Logique & Analyse **231** (2015), 347-351

WEIGHT-REDUCTIONS FOR PARTICULAR UNIFORM STRUCTURES

ROLAND HINNION

ABSTRACT

This article studies possibilities of size-reduction for uniformities on first-order structures, for structures of the particular type "Malitz-structure".

1. Introduction

The kind of uniform structures that we will consider here are so-called "Malitz-structures", investigated namely in [1] [2]. For the reader familiar with the general concept of "uniform space" (as defined in [3]), a Malitz-structure corresponds simply to a first-order structure, which universe is also a uniform space such that the uniformity admits a basis made of equivalence relations, the proper functions of the structure are uniformly continuous and the induced topology is totally separated but not discrete. In Section 2 we give an alternative (self-contained) presentation, technically much easier to handle.

Our main motivation concerns the possibility of getting countable structures (which is a current wish in Model Theory); but the involved technique here gives also some insight about cases where the universe of the initial structure is not necessarily countable; in [4], some possibilities of size-reduction for the universe of the structure (the size of the uniformity staying unaltered) were studied; here we discuss so to say the "converse problem", i.e. possibilities of weight-reduction for the uniformity (while the first-order structure is not modied). The aim is to get a "reduced version" (of the initial structure) that is still a Malitz-structure, with as much as possible control over its properties (compared to those of the initial structure). We always suppose in this paper that the language of the first-order structure is at most countable.

2. Malitz-structures

As announced in Section 1 we give here a simplified presentation and recall some notions and facts (for which the reader can find much more details in [1] [2]).

doi: 10.2143/LEA.231.0.3141813.

A Malitz-structure is a couple $(M, \mathcal{F})$, where M is a first-order structure (we suppose the language to be at most countable) and $\mathcal{F}$ is a set of equivalence relations on the universe U_M of M, such that the following 4 conditions are satisfied:

Cond 1: $\mathcal{F}$ is directed for the order relation $\supseteq$ ("reverse inclusion")

Cond 2: $=_M \notin \mathcal{F}$ ($=M$ is the equality relation on U_M)

Cond 3: $\forall a, b \in U_M\ (a \neq b \Rightarrow \exists \sim \in \mathcal{F}\ \neg a \sim b)$

Cond 4: for each F_M (proper function of M):

$$\forall \sim \in \mathcal{F}\ \exists \sim' \in \mathcal{F} \quad \forall \vec{x}, \vec{y} \text{ in } U_M : \vec{x} \sim' \vec{y} \Rightarrow F_M(\vec{x}) \sim F_M(\vec{y})$$

Of course does "$\vec{x}$" stand for an n-tuple "$x_1, x_2, \ldots, x_n$" and "$\vec{a} \sim' \vec{b}$" for "$\forall i\ a_i \sim' b_i$".

The family $\mathcal{F}$ is called a "Malitz-family" on M, and $\mathcal{F}$ is a basis for a uniformity that has all the desired properties described in Section 1.

Notice at once that U_M and $\mathcal{F}$ are necessarily infinite sets. Notice also that not any infinite first-order structure M admits necessarily a Malitz-family (see [2], 5.2.1)!

The present study focalizes on those such M that do admit Malitz-families, where the uniformity-weight is greater than the size of he universe of M; the aim being to find out whether one can realize equality (weight = size). Our theorem (Section 5) specifies cases where it can be done, and in particular guarantees that any countable first-order structure that admits Malitz-families admits necessarily a countable one.

The following two parameters play several important roles w.r.t. Malitz-structures:

- the "characteristic" (or "additivity") of the directed set $(\mathcal{F}, \supseteq)$:
  $\delta_{\mathcal{F}}$:= the strict supremum of the cardinals of the upperly bounded subsets of $(\mathcal{F}, \supseteq)$
- the "index" of $\mathcal{F}$:
  $\kappa_{\mathcal{F}}$:= the strict supremum of the cardinals $|U_M/\sim|$, for $\sim \in \mathcal{F}$

The Cauchy-completion of M w.r.t. the uniformity induced by the basis $\mathcal{F}$ can be presented as the set (adequately quotiented, of course) of the $\mathcal{F}$-nets in U_M (i.e. the objects of type $(x_\sim)_{\sim \in \mathcal{F}}$ with each $x_\sim \in U_M$) that are "uniform-Cauchy-nets" (i.e. satisfy the rule: $\forall \sim, \sim' \in \mathcal{F} : \sim \supseteq \sim' \Rightarrow x_\sim \sim x_{\sim'}$).

When two Malitz-families $\mathcal{F}_1$, $\mathcal{F}_2$ are involved, with $\mathcal{F}_1 \subseteq \mathcal{F}_2$, one can define a canonical map:

$$\overline{M}_{\mathcal{F}_2} \to \overline{M}_{\mathcal{F}_1} : (x_\sim)_{\sim \in \mathcal{F}_2} \mapsto (x_\sim)_{\sim \in \mathcal{F}_1}$$

(where "$\overline{M}_{\mathcal{F}}$" is the Cauchy-completion of M, corresponding to $\mathcal{F}$).

This map is always a uniformly continuous function and also a morphism of first-order structures. Under some circumstances it is also surjective (see [2] and Section 4).

The adequate notion of "compactness" in this context is the one of δ-cover-compactness, i.e. the property that any covering by open sets contains a sub-covering of size $< \delta$. Further do we say that $\mathcal{F}$ is "compactifying" when $\overline{M}_{\mathcal{F}}$ is $\delta_{\mathcal{F}}$-cover-compact.

Notice that "ordinary" compactness is $\aleph_0$-cover-compactness.
Some basic facts:

- $\delta_{\mathcal{F}} \leq |U_M|$
- $\aleph_0 \leq \delta_{\mathcal{F}} \leq \kappa_{\mathcal{F}}$
- when $\mathcal{F}$ is compactifying: $\delta_{\mathcal{F}} = \kappa_{\mathcal{F}}$

3. Weight-reduction: the construction

The "weight" of a uniformity is classically defined as the minimum of the sizes of its bases. Here, reducing that "weight" will correspond to the construction of a Malitz-family $\mathcal{F}'$ on M, with $\mathcal{F}' \subseteq \mathcal{F}$ and the expectation that $|\mathcal{F}'|$ is as low as possible (where $\mathcal{F}$ is the "initial" Malitz-family on M). Notice that (of course) only the case $|\mathcal{F}| > |U_M|$ is really of interest here (while [4] was obviously concerned by the case $|\mathcal{F}| < |U_M|$).

The construction:

1. Choose, for each pair $\{a, b\}$ of distinct elements of U_M, one equivalence $\sim \in \mathcal{F}$, such that $\neg\, a \sim b$ (see Cond 3, Section 2); call this equivalence $\sim_{\{a,b\}}$.
2. Choose, for each couple $(F_M, \sim)$, where F_M is a proper function of the first-order structure M and $\sim \in \mathcal{F}$, one equivalence $\sim' \in \mathcal{F}$, satisfying Cond 4 (Section 2); call this equivalence $\sim' [F_M]$.
3. Choose, for each pair $\{\sim_1, \sim_2\} \subseteq \mathcal{F}$ one upper bound (in the sense of $(\mathcal{F}, \supseteq)$), and call that equivalence $\sim [\sim_1, \sim_2]$
4. Define: $\mathcal{F}_0 := \{\sim_{\{a,b\}} |\ a, b \in U_M \text{ and } a \neq b\}$
5. Define (for k a natural number):

   $$\mathcal{F}_{k+1} := \mathcal{F}_k \cup \{\sim' [F_M]\ | \sim \in \mathcal{F}_k \text{ and } F_M \text{ is a proper function of } M\} \cup \{\sim [\sim_1, \sim_2]\ | \sim_1, \sim_2 \in \mathcal{F}_k\}$$

6. Consider $\mathcal{F}^{\star} := \cup \{\mathcal{F}_k \mid k \text{ is a natural number}\}$

An elementary verification shows that $\mathcal{F}^{\star}$ is a Malitz-family on M.

Further do we (obviously) have:

$$|\mathcal{F}^{\star}| \leq |U_M|.$$

So we have indeed a “weight-reduction” result; but with rather few control over the parameters $\delta^\star := \delta_{\mathcal{F}^\star}$ and $\kappa^\star := \kappa_{\mathcal{F}^\star}$.

All that can be said here in general follows from the “basic facts” (Section 2) and the fact that $\mathcal{F}^\star \subseteq \mathcal{F}$ (obvious convention: $\delta := \delta_{\mathcal{F}}$ and $\kappa := \kappa_{\mathcal{F}}$):

$$\delta^\star \leq \kappa^\star \leq \kappa.$$

This suffices however to get some more information in particular cases.

Example: if $\kappa_{\mathcal{F}} = \aleph_0$, then $\delta^\star = \kappa^\star = \aleph_0$. Notice also (see [1] [2]) that then $\mathcal{F}$ and $\mathcal{F}^\star$ are both “compactifying”.

In the next section we show how to get more control over the parameters, but at the price of an extra hypothesis…

4. A variant of the construction

We construct now an increasing chain $\mathcal{F}_\alpha$, indexed by ordinals this time. Take $\mathcal{F}_0$ as in Section 3, and define $\mathcal{F}_\gamma := \bigcup_{\beta<\gamma} \mathcal{F}_\beta$ for γ a limit ordinal. Further modify the step from $\mathcal{F}_\alpha$ to $\mathcal{F}_{\alpha+1}$ like this:

- choose, for each $X \in \mathcal{P}_\delta(\mathcal{F}_\alpha)$ (where $\mathcal{P}_\delta(A)$ is the set of the subsets $B \subseteq A$ such that $|B| < \delta$), one upper bound (called “$\sim_X$”) in the sense of $(\mathcal{F}, \supseteq)$.
- define:
$$\mathcal{F}_{\alpha+1} := \mathcal{F}_\alpha \bigcup \{\sim' [F_M] \mid \sim \in \mathcal{F}_\alpha \text{ and } F_M \text{ is a proper function of } M\} \cup \{\sim_X \mid X \in \mathcal{P}_\delta(\mathcal{F}_\alpha)\}$$
- at last, define $\mathcal{F}^\star := \mathcal{F}_\delta$ (with still $\delta := \delta_{\mathcal{F}}$!)

Again is $\mathcal{F}^\star$ a Malitz-family on M, and this time obviously δ-directed (i.e. any $X \subseteq \mathcal{F}^\star$, such that $|X| < \delta$, is upperly bounded in $(\mathcal{F}^\star, \supseteq)$).

So (obviously): $\delta \leq \delta^\star \leq \kappa^\star \leq \kappa$, which gives us a better control over the parameters.

But, in order to still control also the size of $\mathcal{F}^\star$, we have to make an extra hypothesis:

$$|\mathcal{P}_\delta(U_M)| \leq |U_M|$$

Under that hypothesis we can prove by induction on α that

$$\alpha \leq \delta \Rightarrow |\mathcal{F}_\alpha| \leq |U_M|$$

and in particular :

$$|\mathcal{F}^\star| \leq |U_M|.$$

In the case where $\mathcal{F}$ is compactifying, we can get even more information about $\mathcal{F}^\star$, via the Theorem 7.2 in [2], which states that:

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are Malitz-families on M, such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{F}_2$ is compactifying,
Then:

(i) $\delta_1 \leq \delta_2$
(ii) if $\delta_1 = \delta_2$, then $\mathcal{F}_1$ is also compactifying
(iii) if $\delta_1 = \delta_2$, then the canonical map: $\overline{M}_{\mathcal{F}_2} \to \overline{M}_{\mathcal{F}_1}$ is surjective

If we apply that here, we get: if $\mathcal{F}$ is compactifying, then $\delta = \delta^\star = \kappa^\star = \kappa$, $\mathcal{F}^\star$ is also compactifying and the canonical map: $\overline{M}_{\mathcal{F}} \to \overline{M}_{\mathcal{F}^\star}$ is surjective.

5. Synthesis of the main result

Theorem. *Any Malitz-family $\mathcal{F}$ on M admits some $\mathcal{F}^\star \subseteq \mathcal{F}$, such that $|\mathcal{F}^\star| \leq |U_M|$ and $\mathcal{F}^\star$ is also a Malitz-family on M. Under the hypothesis $|\mathcal{P}_\delta U_M| \leq |U_M|$ one can take $\mathcal{F}^\star$ realizing also that $\delta \leq \delta^\star \leq \kappa^\star \leq \kappa$; in this last situation: if $\mathcal{F}$ is compactifying, then so is $\mathcal{F}^\star$, and the canonical map: $\overline{M}_{\mathcal{F}} \to \overline{M}_{\mathcal{F}^\star}$ is surjective.*

Corollary. *Any countable first-order structure that admits Malitz-families admits necessarily a countable Malitz-family.*

Comment about the "extra-hypothesis" (introduced in Section 4 and used in our Theorem) : however that kind of condition is (of course!) not generally satisfied, are there some propitious cases (among which of particular interest w.r.t. our motivations); for example:

- the case where the cardinal of the universe of M is strongly inaccessible (so in particular when M is countable);
- the case where the characteristic is countable (so again, a fortiori, when M itself is countable).

Roland Hinnion

References

[1] Hinnion, R., "A general Cauchy-completion process for arbitrary first-order structures", (2007), Logique & Analyse 197, 5-41.
[2] Hinnion, R, "Directed sets and Malitz-Cauchy completions", (1997), Math.Log. Quart. 43, 465-484.
[3] Kelley, J.L., "General Topology", (1955), Van Nostrand.
[4] Hinnion, R., A Downwards Löwenheim-Skolem-Tarski Theorem for specific uniform structures, Logique & Analyse (2013): Vol. 56, n° 222, 149-156.