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intuitively seems to be the assertability of B given A. Using the usual definition of conditional probabilities, some postulates on absolute probability functions and general laws of logic, Lewis showed that this project of defining a propositional language containing such a conditional is "futile".

Assume a language with the usual connectives $\wedge$, $\vee$ and $\neg$ augmented with a conditional "$\rightarrow$".

The notion of *possible world* is taken as a primitive and it is assumed that a sentence has a truth value in each and every possible world. The conditional probability is defined by:

(1) $P(C/A) =_{def} \dfrac{P(C \wedge A)}{P(A)}$ if $P(A)$ is positive.

The postulates on absolute probability functions are:

(2) $1 \geq P(A) \geq 0$

(3) if A and B are logically equivalent (for the logic in question), then $P(A) = P(B)$

(4) if A and B are incompatible, then $P(A \vee B) = P(A) + P(B)$

(5) if A is true in all possible worlds, then $P(A) = 1$.

for any P, A and B.

The specific postulate for the conditional probability is:

(6) There are *universal probability conditionals*, i.e., conditionals such that $P(A \rightarrow C) = P(C/A)$

for any P, A and C such that $P(A) > 0$.

In addition to these assumptions, Lewis also uses the following elementary result: If P is a probability function, then the function $P'(C) = P(C/A)$ where $P(A) > 0$ is also a probability function, meaning that P' satisfies (2)-(5). From this, we can easily prove that:

(7) $P(A \rightarrow C / B) = P(C / A \wedge B)$ for $P(A) > 0$ and $P(B) > 0$.

Finally, Lewis's proof uses the well-known expansion by cases.

(8) For any A, and B, A is logically equivalent to $((A \wedge B) \vee (A \wedge \neg B))$

Let us suppose, following Lewis, that $\rightarrow$ is a universal probability conditional.

Assuming that there is a probability function P satisfying (1)-(7), Lewis proved that the language is trivial, meaning that if $P(A \wedge C) \neq 0$ and $P(A \wedge \neg C) \neq 0$, then $P(C / A) = P(C)$.

Proposition (Lewis')

A language in which there is a probability function P satisfying (1)-(7) and for which (8) holds is trivial.

Proof

Suppose that $P(A \wedge C) \neq 0$ and $P(A \wedge \neg C) \neq 0$. From (7) and (3) we can derive (9) and (10).

$$(9)\quad P(A \rightarrow C \,/\, C) = P(C \,/\, A \wedge C) = \frac{P(C \wedge A \wedge C)}{P(A \wedge C)} = 1$$

$$(10)\quad P(A \rightarrow C \,/\, \neg C) = P(C \,/\, A \wedge \neg C) = \frac{P(C \wedge A \wedge \neg C)}{P(A \wedge \neg C)} = 0$$

Using the expansion by cases (8), (3), (4) and (1), we have

$$(11)\quad \begin{aligned} P(D) &= P((D \wedge C) \vee (D \wedge \neg C)) = P(D \wedge C) + P(D \wedge \neg C) \\ &= P(D \,/\, C) \cdot P(C) + P(D \,/\, \neg C) \cdot P(\neg C) \end{aligned}$$

Using (9), (10) and (11), we can derive:

$$(12)\quad \begin{aligned} P(C \,/\, A) &= P(A \rightarrow C) = P((A \rightarrow C) \wedge C) + P((A \rightarrow C) \wedge \neg C) \\ &= P(A \rightarrow C \,/\, C) \cdot P(C) + P(A \rightarrow C \,/\, \neg C) \cdot P(\neg C) \\ &= 1 \cdot P(C) + 0 \cdot P(\neg C) = P(C) \end{aligned}$$

Thus A and C are independent.

Q.E.D.

This is the first triviality result.

3. The scope of the triviality result

The strength of a conclusion relies on the strength of the hypotheses used in the proof. The first hypothesis used by Lewis is that:

$$(1)\quad P(C \,/\, A) = \frac{P(C \wedge A)}{P(A)} \text{ if } P(A) \text{ is positive}$$

This definition is universally accepted in the framework of absolute probability functions.

Hypotheses (2), (3) and (4) are equivalent to Kolmogorov's axioms and (5) does not have any credible alternative.

The last hypothesis is:

(6) There are conditionals such that $P(A \rightarrow C) = P(C / A)$ for any P, A and C such that $P(A) > 0$.

In Lewis' view, it is this specific hypothesis that trivializes any language having a universal probability conditional.

It is noteworthy, though, that there is another hypothesis at play in the proof, which is taken for granted by Lewis. Indeed, the proof relies upon a metalogical hypothesis: the law of expansion by cases. Equivalently, using the expansion by cases in conjunction with definition (1) yields proposition (11).

(11) $P(D) = P(D / C) \cdot P(C) + P(D / \neg C) \cdot P(\neg C)$

When applied to a universal probability conditional, we get

$$P(A \rightarrow C) = P(A \rightarrow C / C) \cdot P(C) + P(A \rightarrow C / \neg C) \cdot P(\neg C)$$

Which in turn yields the independence between A and C:

(12) $P(C / A) = 1 \cdot P(C) + 0 \cdot P(\neg C) = P(C)$.

This metalogical hypothesis, however, is not a universally valid rule. For instance, the expansion by cases is invalid in intuitionistic logic.

Proposition

$A \leftrightarrow ((A \wedge B) \vee (A \wedge \neg B))$ is not a theorem of intuitionistic logic

Proof

Let us suppose it is. Let T be an intuitionistic tautology (for example $p \rightarrow p$). We know that, for any C, $C \leftrightarrow T \wedge C$ is a theorem of intuitionistic logic. If we take T for A, we get $T \leftrightarrow ((T \wedge B) \vee (T \wedge \neg B))$. From there, one can easily prove that $T \leftrightarrow (B \vee \neg B)$ is a theorem of the new system.

Q.E.D.

Actually, this proves that the system obtained by adding $A \leftrightarrow ((A \wedge B) \vee (A \wedge \neg B))$ as an axiom is classical logic.

Curiously, a weaker form of the law of expansion by cases holds in intuitionistic logic. If $(B \vee \neg B)$ is proved, then the equivalence between A and $((A \wedge B) \vee (A \wedge \neg B))$ can be proved. In fact, $(B \vee \neg B) \rightarrow (A \leftrightarrow ((A \wedge B) \vee (A \wedge \neg B)))$ is a theorem of intuitionistic logic, and so are $B \rightarrow (A \leftrightarrow ((A \wedge B) \vee (A \wedge \neg B)))$ and $\neg B \rightarrow (A \leftrightarrow ((A \wedge B) \vee (A \wedge \neg B)))$.

4. Conclusion

As a result of my analysis of Lewis' argument, we can conclude that:

(a) If a formal language has a conditional satisfying (6), then for any probability function satisfying (1)-(5) the formal language is trivial unless the rule of expansion by cases does not hold;

(b) The expansion by cases does not hold in intuitionistic logic;

(c) $P(A \rightarrow C) = P(C)$ is not valid in intuitionistic logic because even if $P(A \rightarrow C) = P(C / A)$, in general, $P(((A \rightarrow C) \wedge C) \vee ((A \rightarrow C) \wedge \neg C)) \neq P(A \rightarrow C)$.

The intuitionistic implication is an indicative conditional, and yet we can define its probability as a conditional probability. So, following Morgan and Mares, the language of the intuitionistic propositional calculus with a universal probability conditional is not trivial. Lewis' triviality result does not hold. The flaw in the proof is the use of the expansion by cases, which is not a valid principle of intuitionistic logic.

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