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ON THE LOGICAL CONTENT OF THE WEAK LAW OF EXTENSIONALITY AND ITS RELATION TO THE SUCCESSIVE SIMPLIFICATION OF THE ORIGINAL AXIOM OF LEŚNIEWSKI'S ONTOLOGY II

TOSHIHARU WARAGAI

— To the memory of Paul Gochet —

ABSTRACT

This paper treats the logical character and content of the weak form of the law of extensionality which I named WΩ (weak omega) and the role it plays in the so-called *successive simplification* of the sole axiom of Leśniewski's Ontology. In doing so, the present author defined singular sentences of the form '$a\varepsilon b$' by appealing to the idea we find in Leibniz and Garlandus Compotista.

KEYWORDS

Law of Extentionality · Simplification of the Axiom of Leśniewski's Ontology · Leibniz · Garlandus Compotista.

1. Introduction

This introduction should begin with a justification for writing an article with title 'On the Logical Content of the Weak Law of Extensionality and its Relation to the Successive Simplification of the Original Axiom of Leśniewski's Ontology II'. Indeed, an article of mine with title 'On the Logical Content of the Weak Law of Extensionality and its Relation to the Successive Simplification of the Original Axiom of Leśniewski's Ontology' has appeared in the Festschrift dedicated to the late Professor Hubert Hubien.[1]

1.1. History of Version II

When that Festschrift was published, it appeared that for reasons still unknown to me, my contribution did not correspond to the one I had submitted. Since then, I have not been able to refer to *my own* article as

[1] Beets and Gavray (2005, pp. 95-113).

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Waragai (2005), for it was not quite the intended article. Meanwhile, several papers had cited the paper in its correct submitted version. To mention two of them, Kulicki (2011) and Trypuz (2014) both refer to the paper as *Technical Report 2003-2*. I have no hesitation in asserting that the works by Kulicki and Trypuz are beyond standard, and I am very glad that they used the correct version of the paper. Considering this situation, I decided to let my paper appear in an adequate form. Meanwhile I have written three papers on Leśniewski's Ontology. Two of them deal with a relationship between the so-called law(s) of Leibniz as well as the singular syllogism developed by Garlandus Compotista and the single axiom of Leśniewki's Ontology, and the third of them which I obtained on the 18th March 2014 at my last professorate deals with a syllogistic system in which we can derive the single axiom of Ontology and which is at the same time an intensional extension of Leśniewski's Ontology.

I have been in personal touch with the late Prof. Paul Gochet since 1980 when he wrote an article in which he made a long quotation of my article Waragai (1979). Originally I had planned to submit to this journal one of the three results but the editors of this memorial issue suggested to take the opportunity of this issue to publish the original paper with various additional improvements on the correct version of 2005, some of which are of importance in understanding Leśniewski's Ontology. I hope that the late Professors H. Hubien and P. Gochet would be happy with this humble dedication to them. They were active in Liège, which is said to be the place where Garlandus Compotista was born. What a marvelous historical coincidence!

1.2. Content of Version II

The first purpose of this paper is to make explicit the logical content of the weak form of the law of extensionality in Ontology.[2] It will be referred to as WΩ. We shall refer to the original law of extensionality in Ontology as Ω. Their formulations will be given soon below. WΩ is essentially weaker than Ω in its logical power. It may be said that it is of syllogistic character.

The second purpose of this paper is the analysis of the simplification procedure of the original axiom of Ontology, which was carried out by Leśniewski, Tarski and Sobociński.[3] As Sobociński (1934) is rather complicated, we shall try to make the relationship between WΩ and the simplification procedure as explicit as possible on the basis of the results concerning WΩ.

Then we shall establish several equivalent forms of the law of weak extensionality WΩ which are remarkable in their forms and content. Ω is, to my mind, too extensionalistic, being identical in its logical essence to

[2] For a general information on Ontology, cf. Lejewski (1958) and Urbaniak (2014).

[3] Cf. on this point Sobociński (1934).

that of set theory. Namely, it claims that if two names '*a*' and '*b*' have the same extension, then they play logically the same role. But is this view acceptable in general setting, if we side with the view that Ontology provides us with a suitable logical system to cope with the logical structure of *natural* language that is by no means extensionalistic? If we take into consideration the intensional aspects, this extremely extensionalistic standpoint seems hardly acceptable. In this regard, it is desirable to examine the logical content of Ω, but as the analysis of Ω itself is a matter that should be treated in a wider context, we shall confine our concern here just to the analysis of WΩ, leaving the analysis of Ω itself as a future task.

First, we shall establish some equivalent forms of WΩ which make explicit the logical content of WΩ. Especially WΩ5 reveals the role of the weak form of the law of extensionality that is frequently used in everyday syllogistic logical reasonings; WΩ5 expresses the logical content of WΩ more explicitly from one side, while as we shall see in section 5, WΩ6 is *directly* concerned with the derivation of the original axiom of Ontology.

Then, in sections 6 and 8, we shall make explicit what the procedure of simplification looks like, if we carry it by appealing to WΩ.

Now we enumerate some theses with which we shall be concerned in this paper.

D0	$[ab](a \subset b \equiv [x](x\varepsilon a \supset x\varepsilon b))$
D1	$[ab](a \circ b \equiv a \subset b \wedge b \subset a)$
D2	$[ab](a = b \equiv a\varepsilon b \wedge b\varepsilon a)$
D3	$[ab](!(a) \equiv [\exists x](x\varepsilon a))$
D4	$[a](\rightarrow(a) \equiv [xy](x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y))$
AO	$[ab](a\varepsilon b \equiv [\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y) \wedge [x](x\varepsilon a \supset x\varepsilon b))$
Ω	$[ab](a \circ b \supset (\phi(a) \supset \phi(b))$
H	$[ab](a\varepsilon b \equiv [\exists c](a\varepsilon c \wedge c\varepsilon b))$
WΩ	$[abc](a \circ b \supset (a\varepsilon c \supset b\varepsilon c))$
SOL	$[ab](a\varepsilon b \supset \rightarrow(a))$
T	$[abc](a\varepsilon b \supset (b\varepsilon c \supset a\varepsilon c)) \supset [ab](a\varepsilon b \supset \rightarrow(a)))$
AS	$[a][\exists b][x](x\varepsilon b \equiv x\varepsilon a \wedge \phi(a))$
E	$[ab](a\varepsilon b \equiv a\varepsilon a \wedge a \subset b)$

Hereafter we make a stipulation that a name of a formula is equivalent to the formula itself.[4]

[4] An example is:

D1 $a \circ b \equiv [x](x\varepsilon a \equiv x\varepsilon b)$

which is understood as:

D1 $\equiv$ $([ab](a \circ b \equiv [x](x\varepsilon a \equiv x\varepsilon b))$

D0 defines the functor $\subset$ for *weak inclusion* in terms of ε.

D1 defines the functor $\circ$ for *coextensionality* in terms of $\subset$, so in terms of ε.

D2 defines the functor for *identity* in terms of ε.

D3 states that *a* exists if and only if some individual object is (a) *a*

D4 states that there is at most one *a*, if and only if, for all objects *x* and *y*: if *x* is *a* and *y* is *a* then *x* is (identical with) *y*.

AO is the original and sole axiom of Leśniewski's Ontology.

Ω is the law of extensionality introduced into Ontology in the 1920's.[5] It is evident that Ω corresponds exactly to the law of extensionality in set theory both in content and formulation.

H is the shortest known axiom of Ontology. Indeed Leśniewski, Tarski and Sobociński succeeded in proving that AO and H were equivalent under some conditions. Usually it is thought that Ω is the condition, but this is too strong, and not needed in the simplification procedure. We shall show this fact in section 6 of this paper.

WΩ is the weak law of extensionality. This is a special case of Ω. It should be noticed that it is of syllogistic character. It will be proved that WΩ has two notably important equivalent forms, i.e. WΩ5 and WΩ6 that will appear in the sections to follow.

SOL states that the subject of a sentence of the form '$a\varepsilon b$' is singular.

T is the formula obtained by Tarski, and plays an essential role together with the weak law of extensionality in the procedure of simplification.

AS is the axiom of separation at the level of the calculus of names. As is easy to see, it is completely parallel to the axiom of separation in set theory.[6]

A fact concerning the simplification procedure is that a careful examination of the proof we find in Sobociński (1934) shows that to establish the equivalence between H and AO, we only need WΩ, and not the full power of Ω.[7]

[5] Cf. Sobociński (1934).

[6] In Ontology, AS has the form:

[AS] $[\exists a][x](x\varepsilon a \equiv x\varepsilon x \wedge \phi(x))$

the verbal rendering of which is:

(AS) for every ϕ, there is a name that has as its extension only those *objects* that satisfy ϕ.

AS and [AS] are equivalent to each other in the presence of A1 to be stated in the section 3.

To be exact, I add that what is given in Ontology is not the axiom AS or [AS] but a *rule* for introducing a name for a given predicate. AS in Ontology is equivalent to [AS]. A question will be raised concerning the adequacy of the reading of quantifiers, which will not be touched in this paper.

[7] The reader is asked to analyse the proof presented in Sobociński (1934), and check that it is indeed so. The part concerned is the section 5 of Sobociński (1934).

E will be used to show another simplification that makes use of only a form of the weak law of extensionality (WΩ6). It should be remarked that, according to our semantic intuition (more exactly that of Garlandus and Leibniz), E might seem to serve as an axiom for *singular sentences* that should govern the copula of the type individual/generic (ε). But as we shall see in section 8, this is not the case. E is too weak to work as an appropriate axiom for ε due to its extensional definition. Cf. the definition of E and proposition 23 in section 7.2. We need a condition, i.e., the thesis that will be referred to as WΩ6. The essential role of E will be discussed in section 8. On this point, see also the system W in section 3.

2. Intended Reading of Sentences of the Form '*aεb*'

In order to make explicit the logical content of the sole axiom of Leśniewski's Ontology AO, we shall clarify the logical intention of sentences of the form '$a\varepsilon b$'. Since an appropriate interpretation of Ontology is to regard it as an extention of syllogistic by means of 1) singular sentences that we can date back to Garlandus Compotista and Leibniz and 2) quantifiers that we can date back to Frege, AO is obtainable within the framework of syllogistic equipped with quantificational calculus. This will not be carried out in this paper. I postpone it to Waragai (201a). Ontology is sometimes said to be an ununderstandable and even *esoteric* system. This is of course wrong. Let me summarize my opinion on the naturalness and logical status of Ontology:

Manifesto

> Ontology is a system of syllogistic extended by means of 1) singular sentences[8] and 2) the quantification theory. And everybody who understands syllogistic, singular sentence and the theory of quantification is obliged to accept Ontology as a genuine system of logic. Ontology is (one of) the most basic logical system(s) with which we uncounsiously have been accustomed since the creation of the system of syllogistic (and the theory of quantification by Frege).[9] There is nothing *esoteric* in Ontology.

For further discussion, it is necessary to explain the intended reading of a sentence of the form '$a\varepsilon b$'. The intended reading of '$a\varepsilon b$' is given by the following condition:

TC '$a\varepsilon b$' is true iff *a* is an object and every *a* is *b*.

[8] Concerning the specific logical status of singular sentences in syllogistic, cf. Leibniz (1690), Leibniz (1996) and Łukasiewicz (1958).

[9] The place of Garlandus Compotista is between them. On Garlandus, see Henry (1977) and Waragai (2008).

That is, a singular sentence is a universal affirmative sentence the subject term of which happens to be singular. This is the central idea of Gerlandus and Leibniz.

One may well take notice of the fact that this condition immediately offers the Ontological Table given in Lejewski (1958) and this may be the most adequate reading of the singular sentence '$a\varepsilon b$'. In fact, this reading offers us a justification of Ontology. This will be carried out in Waragai (201a). Further, this reading reflects well and is in completely good harmony with the very idea of Garlandus Compotista and Leibniz.[10] I give here some examples:

Let us take the following meta-logical axiom:

ATC1 every *a* is *a*

Then we obtain:

TC1 '$a\varepsilon a$' is true iff *a* is an object.

which states that the semantical content of '$a\varepsilon a$' is just that *a* is an object (individual entity).

TC2 '$a\varepsilon b \supset a\varepsilon a$' is true.

Pr.

1	'$a\varepsilon b$' is true	[sup.]
2	*a* is an object and every *a* is *b*	[1,TC.]
3	*a* is an object	[2]
4	*a* is an object and every *a* is *a*	[3,ATC1]
5	'$a\varepsilon a$' is true	[4,TC.]

On this semantical basis, let us introduce a symbol '$\bigvee$' which is meant to correspond to 'object'. Let us set a meta-logical axiom:

ATC2 '$a\varepsilon\bigvee$' is true iff '$a\varepsilon a$' is true.

Evidently we have the following:

TC3 '$a\varepsilon\bigvee$' is true iff *a* is an object. [TC1,ATC2]

hence we have:

TC4 '$a\varepsilon b\supset a\varepsilon\bigvee$' is true. [TC2,TC3]

TC1, TC2, TC3 and TC4 all express Leśniewski's logical intuition.

For a full treatment of the justification of Leśniewski's Ontology, the reader is asked to refer to Waragai (201a).

[10] Compotista (1959), Leibniz (1690), Leibniz (1996).

3. Leśniewski's Derivation of AO

Now let us deduce AO following Leśniewski's method. Here we will follow the proof we find in Leśniewski (1931).

Let $\subset$, Δ, $\sqsubset$, $=$, $\rightarrow$, ! and ε be syllogistic functors that respectively represent the universal-affirmative sentence formative operator *without existential import*, the particular-positive sentence formative operator, the universal-positive operator *with existential import*, identity sentence formative functor, the uniqueness-declarative operator, the existence-declarative operator and the singular sentence formative operator. Note that Leśniewski here assumes that they are all definable in terms of ε and logical connectives, thus extensionally.

3.1. A Short Remark on 'every *a* is *b*'

Before going into the topic of this section, I would like to make an excursion. I just mentionned that such an extensional treatment seems to lead us unavoidably to treat the *universal affirmative* proposition in a conditional, extensional, way: i.e.

(I) every *a* is *b*

is understood as

(II) $[x](x \text{ is-}a \supset x \text{ is-}b)$

Now is this an acceptable view? So far as I know, there are some who think that this is not the case, and I have been sharing the same opinion for a long time. On this point, e.g., cf. Pelletier (1972). A very naive question that has been occupying my mind for a long time is: where is an implication (or implicational element) to be found in (I)? Does it have a *direct* relationship with *implication*? If not *directly*, then *how*? The scope of the quantifying word 'every' seems to play the role of restricting the domain of discourse just to '*a*', while in (II) we see an implication essentially used and the domain of discourse is the whole universe. But isn't it rather the fact that we constantly change the domain of quantifying phrases in everyday syllogistic reasonings, implication being thereby a secondary logical phenomenon if any? I will discuss this important, though until now not fully examined problem in the nearest future. The system W that I will mention briefly here was constructed partly for this purpose.[11]

Before coming to Leśniewski's proof, I wish to make an excursion pointing out an interesting (and important) technical fact.[12] It is concerned with

[11] Cf. Waragai (201b).
[12] This is what I realized on my last professorate.

the definition of singular sentences by means of universal affirmative sentences.

If we are allowed to take (I) and (II) as equivalent, it may well be expected that E should be able to function as a proper axiom for sentences of the form '$a\varepsilon b$'. But this view is unsustainable. The inconvenience with E is that the functor 'every . is ..' is defined in an extensional way. Now the fact is that if we introduce a non-extensional functor for 'every *a* is *b*' which we denote as *Aab*, and set up as an axiom:

Aε $\quad a\varepsilon b \equiv a\varepsilon\bigvee \wedge Aab$

with some appropriate syllogistic axioms that regulate the behaviour of the functor *A* and ε, we can obtain a system of Ontology that can deal with names/concepts in an adequate way. To make the system explicit, I will call it tentatively W, listing up its axioms. They run as follows:

Primitive symbols: ε (read: is), $\bigvee$ (read: *object*), *Aab* (read: every *a* is *b*),

WA1 $\quad a\varepsilon b \equiv a\varepsilon\bigvee \wedge Aab$

WA2 $\quad Aaa$

WA3 $\quad Aab \wedge Abc \supset Aac$

WA4 $\quad Aab \wedge a\varepsilon\bigvee \wedge b\varepsilon\bigvee \supset Aba$

WA5 $\quad a \circ b \wedge a\varepsilon\bigvee \supset b\varepsilon\bigvee$[13]

Once we know the first result of this article concerning WΩ, it is not difficult to show that we can deduce from WA1-WA5 the original axiom of Ontology and also it is easy to check that:

WT1 $\quad Aab \supset a \subset b$

which is (a form of) the law of *Dictum de Omni*, one of the basic laws of syllogistic logical systems, follows from *Aab*, while the opposite direction does not hold. That is:

WT2 $\quad [abx]\,(Aab \supset (x\varepsilon a \supset x\varepsilon b))$

while

(III) $\quad$ '$a \subset b$' does not imply '*Aab*',

nor (what is the same):

(IV) $\quad$ '$[x]\,(x\varepsilon a \supset x\varepsilon b)$' does not imply '*Aab*'.

[13] $a \circ b$ is *not* $Aab \wedge Aba$.

and (III) and (IV) are logically provable facts. This will be treated in Waragai (201b).

W seems us to allow to treat names or concepts in an intensional way, for example we can treat in this extended Ontology ‘Chimaera buzzing in vacuum’ (Chimaera in vacuo bombinans) as different from ‘man there being no man (homo nullo homine existente)’. This seems to be a big advantage in regarding Leśniewski’s Ontology as an appropriate logical tool for analysing everyday logical reasonings. As this is a matter that exceeds the scope of this paper, I have to leave it for another occasion.

3.2. *Leśniewski’s Derivation*

Now let us proceed to the derivation of AO proposed by Leśniewski himself.

1	$a\varepsilon b \supset a\varepsilon\bigvee$	[sup.(=TC2)]
2	$a\Delta b \equiv [\exists x]\,(x\varepsilon a \wedge x\varepsilon b)$	[def.]
3	$a \subset b \equiv [x]\,(x\varepsilon a \supset x\varepsilon b)$	[def.]
4	$a \sqsubset b \equiv \bigvee\Delta a \wedge a \subset b$	[def.]
5	$a = b \equiv a\varepsilon b \wedge b\varepsilon a$	[def.]
6	$\rightarrow(a) \equiv [xy]\,(x\varepsilon a \wedge y\varepsilon a \supset x = y)$	[def.]
7	$a\varepsilon b \equiv a \sqsubset b \wedge \rightarrow(a)$	[def.]
8	$\bigvee\Delta a \equiv [\exists x](x\varepsilon a)$	[1,2]
9	$a \sqsubset b \equiv [\exists x]\,(x\varepsilon a) \wedge [x]\,(x\varepsilon a \supset x\varepsilon b)$	[4,8,3]
10	$[xy]\,(x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y) \supset [xy]\,(x\varepsilon a \wedge y\varepsilon a \supset x = y)$	[4]
11	$[xy]\,(x\varepsilon a \wedge y\varepsilon a \supset a = b) \equiv [xy]\,(x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y)$	[9,4]
12	$\rightarrow(a) \equiv [xy]\,(x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y)$	[5,10]
13	$a\varepsilon b \equiv [\exists x]\,(x\varepsilon a) \wedge [x]\,(x\varepsilon a \supset x\varepsilon a) \wedge [xy]\,(x\varepsilon a \wedge x\varepsilon a \supset x\varepsilon y)$	[6,8,11]
14	$a\varepsilon b \equiv [\exists x]\,(x\varepsilon a) \wedge [xy]\,(x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y) \wedge [x]\,(x\varepsilon a \supset x\varepsilon b)$	[12]

As the supposition is true by TC4, so we obtain the following theorem:

(Theorem 1): AO is true under TC.

Notice that in this derivation, Leśniewski took the singular sentence in the following way:

(A01) $a\varepsilon b \equiv \bigvee\Delta a \wedge \rightarrow(a) \wedge a \subset b$

which is equivalent to:

(A02) $a\varepsilon b \equiv !\,(a) \wedge \rightarrow(a) \wedge a \subset b$ [4,7,8]

from which we directly arrive at AO by appealing to appropriate definitions. A singular sentence is here regarded as a kind of universal-positive sentence with the restriction that the subject is singular.[14]

The logical status of singular sentences in syllogistic in the traditional syllogistical system is said to be very unclear. It is worth mentioning some passages from Łukasiewicz's:

> A third inexactitude concerns the conclusion drawn by Aristotle from this classification of terms. It is not true that our arguments and inquiries deal as a rule with such universal terms as may be predicated of others and others of them. It is plain that individual terms are as important as universal, not only in everyday life but also in scientific researches. *This is the greatest defect of the Aristotelian logic, that singular terms and propositions have no place in it.* (Łukasiewicz 1958, p. 6, my italics).

Further he states:

> It is essential for the Aristotelian syllogistic that the same term may be used as a subject and as a predicate without any restriction. In all three syllogistic figures known to Aristotle there exists one term which occurs once as a subject and then again as a predicate: in the first figure it is the middle term, in the second figure the major term, and in the third figure the minor term. In the fourth figure all three terms occur at the same time as subjects and as predicates. *Syllogistic as conceived by Aristotle requires terms to be homogeneous with respect to their possible positions as subjects and predicates*. This seems to be the true reason why singular terms were omitted by Aristotle. (Łukasiewicz 1958, p. 7, my italics).

There is another way to derive AO from the Russellian theory of description. This is to find in Waragai (1990) and Waragai (2000). Hiż (1977) offers a similar analysis, but I presume that Leśniewski would be rather reluctant to his analysis. As for his philosophical standpoint, cf. Hiż (1971).

4. The System WLO

One aim of the present paper is to establish some equivalent formulas of WΩ. For this purpose, let us consider a system with the following axioms (with appropriate definitions):

A1 $[ab](a\varepsilon b \supset a\varepsilon a)$
A2 $[ab](a\varepsilon b \wedge b\varepsilon b \supset b\varepsilon a)$
A3 $[abc](a\varepsilon b \wedge b\varepsilon c \supset a\varepsilon c)$

[14] On this point, cf. Leibniz (1960), p. 182, Leibniz (1996), p. 568, Garlandus (1959), *De sillogismis extra Librum*, pp. 124-126, Liber Sextus, *De Sillogismis Hipoteticis*, especially <*De equipolentia hipoteticarum et cathegoricarum quarundam*>, pp. 131-133. Also Henry (1977) and Waragai (2008).

We shall refer to the system <A1,A2,A3> as WLO.[15] This system may seem to be simple and weak in logical power. But the fact is that it is, when equipped with the weak law of extensionality, sufficiently strong for developing syllogistic that allows singular sentences.

4.1. The Intended Meaning of the Axioms of WLO

The intended meaning of A1, A2 and A3 may be clear in the light of TC and some auxiliary conditions. The following theorem holds:

(Theorem 2): A1, A2 and A3 are true under TC.

Proof:
Because A1, A2, A3 are deducible from AO that is true under TC, and a proof is truth-preserving, they are also true under TC. For their proofs, the readers are asked to refer to Waragai and Oyamada (2007). A full treatment is to be found in Waragai (201a).

4.2. Some Systems Equipped with WΩ

Let us enumerate the systems we are hereafter concerned with in this paper. They are:

$\mathrm{WLO} = < \mathrm{A1}, \mathrm{A2}, \mathrm{A3} >$
$\mathrm{WLO1} = < \mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{W\Omega} >$
$\mathrm{WLO2} = < \mathrm{AO}, \mathrm{W\Omega} >$
$\mathrm{WLO3} = < \mathrm{H}, \mathrm{SOL}, \mathrm{W\Omega} >$
$\mathrm{WLO4} = < \mathrm{T}, \mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{W\Omega} >$
$\mathrm{WLO5} = < \mathrm{T}, \mathrm{AO}, \mathrm{W\Omega} >$
$\mathrm{WLO6} = < \mathrm{T}, \mathrm{H}, \mathrm{SOL}, \mathrm{W\Omega} >$
$\mathrm{WLO7} = < \mathrm{T}, \mathrm{H}, \mathrm{W\Omega} >$
$\mathrm{WLO8} = < \mathrm{AS}, \mathrm{A1}, \mathrm{A2}, \mathrm{A3}, \mathrm{W\Omega} >$
$\mathrm{WLO9} = < \mathrm{AS}, \mathrm{A0}, \mathrm{W\Omega} >$
$\mathrm{WLO10} = < \mathrm{AS}, \mathrm{H}, \mathrm{SOL}, \mathrm{W\Omega} >$
$\mathrm{WLO11} = < \mathrm{AS}, \mathrm{H}, \mathrm{W\Omega} >$

with appropriate definitions.

To state the results in advance, WLO1, WLO2 and WLO3 are inferentially equivalent to each other. WLO4, WLO5, WLO6 and WLO7 are inferentially equivalent to each other. WLO8, WLO9, WLO10 and WLO11 are equivalent to each other.

[15] <A1, A2, A3> is a slight modification of the system studied in Ishimoto (1977). Both systems are inferentially equivalent

4.3. Some Theorems of WLO

T1	$[ab]\,(a \circ b \wedge x\varepsilon b \supset x\varepsilon a)$	[D1]
T2	$[ab]\,(a \circ b \wedge x\varepsilon a \supset x\varepsilon b)$	[D1]

A point that merits a mention concerning WLO is that it is the propositional (quantifier-free) fragment of Ontology. A1, A2 and A3 suffice to prove one half of AO (from right to left), and they are strong enough to prove some essential equivalence between formulas expressing WΩ. WΩ(6) plays, as we shall see, the essential role in proving the other half of AO (from left to right).

T3	$[ab]\,(a = a \equiv a\varepsilon a)$	[D2(b/a)]
T4	$[ab]\,(a = b \supset b = b)$	[D2, A1(a/b, b/a), T3]
T5	$[ab]\,(a\varepsilon b \wedge b\varepsilon c \supset b\varepsilon a)$	[A1(a/b, b/c), A2]

4.4. The Law of Weak Extensionality and some Formulas that Are Equivalent to It

Let us consider the following formulas.

WΩ1	$[ab]\,(a \circ b \supset [c]\,(a\varepsilon c \supset b\varepsilon c))$
WΩ2	$[ab]\,(a \circ b \supset [c]\,(a = c \supset b = c))$
WΩ3	$[ab]\,(a \circ b \supset (a = a \supset a = b))$
WΩ4	$[ab]\,(a \circ b \supset (a = a \supset b = b))$
WΩ5	$[ab]\,(a \circ b \supset (a\varepsilon a \supset b\varepsilon b))$

Notice that WΩ1 is a special case of the axiom of extensionality Ω, WΩ2 is a weakened form of WΩ1, WΩ3 is a special case of WΩ2, WΩ4 is a weakened form of WΩ3, and WΩ5 is equivalent to WΩ4 in the presence of T3.

We now pass on to the proof that WΩ1, WΩ2, WΩ3, WΩ4 and WΩ5 are in fact equivalent to each other in WLO.

T6 $[ab]\,((a \circ b \supset [c]\,(a\varepsilon c \supset b\varepsilon c)) \supset (a \circ b \supset [c]\,(a = c \supset b = c)))$

Pr.

1	$a \circ b \supset [c]\,(a\varepsilon c \supset b\varepsilon c)$	[sup]
2	$a \circ b$	[sup]
3	$a = c$	[sup]
4	$a\varepsilon c$	[3, D2(b/c)]
5	$b\varepsilon c$	[1(c/c), 2, 4]
6	$c\varepsilon c$	[3, D2(b/c), A1(b/c)]

7	$c \varepsilon b$	[5,6,A2($a/b, b/c$)]
8	$b = c$	[5,7,D2($a/b, b/c$)]

From T6 it follows immediately that

$\vdash$ WΩ1 $\supset$ WΩ2

T7 $[ab]\,((a \circ b \supset [c]\,(a = c \supset b = c)) \supset (a \circ b \supset (a = a \supset a = b)))$

Proof is evident.

T8 $[ab]\,((a \circ b \supset (a = a \supset a = b)) \supset (a \circ b \supset (a = a \supset b = b)))$

Pr.

1	$a \circ b \supset (a = a \supset a = b)$	[sup.]
2	$a \circ b$	[sup.]
3	$a = a$	[sup.]
4	$a = b$	[1,3]
5	$b = b$	[4,T4]

T9 $[ab]\,((a \circ b \supset (a = a \supset b = b)) \supset (a \circ b \supset (a \varepsilon a \supset b \varepsilon b)))$

Proof is evident. (Use T3.)

T10 $[ab]\,((a \circ b \supset (a \varepsilon a \supset b \varepsilon b)) \supset (a \circ b \supset [c]\,(a \varepsilon c \supset b \varepsilon c)))$

Pr.

1	$a \circ b \supset (a \varepsilon a \supset b \varepsilon b)$	[sup.]
2	$a \circ b$	[sup.]
3	$a \varepsilon c$	[sup]
4	$a \varepsilon a$	[3,A1(b/c]
5	$b \varepsilon b$	[1,2,4]
6	$b \varepsilon a$	[2,5,D1(x/b)]
7	$b \varepsilon c$	[3,6,A3($a/b, b/a$)]

4.5. Intended Reading of WΩ: I

We summarize the result concerning the equivalences established above. We have proved the following theorem:

> Theorem 3: WΩ1, WΩ2, WΩ3, WΩ4 and WΩ5 are equivalent to each other in WLO.

Due to this fact, we may hereafter refer to any of WΩi (i=1,...,5) as WΩ as far as A1, A2 and A3 hold.

As the original WΩ, i.e. WΩ1, is equivalent to WΩ5, the substantial content of the weak law of extensionality in Ontology is in the light of TC the following:

> (WΩ) If *a* and *b* are coextentional[16] and *a* is an individual object, then *b* is an individual object, too.

Here we appealed to TC. It should be noticed that this is one of the very basic laws in syllogistic inferences which contain as their components singular propositions. It should also be remarked that this law is difficult or impossible to express in the standard first-order logical systems, while in everyday life reasonings as well as in scientific reasonings, such examples abound.

5. An Important Formula Equivalent to WΩ

Now let us pass on to another quite important equivalence which, we might say, reveals the very nature of the weak form of the law of extensionality in Ontology. The formula that we are going to focus on is the following:

$$(*) \qquad !(a) \wedge \rightarrow (a)$$

and we shall establish that, in WLO, it is equivalent to:

$$(**) \qquad a\varepsilon a$$

that is, the following is inferentially equivalent to WΩ in WLO:

$$\text{W}\Omega 6 \quad !(a) \wedge \rightarrow (a) \equiv a\varepsilon a$$

The following holds in WLO:

$$\text{T11(=SOL)} \quad [ab]\,(a\varepsilon b \supset \rightarrow (a))$$

Pr.

1	$a\varepsilon b$	[sup.]
2	$x\varepsilon a$	[sup.]
3	$y\varepsilon a$	[sup.]
4	$a\varepsilon a$	[1,A1]
5	$a\varepsilon x$	[2,4,A2$(a/x, b/a)$]
6	$y\varepsilon x$	[3,5,A3$(a/x, b/a)$]

from which we deduce:

$$\text{T12} \quad [a]\,(a\varepsilon b \supset !(a)) \qquad\qquad [\text{D3,A1}]$$

[16] That is: ‘$a \circ b$’ and *not* ‘$Aab \wedge Aba$’.

T13	$[ab]\,(a\varepsilon b \supset !(a) \wedge \rightarrow (a))$	[T11,T12]
T14	$[ab]\,(a\varepsilon a \supset !(a) \wedge \rightarrow (a))$	[T13(b/a)]
T15	$[ab]\,(a = a \supset !(a) \wedge \rightarrow (a))$	[T14,T3]

T16 $[ab](a \circ b \supset (\rightarrow (a) \supset \rightarrow (b)))$

Pr.

1	$a \circ b$	[sup.]
2	$\rightarrow (a)$	[sup.]
3	$x\varepsilon b$	[sup.]
4	$y\varepsilon b$	[sup.]
5	$x\varepsilon a$	[3,1,T2]
6	$y\varepsilon a$	[2,5,T2]
7	$y\varepsilon x$	[5,6,A3(a/y, b/a)]

T17 $[a]\,(!(a) \wedge \rightarrow (a) \supset a = a) \supset [ab]\,(a \circ b \wedge a = a \supset b = b))$

Pr.

1	$[a]\,(!(a) \wedge \rightarrow (a) \supset a = a)$	[sup.]
2	$a \circ b$	[sup.]
3	$a = a$	[sup.]
4	$!(a)$	[3,T15]
5	$\rightarrow (a)$	[3,T15]
6	$\rightarrow (b)$	[2,5,T16]
7	$a\varepsilon a$	[3,T1]
8	$a\varepsilon b$	[2,7,D1(x/a)]
9	$!(b)$	[8,D2]
10	$!(b) \wedge \rightarrow (b)$	[6,9]
11	$b = b$	[1(a/b),10]

T18 $[ab]\,(a \circ b \wedge a = a \supset b = b) \supset [a]\,(!(a) \wedge \rightarrow (a) \supset a = a)$

Pr.

1	$[ab]\,(a \circ b \wedge a = a \supset b = b)$	[sup.]
2	$!(a)$	[sup.]
3	$\rightarrow (a)$	[sup.]
4	$[xy]\,(x\varepsilon a \wedge y\varepsilon a \supset y\varepsilon x)$	[3,D4]
5	$x\varepsilon a$	[2,D2]
6	$[y]\,(y\varepsilon a \supset y\varepsilon x)$	[4,5]
7	$[y]\,(y\varepsilon x \wedge x\varepsilon a \supset y\varepsilon a)$	[A3]

8	$[y](y\varepsilon x \supset y\varepsilon a)$	[7,5]
9	$x \circ a$	[6,8,D1(a/x, b/a)]
10	$x\varepsilon x$	[5,A1(a/x, b/a)]
11	$x = x$	[10,T3(a/x)]
12	$a = a$	[1(a/x, b/a)]

T19	$[ab](a \circ b \wedge a\varepsilon a \supset b\varepsilon b) \equiv [a](!(a) \wedge \rightarrow (a) \supset a = a)$	[T16,T3]
T20	$[ab](a \circ b \wedge a\varepsilon a \supset b\varepsilon b) \equiv [a](!(a) \wedge \rightarrow (a) \equiv a = a)$	[T19,T15]
T21	$[ab](a \circ b \wedge a\varepsilon a \supset b\varepsilon b) \equiv [a](!(a) \wedge \rightarrow (a) \equiv a\varepsilon a)$	[T20,T3]

5.1. Intended Reading of WΩ: II

From the results we obtained in the previous chapter, we have the following theorem:

> Theorem 4: The weak law of extensionality in Ontology is, in WLO, equivalent to
>
> WΩ6 $[a](!(a) \wedge \rightarrow (a) \equiv a\varepsilon a)$

Due to this fact, we may hereafter refer to WΩi (i=1,...,6) as WΩ. The reading of WΩ6 is in the light of TC5 as follows:

> (WΩ6) *a* exist and there is at most one *a* if and only if *a* is an individual object.

The meaning of WΩ should intuitively be clear. It should be noticed that we constantly make use of this law [(WΩ6)] in everyday as well as in scientific inference activities at the level of syllogistic reasoning, while this law is impossible to state in the usual first-order predicate logic calculus, for there is no room in it for general names. This fact suggests the logical elegance and naturalness of Ontology. That this holds in WLO leads us to natural systems of Ontology, which is a matter of the sections to follow.

6. Systems of Ontology Equipped with WΩ

We will be concerned in this section with the construction and comparison of some systems of Ontology that are equipped with the weak law of extensionality.

It should be noticed that the result obtained in the previous section, esp. T21, is used in the next subsection.

Indeed once we realize that WΩ1 is equivalent to WΩ6 (in WLO), we are led almost automatically to AO in the presence of A1 and A3.

6.1. WLO1 is inferentially Equivalent to WLO2: I

Now let us consider the system WLO1 and WLO2. We will show that they are inferentially equivalent to each other. To show this, we shall be working within WLO1, i.e. WLO equipped with WΩ.

T22 $[ab](a\varepsilon b \supset !(a) \wedge \rightarrow (a) \wedge [x](x\varepsilon a \supset x\varepsilon b))$ [T13, A3]

T23 $[ab](!(a) \wedge \rightarrow (a) \wedge [x](x\varepsilon a \supset x\varepsilon b) \supset a\varepsilon b)$

Pr.

1	$!(a) \wedge \rightarrow (a) \wedge [x](x\varepsilon a \supset x\varepsilon b)$	[sup.]
2	$a\varepsilon a$	[1, WΩ6]
3	$a\varepsilon a \supset a\varepsilon b$	[1(x/a)]
4	$a\varepsilon b$	[2, 3]

T24 $[ab](a\varepsilon b \equiv !(a) \wedge \rightarrow (a) \wedge [x](x\varepsilon a \supset x\varepsilon b))$ [T22, T23]

that is:

T25 $[ab](a\varepsilon b \equiv [\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y) \wedge [x](x\varepsilon a \supset x\varepsilon b))$
[T24, D3, D4, D5]

which is AO, the sole axiom of Ontology.

Let us call a system of Ontology one that contains AO as its thesis. Thus we have the following theorem:

Theorem 5: WLO1 is a system of Ontology.

Thus we arrived at a system of Ontology where the weak law of extensionality appears explicitly.

From theorem 6 we have:

Proposition 1: A1 ∧ A2 ∧ A3 ∧ WΩ ⊃ AO [Theorem 6]

hence:

Proposition 2: A1 ∧ A2 ∧ A3 ∧ WΩ ∧ ⊃ AO ∧ WΩ [Proposition 1]

6.2. An Excursion

Another way to show T24 is the following:

Pr:

1	$a\varepsilon b \equiv a\varepsilon a \wedge [x](x\varepsilon a \supset x\varepsilon b)$	[A1, A3]
2	$a\varepsilon a \equiv [\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y)$	[WΩ6]
3	$a\varepsilon b \equiv [\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset x\varepsilon y) \wedge [x](x\varepsilon a \supset x\varepsilon b)$	[1, 2]

We remark that the proof above is based on WLO1, for A2 was needed to prove WΩ6.

A careful look at the result in this excursion suggests us a specific and in a sense distinguished role of WΩ6. We shall return to this problem later.

6.3. WLO1 is Inferentially Equivalent to WLO2: II

To show the reverse direction of Proposition 2, we prove that A1, A2 and A3 follow from AO.

Proposition 3: AO $\supset$ A1

Pr.

1	AO	[sup.]
2	$a\varepsilon b$	[sup.]
3	$[\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset y\varepsilon x) \wedge [x](x\varepsilon a \supset x\varepsilon b))$	[1,2]
4	$[\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset y\varepsilon x)$	[3,AO]
5	$[\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset y\varepsilon x) \wedge [x](x\varepsilon a \supset x\varepsilon a))$	[4]
6	$a\varepsilon a \equiv [\exists x](x\varepsilon a) \wedge [xy](x\varepsilon a \wedge y\varepsilon a \supset y\varepsilon x) \wedge [x](x\varepsilon a \supset x \supset a)$	[1(b/a)]
7	$a\varepsilon a$	

Proposition 4: AO $\supset$ A2

1	AO	[sup.]
2	$a\varepsilon b$	[sup.]
3	$b\varepsilon b$	[sup.]
4	$[xy](x\varepsilon b \wedge y\varepsilon b \supset y\varepsilon x)$	[2,T25]
5	$a\varepsilon b \wedge b\varepsilon b \supset b\varepsilon a$	[4($x/a, y/a$)]
6	$b\varepsilon a$	[5,2,3]

Proposition 5: AO $\supset$ A3

1	AO	[sup.]
2	$a\varepsilon b$	[sup.]
3	$b\varepsilon c$	[sup.]
4	$!(a) \wedge \rightarrow (a)$	[2,T22]
5	$[xy](x\varepsilon a \supset x\varepsilon b)$	[2,T25]
6	$[xy](x\varepsilon b \supset x\varepsilon c)$	[3,T25]
7	$[xy](x\varepsilon b \wedge y\varepsilon c)$	[3,T25]
8	$!(a) \wedge \rightarrow (a) \wedge [xy](x\varepsilon b \wedge y\varepsilon c)$	[3,T25]
9	$a\varepsilon c$	[8,T23]

Therefore we have the proposition:

Proposition 6: $\mathrm{AO} \supset \mathrm{A1} \wedge \mathrm{A2} \wedge \mathrm{A3}$

so that we have:

Proposition 7: $\mathrm{AO} \wedge \mathrm{W}\Omega \supset \mathrm{A1} \wedge \mathrm{A2} \wedge \mathrm{A3} \wedge \mathrm{W}\Omega$ [Proposition 6]

therefore:

Proposition 8: $\mathrm{A1} \wedge \mathrm{A2} \wedge \mathrm{A3} \wedge \mathrm{W}\Omega \equiv \mathrm{AO} \wedge \mathrm{W}\Omega$ [Propositions 2,7]

Therefore we have the following theorem:

Theorem 6: WLO1, i.e. <A1,A2,A3,WΩ> is inferentially equivalent to WLO2, i.e., <AO,WΩ>.

As many logicians have been deploring, it is quite hard to capture the intention of AO, while A1, A2, A3 and WΩ5, WΩ6 are easy to capture in the light of TC, so that those who have difficulties with AO should begin with WLO1, for it is easy enough to capture its intuitive sense. A reader who has still difficulties with AO or WLO1 should consult Waragai (201a) where the present author tries to give a full justification of WLO1. He will see that there is nothing esoteric with Leśniewski's Ontology.

6.4. WLO1 is Inferentially Equivalent to WLO3

Now let us consider the systems WLO1 and WLO3. We have the following propositions:

Proposition 9: $\mathrm{H} \supset \mathrm{A3}$

Pr.

1	$[ab]\,(a\varepsilon b \equiv [\exists c]\,(a\varepsilon c \wedge c\varepsilon b))$	[sup.]
2	$[ab]\,([\exists c]\,(a\varepsilon c \wedge c\varepsilon b) \supset a\varepsilon b)$	[1]
3	$[abc]\,(a\varepsilon c \wedge c\varepsilon b \supset a\varepsilon b)$	[2]

Proposition 10: $\mathrm{SOL} \supset \mathrm{A2}$

Pr.

1	SOL	[sup.]
2	$a\varepsilon b$	[sup.]
3	$b\varepsilon b$	[sup.]
4	$\rightarrow (b)$	[3, SOL]
5	$[xy]\,(x\varepsilon b \wedge y\varepsilon b \supset y\varepsilon x)$	[4, D4]

6	$a\varepsilon b \wedge b\varepsilon b \supset b\varepsilon a$	$[4(x/a, y/b]$
7	$b\varepsilon a$	[6,2,3]

Further we have:

Proposition 11: $\mathrm{H} \wedge \mathrm{SOL} \supset \mathrm{A1}$

Pr.

1	H	[sup.]
2	SOL	[sup.]
3	$a\varepsilon b$	[sup.]
4	$a\varepsilon c \wedge c\varepsilon b$	[3,1]
5	$c\varepsilon b$	[4]
6	$\rightarrow (c)$	[5,SOL]
7	$[xy]\,(x\varepsilon c \wedge y\varepsilon c \supset y\varepsilon x)$	[6,D4]
8	$a\varepsilon c \wedge a\varepsilon c \supset a\varepsilon a$	$[7(x/a, y/a)]$
9	$a\varepsilon c$	[4]
10	$a\varepsilon a$	[8,9]

Therefore we have:

Proposition 12: $\mathrm{H} \wedge \mathrm{SOL} \supset \mathrm{A1} \wedge \mathrm{A2} \wedge \mathrm{A3}$ [Propositions 9, 10, 11]

therefore:

Proposition 13: $\mathrm{H} \wedge \mathrm{SOL} \wedge \mathrm{W}\Omega \supset \mathrm{A1} \wedge \mathrm{A2} \wedge \mathrm{A3} \wedge \mathrm{W}\Omega$ [Proposition 12]

Proposition 14: $\mathrm{H} \wedge \mathrm{SOL} \wedge \mathrm{W}\Omega \supset \mathrm{AO}$ [Propositions 13,1]

Now we shall show the reverse of Proposition 13. It is easy to check that H and SOL are derivable from A1, A2, A3. SOL was already proved to be a thesis of WLO(T11). To see that H is a thesis in WLO, we only need to combine A1 and A3.

T26 $[ab]\,(a\varepsilon b \equiv [\exists c]\,(a\varepsilon c \wedge c\varepsilon b))$

Pr.

1.1	$a\varepsilon b$	[sup.]
1.2	$a\varepsilon a$	[1,A1]
1.3	$[\exists x]\,(a\varepsilon x \wedge x\varepsilon b)$	[1.2,1.1]
1	$a\varepsilon b \supset [\exists x]\,(a\varepsilon x \wedge x\varepsilon b)$	[1.1.,1.3]
2.1	$[\exists x]\,(a\varepsilon x \wedge x\varepsilon b)$	[sup.]
2.2	$a\varepsilon x \wedge x\varepsilon b$	[2.1]
2.3	$a\varepsilon b$	$[2.2, \mathrm{A3}(b/x, c/b)]$

2	$[\exists x](a\varepsilon x \wedge x\varepsilon b) \supset a\varepsilon b$	[2.1,2.3]
3	$a\varepsilon b \equiv [\exists x](a\varepsilon x \wedge x\varepsilon b)$	[1,2]

Therefore we have:

Proposition 15: A1 ∧ A3 ⊃ H [T26]

From T11 we have:

Proposition 16: A1 ∧ A2 ∧ A3 ⊃ SOL [T11]

Thus we have:

Proposition 17: A1 ∧ A2 ∧ A3 ⊃ H ∧ SOL [Propositions 15,16]

Therefore:

Proposition 18: A1 ∧ A2 ∧ A3 ∧ WΩ ⊃ H ∧ SOL ∧ WΩ [Proposition 17]

Hence:

Proposition 19: A1 ∧ A2 ∧ A3 ∧ WΩ ≡ H ∧ SOL ∧ WΩ [Propositions 13,18]

Thus we arrived at the following theorem:

Theorem 7: WLO1, i.e., <A1,A2,A3,WΩ> and WLO3, i.e., <H,SOL,WΩ> are inferentially equivalent to each other, and are systems of Ontology.

6.5. WLO1, WLO2 and WLO3 are Equivalent to Each Other

From Theorem 6 and Theorem 7, we obtain the following theorem:

Theorem 8: WLO1, WLO2 and WLO3 are inferentially equivalent to each other, and are systems of Ontology.

If we take into consideration the fact that A1, A2, A3 and H are all derivable from AO without appealing to WΩ, WLO2 may seem to be stronger than WLO1 and WLO3, but this is not the case, so that we are forced to ackowledge that WΩ is endowed with an essentially strong potentiality in the syllogistic systems that are able to deal with singular sentences in an explicit manner. For a syllogistic system to be able to treat singular sentences properly, WΩ seems to be unavoidably required.

7. The Logical Structure of Simplification

In this section, we shall cope with the logical structure of the simplification procedure. Due to the inferential equivalence between WLO1, WLO2 and

WLO3, we tentatively fix WLO3 as our official system for the purpose of this section.

As the whole simplification procedure is essentially dependent on the result due to Tarski, we shall be concerned with Tarski's result in the next subsection.

7.1. Tarski's Result

We shall refer to the result obtained by Tarski as T. We shall prove T after the manner taken by Tarski, stating explicitly the supposition needed for its proof. T is the following thesis:

$$\text{T} \quad [abc]\,(a\varepsilon b \supset (b\varepsilon c \supset a\varepsilon c)) \supset [ab]\,(a\varepsilon b \supset \rightarrow (a)))$$

We have:

$$\text{(T)} \quad \text{A3} \supset \text{SOL}$$

To prove T, we need the following axiom schema:

$$\text{AS} \quad [a]\,[\exists b]\,[x]\,(x\varepsilon b \equiv x\varepsilon a \wedge \phi(x))$$

which is a complete analog of the axiom of separation in set theory. If we accept the view that a predicate can be converted into a name that corresponds to the predicate, then it is a plausible axiom in the calculus of names as well. (An example: 'run' can be converted into 'that which runs'.) It is not out of place to remark that this confers syllogistic a considerably strong inferential power.[17]

Here, we just repeat Tarski's ingenious proof of T.[18] To prove T, Tarski introduced the following functor $*$ by appealing to AS stated as a rule.

$$Df\text{A} \quad [xab]\,(x\varepsilon * [ab] \equiv x\varepsilon a \wedge b\varepsilon x) \qquad [\text{AS}(\phi(x)\,/\,b\varepsilon x)]$$

Under AS and A3 we are able to deduce the result of Tarski.

$$\text{T} \quad [abc]\,(a\varepsilon b \supset (b\varepsilon c \supset a\varepsilon c)) \supset [ab]\,(a\varepsilon b \supset \rightarrow (a))$$

Pr.

1	$[xab]\,(x\varepsilon * [ab] \equiv x\varepsilon a \wedge b\varepsilon x)$	[DfA]
2	A3	[sup.]
3	$a\varepsilon b$	[sup.]

[17] An example is to find in Waragai (1998), where the so-called law of Leibniz was reduced to a special case of *Barbara*.

[18] Cf. Sobociński (1934), section 2.

4	$x\varepsilon a$	[sup.]
5	$y\varepsilon a$	[sup.]
6	$a\varepsilon *[bx]$	[3,4,1 $(x/a, a/b, b/x)$]
7	$y\varepsilon *[bx]$	[2,5,6]
8	$y\varepsilon b \wedge x\varepsilon y$	[7,1]
9	$x\varepsilon y$	[8] [End of the proof of T]

Now let us define AS$*$ as follows:

AS$*$ $[\phi][a][\exists b][x](x\varepsilon b \equiv x\varepsilon a \wedge \phi(x))$

From the proof of T shown above we have the following proposition:

Proposition 20: AS* $\wedge$ A3 $\supset$ SOL
Proposition 20°: AS* $\supset$ T

Although Tarski used the proper form of definition *Df*A instead of introducing a name without parameters, there is no difference in the result in this case. Thus seen, it should be mentioned that the Axiom of Separation AS, here in the calculus of names as well as in set theories, should be regarded as of definitional character.

Let us examine the proof of T to see that it is creative. Clearly the following holds:

(I) AS* $\supset$ *Df*A [Instantiation]

then from the suppositions:

(*) *Df*A, A3, $a\ \varepsilon\ b$, $x\ \varepsilon\ a$, $y\ \varepsilon\ a$

we obtain:

(**) $x\ \varepsilon\ y$

therefore :

(II) *Df*A $\wedge$ A3 $\supset$ SOL

therefore:

(III) *Df*A $\supset$ (A3 $\supset$ SOL)

that is:

(IV) *Df*A $\supset$ T

so that if we accept AS*, T becomes a thesis. Now what should be recognized is that T does not hold universally. Not any transitive relation does satisfy SOL. What did, then, establish such a formula T as valid?

It is, as we see in (IV), *Df*A. It is *Df*A that has created the thesis T. In other words, without *Df*A we couldn't have reached T. Thus *Df*A is a creative definition. Hence we have to conclude that AS(*) itself is of creative character. It is on T and AS*, theses of creative character, that the simplification is carried out.

If we have the following fact established:

(*) A1, A2, A3 and WΩ are mutually independent,

we can have as its consequence that T is not a thesis of WLO1(2,3). Further the fact that AS is also of creative character seems to cause several serious logical problems which are worth a good deliberation. But all these are beyond the scope of this paper. I will discuss fully these subjects and related problems in Waragai (201a).

7.2. The Structure of Simplification

At this stage, we are not far from the aimed simplification. We have:

Proposition 21: $\mathrm{AO} \wedge \mathrm{W}\Omega \equiv \mathrm{H} \wedge \mathrm{SOL} \wedge \mathrm{W}\Omega$ [Proposition 19, 8]

whence:

Proposition 22: $\mathrm{W}\Omega \supset (\mathrm{AO} \equiv \mathrm{H} \wedge \mathrm{SOL})$ [Proposition 21]
Proposition 23: $\mathrm{T} \wedge \mathrm{H} \supset \mathrm{SOL}$ [Propositions 9, Def. of T]
Proposition 24: $\mathrm{T} \supset (\mathrm{H} \wedge \mathrm{SOL} \equiv \mathrm{H})$ [Propositions 20, 9]

so that we have:

Proposition 25: $\mathrm{T} \wedge \mathrm{W}\Omega \supset (\mathrm{AO} \equiv \mathrm{H})$ [Propositions 24, 22]

This is our first simplification result. In contrast to Proposition 37, it is a genuine simplification.

In this way, T and the weak law of extensionality go hand in hand in the procedure of simplification.

Proposition 26: $\mathrm{AS}^* \wedge \mathrm{W}\Omega \supset (\mathrm{H} \equiv \mathrm{AO})$ [Propositions 25, 20°]

Thus we obtain the following theorem:

Theorem 9: Under AS and WΩ, H is equivalent to AO.

and this is the substantial and correct formulation of the simplification result. It is in this way that AS and the law of extensionality go hand in hand in the simplification procedure. Since AS is accepted in Ontology, theorem 12 takes the following form:

Theorem 9°: Under WΩ, H is equivalent to AO.

Since they, i.e. Leśniewski, Tarski and Sobociński, took Ω instead of WΩ, theorem 9 is stated as follows:

Theorem 9°°: Under Ω, H is equivalent to AO.

Or simply,

Theorem 9°°°: H is equivalent to AO.

8. On E and Some Simplification Results

8.1. Aim of this Section

In this section we shall be concerned with some results of simplification that are related to a specific logical character of WΩ6. The sentence we shall be concerned with is the following one which we have stipulated to refer to as E:

E $[ab](a\varepsilon b \equiv a\varepsilon a \wedge a \subset b)$

(E) $[ab](a\varepsilon b \equiv a\varepsilon \bigvee \wedge Aab)$

It should be remarked that E breaks down due to its extensionalistic definition, A1 ∧ A3, which will not be the case with (E). Cf. on this point, WA1 in subsection 3.1.

It should also be remarked that E might look as if it were able to function as an axiom for *singular sentences*. This holds, however, only under some conditions, i.e., WΩ and AS, as we will show soon.[19]

The systems we shall be concerned with are the following:

WLO12 = < E, SOL, WΩ >
WLO13 = < T, E, SOL, WΩ >
WLO14 = < T, E, WΩ >
WLO15 = < AS, E, SOL, WΩ >
WLO16 = < AS, E, WΩ >

8.2. E and Simplification

It is not difficult to realize that E is equivalent to the product of A1 and A3.

Proposition 27: E ≡ A1 ∧ A3

Pr.

A1 $[ab](a\varepsilon b \supset a\varepsilon a)$

A3 $[abc](a\varepsilon b \wedge b\varepsilon c \supset a\varepsilon c)$

[19] Cf. section 3. This is closely connected with the system W.

1.1	$a\varepsilon b$	[sup.]
1.2	$a\varepsilon a$	[1.1, A1]
1	$a\varepsilon b \supset a\varepsilon a$	[1.1, 1.2]
2.1	$a\varepsilon b$	[sup.]
2.2	$x\varepsilon a$	[sup.]
2.3	$x\varepsilon b$	[2.2, A3 $(a/x, b/a, c/b)$]
2	$a\varepsilon b \supset (x\varepsilon a \supset x\varepsilon b)$	[2.1, 2.2, 2.3]
3	$a\varepsilon b \supset [x]\,(x\varepsilon a \supset x\varepsilon b)$	[2]
4	$a\varepsilon b \supset a\varepsilon a \wedge [x]\,(x\varepsilon a \supset x\varepsilon b)$	[1, 3]
5	$a\varepsilon a \wedge [x]\,(x\varepsilon a \supset x\varepsilon b) \supset a\varepsilon b$	[thesis of predicate calculus]
6	$a\varepsilon b \equiv a\varepsilon a \wedge [x]\,(x\varepsilon a \supset x\varepsilon b)$	[4.5]
7	$[ab]\,(a\varepsilon b \equiv a\varepsilon a \wedge [x]\,(x\varepsilon a \supset x\varepsilon b))$	[6]
8	E	[7]

Thus A1 and A3 imply E. It is evident that E implies A1 and A3. (End of Proof.)

Thus, despite its apparent kinship to AO, E is deductively essentially weak and doesn't reach the logical potentiality possessed by the original axiom of Ontology, AO.

Let us enumerate several propositions we shall need for further deductions:

Proposition 28:	AO $\supset$ SOL	[Propositions 6,16]
Proposition 29:	AO $\supset$ E	[Propositions 3,5,27]
Proposition 30:	E $\supset$ H	[Propositions 27,15]
Proposition 31:	AO $\supset$ E	[Propositions 3,5,27]
Proposition 32:	T $\wedge$ E $\supset$ SOL	[Propositions 23,30]
Proposition 33:	AS* $\wedge$ H $\supset$ SOL	[Propositions 32,20°]
Proposition 34:	AS* $\wedge$ E $\supset$ SOL	[Propositions 33,30]
Proposition 35:	E $\wedge$ WΩ $\wedge$ SOL $\supset$ AO	[Propositions 19,8]
Proposition 36:	W Ω $\wedge$ SOL $\supset$ (E $\supset$ AO)	[Proposition 35]

Appealing to Proposition 31, we obtain:

Proposition 37: WΩ $\wedge$ SOL $\supset$ (E $\equiv$ AO) [Propositions 36,31]

It might seem that we have here succeeded in obtaining a simplification result. But it should be noticed that this simplification is an apparent one, for SOL is a component of AO that is to be simplified. It is at best a simplification by means of *petitio principii*.

8.3. E and Generalized Simplification Results

From Proposition 37, we have:

Proposition 38: $W\Omega \wedge SOL \wedge E \equiv W\Omega \wedge SOL \wedge AO$ [Proposition 37]
Proposition 39: $W\Omega \wedge SOL \wedge E \equiv W\Omega \wedge AO$ [Propositions 38,28]
Proposition 40: $W\Omega \wedge SOL \wedge E \equiv W\Omega \wedge A1 \wedge A2 \wedge A3$ [Propositions 39,8]
Proposition 41: $W\Omega \wedge SOL \wedge E \equiv W\Omega \wedge H \wedge SOL$ [Propositions 40,19]

Therefore from Propositions 39, 40, and 41, we have :

Theorem 10: WLO12 is inferentially equivalent to WLO1(2,3)

Further we have:

Proposition 42: $W\Omega \supset (AO \equiv A1 \wedge A2 \wedge A3)$ [Propositions 39,40]
Proposition 43: $W\Omega \supset (AO \equiv H \wedge SOL)$ [Propositions 39,41]

Now we have the following :

Proposition 43: $T \wedge W\Omega \wedge SOL \wedge E \equiv T \wedge W\Omega \wedge AO$ [Proposition 39]
Proposition 44: $T \wedge W\Omega \wedge SOL \wedge E \equiv T \wedge W\Omega \wedge A1 \wedge A2 \wedge A3$ [Proposition 40]
Proposition 45: $T \wedge W\Omega \wedge SOL \wedge E \equiv T \wedge W\Omega \wedge H \wedge SOL$ [Proposition 41]
Proposition 46: $T \wedge W\Omega \wedge SOL \wedge E \equiv T \wedge W\Omega \wedge H$ [Propositions 45,23]
Proposition 47: $T \wedge W\Omega \wedge SOL \wedge E \equiv T \wedge W\Omega \wedge E$ [Propositions 46,32]

Therefore we have:

Theorem 11: WLO13 is inferentially equivalent to WLO14 and WLO4(5,6,7)

Proposition 48: $T \wedge W\Omega \supset (E \equiv AO)$ [Propositions 45,43]
Proposition 49: $T \wedge W\Omega \supset (E \equiv A1 \wedge A2 \wedge A3)$ [Propositions 47,44]
Proposition 50: $T \wedge W\Omega \supset (E \equiv H)$ [Propositions 47,46]
Proposition 51: $T \wedge W\Omega \supset (H \equiv AO)$ [Propositions 40,42]

Now replacing AS* for T, we obtain :

Proposition 52: $AS^* \wedge W\Omega \wedge SOL \wedge E \equiv AS^* \wedge W\Omega \wedge AO$ [Proposition 39]
Proposition 53: $AS^* \wedge W\Omega \wedge SOL \wedge E \equiv AS^* \wedge W\Omega \wedge A1 \wedge A2 \wedge A3$ [Proposition 38]

Proposition 54: $AS^* \wedge W\Omega \wedge SOL \wedge E \equiv AS^* \wedge W\Omega \wedge H \wedge SOL$ [Proposition 39]
Proposition 55: $AS^* \wedge W\Omega \wedge SOL \wedge E \equiv AS^* \wedge W\Omega \wedge H$ [Propositions 46,33]
Proposition 56: $AS^* \wedge W\Omega \wedge SOL \wedge E \equiv AS^* \wedge W\Omega \wedge E$ [Proposition 34]

Therefore we have:

Theorem 12: WLO15 is inferentially equivalent to WLO16 and WLO8 (9,10,11)

Further we have:

Proposition 57: $AS^* \wedge W\Omega \supset (E \equiv AO)$ [Propositions 48,44]
Proposition 58: $AS^* \wedge W\Omega \supset (E \equiv A1 \wedge A2 \wedge A3)$ [Propositions 48,45]
Proposition 59: $AS^* \wedge W\Omega \supset (E \equiv H)$ [Propositions 48,47]
Proposition 60: $AS^* \wedge W\Omega \supset (H \equiv AO)$ [Propositions 49,51]

Now notice that all the Propositions in subsection 8.3 as well as Propositions 36, 37 stem from Proposition 35. If we analyse the logical structure of Propositions after Proposition 35, then we will know what role T (so AS(*)) and WΩ play.

To summarize the results thus far obtained, we have:

Theorem 13: Under AS (or T) and WΩ, E (=$A1 \wedge A3$) and $A1 \wedge A2 \wedge A3$ and H and AO are all equivalent to each other.

This is a generalization of Proposition 25 and Proposition 26.

Now a remark concerning the simplification that was carried out on the basis of Proposition 35 is in order. All of the results in this section depend on Proposition 35. It is indeed a formula which becomes all at once clear if we consider the logical information it conveys.

Consider its equivalent Proposition 40. As was shown, E is nothing more than the conjunction of A1 and A3, and it is easy to prove that under A1 and A3, SOL is equivalent to A2, so that the left hand side of Proposition 40 is just the conjunction of WΩ and A2 and A1 and A3, which is exactly the right hand side of Proposition 40. Proposition 40 holds clearly, even without logical calculation. But the achievement of the simplification results requires to erase SOL, a component formula of AO, from the formulas, and it is this that was accomplished by appealing to T (and AS*). In sum, in either case (T, AS*) the simplification of AO is based on the creative definition proposed by Tarski. They seem to have been unobtainable without using Tarski's creative definition *Df*A.

Now I end up this dedication with a short remark and a theorem. As we have seen by now, WLOn (n=1,...,16) systems can be grouped into three:

G1	WLO1(2,3,10)	[Theorem 10]
G2	WLO4(5,6,7,13,14)	[Theorem 11]
G3	WLO8(9,10,11,15,16)	[Theorem 12]

Now let us take as their representatives WLO1 (=<A1, A2, A3, WΩ>), WLO4 (=<T, A1, A2, A3, WΩ>) and WLO8 (=<AS, A1, A2, A3, WΩ>).

We can prove that A1, A2, A3, and WΩ are mutually independent from each other. As this proof itself exceeds the scope of this paper, I will postpone giving a proof of it until Waragai (201a).

Now suppose their independence, recalling what form T has. That is (A3 ⊃ SOL) and in the presence of A1 and A3, SOL is equivalent to A2, so that in the presence of A1 and A3, T has the form (A3 ⊃ A2), which means that the mutual independence among them gets destroyed with the acceptance of T. Now that the systems belonging to the group G2 have T as an axiom, the mutual independence is destroyed, which is not the case with any system belonging to G1. Thus any system belonging to G2 is stronger than any system belonging to G1.

It is evident that AS* is essentially stronger than T. Therefore, any system belonging to G3 is essentially stronger than any system belonging to G2, whence the theorem:

> Theorem 14: G1 < G2 < G3, where S1 < S2 states that S2 is essentially stronger than S1.

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Toshiharu WARAGAI,
Tokyo Institute of Technology,
Meguro-ku,
Tokyo, Japan,
toshiharuwaragaitit@gmail.com