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WHERE IS LOGICAL KNOWLEDGE LOCATED?

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— To the memory of Paul Gochet (1932-2011) —

ABSTRACT

An attempt is made to distinguish logical from all other kinds of knowledge, especially mathematics. The tradition of stressing forms of proposition is revived, imitating the theory of moments in phenomenology from which it follows that logic is intrinsically dependent on the setting in which it is used. Sites are sought also for several topics that overlap with both logic and foundational branches of mathematics; they include set theory, model theory, axiomatisation and metamathematics. Logic is held exhibit structuralism in a way that mathematics does not. Both logic and its neighbouring topics manifest self-reference, which is an exceptionally ubiquitous but perplexing moment.

KEYWORDS

Logical knowledge · forms · structures · momental logic · self-reference.

1. Introduction

1.1. *Aims*

> […] the attempt to formulate the foundations of logic is rendered arduous by a corresponding "logocentric" predicament. *In order to give an account of logic, we must presuppose and employ logic.*
>
> — Henri Scheffer (1926, 227-228)

The status of **logical knowledge** is notoriously *elusive*: it seems to be "everywhere" and yet it is hard isolate to detach from the (normally) extra-logical contexts in which it is used, even in disciplines such as mathematics and the law where its presence is marked. The characterisation proposed here considers classical two-valued logic, named 'L2', as an example of a logic; the analysis of other logics in a comparable manner is noted at the end. We suggest that L2 has four main **sectors**, centered respectively on propositions, propositional functions, assertion and deduction. It builds upon an examination in Grattan-Guinness, I. (2013) (named 'GGan' below)

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of the assertion of a proposition (that is, the assignment to it of a truth-value), and of the various modes of negating it;[1] so we start with the principal conclusions drawn there. As there, major notions are rendered here in boldface at their introduction.

On assertion, to any proposition R we associate two asserted propositions in the metalogic ML2 of L2:

'It is true that R' (symbolised '+R') is the 'affirmation' of R;
'It is untrue that R' (symbolised '−R') is its 'denial'.

R is a proposition about some state of affairs in a **setting**, which can come from any body of knowledge. When attention is paid to asserted propositions, propositional and functional calculi are developed, which supplement the normal calculi of unasserted propositions. It is a substantial sector of logic, which remains strangely undeveloped; for example Alfred Tarski mentioned 'asserted statements' right at the start of his famous textbook on logic [Tarski, A. (1941), 3], but used them only in part of one chapter.

On modes of negation, a formal logic normally restricts itself to the 'external' mode, where an entire proposition is negated; but we include also the 'internal' negation of a sub-proposition or subsidiary infinitive verb within a proposition. For example, the proposition 'John feels that Jill speaks badly', with the principal verb 'feels' and one subsidiary verb 'speaks', takes not only the external negation 'John does not feel that Jill speaks badly' but also the internal negation 'John feels that Jill does not speak badly' and the external-internal negation 'John does not feel that Jill does not speak badly'. The admission of internal as well as external negations into logic considerably increases its utility and domain of reference; in particular, it is essential to expounding assertion.

These modifications to negations and to assertion are maintained here. However, they are not essential for our characterisation; so readers who prefer to continue to work only with external negation and to leave assertion unarticulated should ignore the appropriate parts of section 2.

1.2. *Assumptions and limitations*

1) We maintain from GGan the assumption that the primary notion of logical knowledge is the proposition and its referentiability rather than the term and its meaning. As Louis Couturat put it (Couturat, L. 1913, 134)

> The Old Logic began with the theory of terms, because it was restricted to the study of the relations between concepts (i.e. judgements of attribution). Modern Logic prefers to take the proposition as its ultimate element.

[1] Grattan-Guinness (2012) is a much more elaborate philosophical and historical version of this article and its predecessor.

Otherwise we hope to be neutral about various philosophical positions and issues. Of especial pertinence is the question of whether logical knowledge is primarily concerned with (in)valid deduction and truth transmission with the processing of information (Sagüillo, J.M. 2009).[2] (The word 'or' denotes exclusive disjunction, 'either-or').

2) Solely for reasons of length, the discussion is usually limited to propositions as such; the consequences for sentences as propositions in a language and to statements as utterances of sentences are not explored. The intentions of the utterer include persuading others to share his beliefs and knowledge of what he knows to be true or untrue, improving the cogency of an argument by converting it to a line of reasoning that is already well known, detecting errors in the logic of an argument, using words such as 'true' and 'untrue' metaphorically, exploiting equivocations, ambiguities and jokes, and even resorting to deliberate lying. These are important actions, called 'argumentations', well captured in Corcoran, J. (1989) and Walton, D. (1989, 1996); they complement the discussion proffered here.

3) In several places we shall distinguish between the base 'object logic' of a logic and its hierarchy of metalogics. Since the word 'object' is already heavily used in logic and philosophy, we shall use 'host logic' instead.

1.3. *The need for well-suitedness*

An important but little-recognised concern in logic and mathematics is the theory of **sortal** and **unsortal terms**. Involving the universe(s) of discourse of a logical deduction and the ranges of significance of propositional functions, it holds that the realm of "objects" and properties that we can consider is too vast and varied for universes and ranges to embrace them all; thus associated propositions, while well-formed and not paradoxical, are **sortally ill-suited** and so cannot be asserted at all. For instance, the pair of propositions 'Winston Churchill is (not) the king of Spain' are well-suited, respectively untrue and true; but 'Winston Churchill is (not) right-angled' are both ill-suited, as also are, say, 'every trapezium is (not) puzzled', '$\sqrt{34}$ is (not) happy', 'Elephants (do not) hate Mondays' and 'The city of Paris is (not) symphonic'. Unless excepted we assume that a proposition is **well-suited**; that is, in a simple proposition its predicate relates to its subject.

[2] Other issues include intension versus extension, Platonism versus empiricism versus nominalism versus a priorism versus psychologism versus formalism, and analysis versus synthesis (Otte, M. and Panza, M. 1997). Some of these issues may well bear less upon logic than upon its settings; compare Weir, A. (2010) on several of them relative to mathematics.

The theory of ‘syntactic relativisation’ allows us to alter the range of significance of propositional functions and the universe(s) of discourse of a logical deduction so that ill-suited propositions can become (un)true. For example, if we are developing a geometrical theory where quantification applies only over trapeziums, then the proposition Z := ‘Every trapezium is puzzled’ must be sortally ill-suited. But if the universe(s) of discourse were increased to include objects of any kind, then Z could be rewritten as Z’ := ‘For any entity, if it is a trapezium then it is puzzled’, which is untrue. Symbolically, using ‘T(x)’ for ‘x is a trapezium’ and ‘t’ for ‘trapezium’ with P(x) as ‘x is puzzled’, the difference is between the ill-suited proposition Z := $(\forall t)P(t)$ and the untrue proposition Z’ := $(\forall x)(T(x) \rightarrow P(x))$. The corresponding difference obtains between not-Z and not-Z’, and also between existentially quantified propositions such as ‘Some trapezium is (not) puzzled’.[3]

Neither of these theories receives the attention that they deserve, though they are invoked in talk of ‘(un)restricted quantifiers’ (for example, Rosser, J. B. (1953, 140-152)). A comprehensive classification of terms into sortal and unsortal is a very formidable task, far from complete elucidation.[4] The discussions of assertion and negations in GGan applied only to well-suited propositions, but ill-suitedness will now play a larger role.

2. Inside logic

> Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features.
>
> — Bertrand Russell Russell, B. A. W. (1919, 169)

2.1. *The importance of moments*

The study of indirect proof-methods in GGan exposed a gap in proofs of asserted theorems. Instead of proving, for example, that ‘It is true that $\sqrt{2}$ is an irrational number’, reduction ad absurdum delivers only ‘It is untrue that $\sqrt{2}$ is not an irrational number’. The gap is caused by the difference between denial and external negation, which is a *semantic* property. It is fillable by assuming that the denial of R is logically equivalent to the affirmation of its external negation,

DN := -R if and only if +(not-R);

[3] Sagüillo, J. M. (2000) distinguishes between its roles in informal and formalised theories whereas I focus upon theories that are formalised enough that their pertaining logic is explicitly studied.

[4] A good survey of Anglo-Saxon theories is given in Lowe, E. J. (2009); some European contributions, such as Husserl’s, are assessed in Simons, P. (1987).

the (il)legitimacy of adopting DN is determined by the *content* of the setting and so itself is extra-logical if the setting is not logic itself. Logic is involved when we consider the different *forms* in which DN may be cast.

Forms have long been emphasised in logic; in particular, the classification of (in)valid syllogistic inferences in Aristotle and many authors later, and the distinction between form and matter that was emphasised by various logicians in the 19th century. Forms are part of the logical furniture; in the new-old version proposed here, they constitute the dining table. We discuss the word 'formal' in sub-section 2.5.

To make more precise the status of forms, we invoke the powerful philosophical distinction between independent and dependent parts (or **moments**) of a totality. An example is between the tail of a dog and her weight; the tail can be considered on its own, but its weight is *necessarily of something else*, in this case of the dog. Both parts and moments may have their own parts and moments (the fur on the tail, the weight of that fur, the owner's surprise over that weight), and so on. While parts (of parts of ...) of a totality can be expressed in set theory using membership and inclusion, moments cannot, for they do not belong to the totality in the same way. However, one can speak of a set of moments for a given totality.

The distinction was applied to logic and mathematics especially by the philosopher Edmund Husserl from the 1890s; it is explored in detail in Smith, B. (1982). We take 'moment' as a synonym for 'form', and also for 'schema(ta)' and the phrase 'logical skeleton', which have been used in logic texts.

2.2. *Propositions*

We propose that one sector of the host logic of L2 is a theory of **propositional moments**. The manner of their determination is well known (for example, Russell, B. A. W. (1919, 194-201)). Take some (well-suited) proposition R, not necessarily asserted, in a setting of L2 (which could be L2 itself), and expose its propositional moment by stripping out its particular propositions leaving behind its connectives in their various modes, and propositional quantifiers. This moment is not itself a proposition.

When possible we strip propositions further to reveal the next sector of the host logic, **functional moments**. They comprise particular sub-propositions, propositional functions (including relations) and quantifiers of first or higher orders, with their modes noted and scopes checked. Not itself a proposition, it does not contain the variables upon which the quantifiers operate, or particular values of those variables; they come from the setting. We also place in the metalogic stripped-out versions of **specifications**, propositions that state the ranges of significance of propositional functions and the universe(s) of discourse of a logical deduction that obtain in the setting, and rules for determining the scopes of connectives and quantifiers.

Finally, we strip out free and quantified variables, constants and parameters. But we do not assume that these notions are *exclusively* logical, for they occur in many other settings; for example from mathematics, the differential and integral calculus, numerical methods and mechanics.

We take as an example the (as it happens, true) unasserted first-order proposition in arithmetic ‘Some integers are not odd’, and strip it to reveal the functional moment ‘Some Xs are not Y’, where ‘X’ and ‘Y’ are schematic letters (not quantifiable variables); a popular alternative writing style in logic texts is ‘Some — are not —’. We can set that moment in, say, biology by putting X = men and Y = mortal to produce the (presumably untrue!) proposition ‘Some men are not mortal’; or in chemistry and choose X = acids and Y = poisonous to produce the true proposition ‘Some acids are not poisonous’; or go back to arithmetic and reconstruct ‘Some integers are odd’. In each setting we may alter the specifications of ranges and universes, maybe setting them very wide (at “all” entities, say).

The numbers of propositional and functional moments are very large; for example, any theorem in the propositional or functional calculus in L2 may be so reduced. It is not practicable to catalogue them all; better is to take a proposition and follow the procedure just described.[5]

2.3. *Asserted propositions and their laws*

While the truth-value of an asserted proposition depends upon the setting involved, whether L2 itself or not, assertion and the pair of truth-values themselves form the third sector of L2. The propositions +R and −R are sited in the metalogic ML2 of L2, which is also the place for assessing the completeness and consistency of L2 and handling its paradoxes. ML2 can have its own metalogic (for example, to handle its own paradoxes); the usual hierarchy of meta-levels is available if desired, with as there is no “final” all-embracing logic, just as there is no largest cardinal or ordinal number.

Any asserted proposition R in any level is stripped down to reveal its **assertional moment**:

‘It is V that X’,

[5] The work could involve coping with adverbs such as ‘hence’, ‘since’, ‘only’, ‘but’ and ‘because’ – and especially the ambiguous ‘if not’ as in ‘Wyn is pedantic if not bureaucratic’, with its contrary senses of ‘although not’ and ‘and even’.

where V takes the values ‘true’ and ‘untrue’ for L2 and ‘X’ is the propositional moment of a proposition sited in the level below. Once again it is not an assertion and not even a proposition.

The affirmation ‘+’ and the denial ‘-’ of R are two iterable operations that constitute an algebra, which also is satisfied by L2. Its basic properties are:

1) ++R = +R, which we follow Benjamin Peirce (Peirce, B. 1870) in calling ‘idempotence’;[6] it is also satisfied by conjunction and inclusive disjunction;
2) --R = +R, which we call ‘dipotence’; it has analogues in internal and in external negation;
3) +-R = -+R = -R, which we call ‘supervention’; and
4) (+-)+R = +(-+)R, an example of associativity and applicable to any trio of adjacent truth-values.

Each equality sign refers to the logical equivalence (‘if and only if’) of the corresponding propositions. These properties are a great relief, because we can iterate both kinds of assertion indefinitely many times, to produce well-formed horrors such as

--+-++-...--++-R;

but they can be reduced to linearity by successively taking each pair <u>or</u> trio of assertions from the right and applying to it the appropriate law. Two inverse algebras of de-assertion can also be developed: they differ from the above algebras in the interesting respect that de-assertion is not iterable.

A feature of assertion about which logicians are often curiously uncurious is the status of the asserted proposition itself. It must be taken to be a guess about some state of affairs in about the setting, and so subject to testing there. This process involves theories of truth, of which there are several, none all-encompassing. Truth theories and testing use a logic, but they are not logic; so unfortunately logicians do not usually discuss them.[7]

2.4. *Deduction*

Deduction is a cluttered sector covering implication, inference, logical consequence and entailment, often not clearly distinguished (Corcoran, J. (1973). As in GGan, we locate implication and the conditional in L2 and confine remarks here to ML2.

[6] Idempotence gradually became known in algebras, especially through American interest around the 1900s in model theory (Grattan-Guinness, I. 1997); now it has a respectable remit in both mathematics and logics (Gunawardena, J. 1998).

[7] See, for example, Quine (1951, 3-5 and ch. 1) and Church, A. (1956, esp. pp. 23-27). Good sources on theories of truth include Haack, S. (1978) and Field, H. (2008).

Deductions (synonymously, 'derivations' or 'demonstrations') of all kinds, whether valid or invalid, are stripped down to reveal their **deductional moments**. Rules of inference constitute an important kind of deduction; for example, the *modus tollendo tollens* has as moment

+X; +(if not-Y, then not-X); hence +Y,

where X and Y are propositional moments. As usual, it is not a rule and not even a proposition.

A further species of deduction is by **rules of substitution**. There are several kinds: of words in propositions (including symbolism in formulae), and of propositions within larger ones, including guarantees that the proposition created by the substitution is well-suited. Substitution also occurs when changing specifications of ranges and universes (sub-section 2.2). The result of substituting in proposition R to produce proposition S is a 'non-formal principle of inference' of S from R, to quote Russell's nice phrase (Russell, B. A. W. 1919, 115) when he apologised for their omission from his earlier logical writings. (Sheffer's predicament, stated at the head of this article, was a reaction to *Principia Mathematica*.) Indeed, logicians often do not clarify the logical character of these rules: mathematicians rarely mention them at all, though an example occurs in the application of 'the algebra of classes' to logic by Saunders MacLane (Birkhoff & MacLane 1953, 343), a mathematician notable for his strong interest in logic (section 4).

2.5. *Momental logic*

We have individuated four kinds of moments: propositional, functional, deductional and assertional. Both 'logical constants' such as propositions, connectives and quantifiers, and the metalogical constants such as truth-values and assertion, belong to its domain of reference. So do rather overlooked features of L2, such as specifications and rules of substitution. Taken all together they constitute **momental logic**, and comprise the core of L2, existing independently of all settings.[8] This explains why logical knowledge is so *elusive*; being momental, it is always dependent upon, and thereby subordinate to, something else somewhere else.

None of these four sectors reduces to any of its companions, and no underlying kind of knowledge is the basis for all of them. On the contrary, deduction depends upon propositional, functional and assertional moments, of which the last itself depends upon the previous two.

[8] A temptation is to call it 'pure' logic, but this name has been used in some theories of logic with connotations that are not upheld here.

Those who are suspicious of creating moments from totalities could replace them with rules of substitution that go from, say, ‘acids’ to ‘integers’ in the example of sub-section 2.2 without invoking any moment; but the loss of moments is a pity. Further, this use of substitution needs careful study; a valuable source of emulation is combinatory logic (Cardone, F. and Hindley, J. R. 2009), which incidentally was motivated in Haskell Curry in the 1920s because he noticed their absence from *Principia mathematica*.

Momental logic can be regarded as ‘formal’. Dutilh Novaes, C. (2011) acutely exhibits around a dozen different senses in which the phrase ‘formal logic’ is used; the senses that fit this construal best are ‘topic-free’ and ‘topic-neutral’, while those that stress abstraction or the dominance of rules have some bearing.

2.6. *Self-settings and paradoxes*

The self-referential process of applying a host logic to itself becomes here the siting of propositional and functional moments in the metalogic ML2 and its assertional and deductional moments in its metametalogic. For example, continuing with the example ‘Some Xs are not Y’ of sub-section 2.2 above, put X = proposition and Y = untrue, and find in ML2 the true proposition ‘Some propositions are untrue’. Situations involving mutual cross-reference between propositions also belong here.

Certain moments will produce the propositional and other paradoxes. In order to formulate them we follow GGan in drawing upon a distinction that is well-known in the philosophy of language (see, for example, [Donnellan 1966]) but seemingly not in logic: between attributive propositions such as ‘the father of Cynthia does not feel that Jill speaks badly’ and referential propositions such as ‘John does not feel that Jill speaks badly’. The liar paradox arises not only in the well-studied unasserted and self-attributive proposition ‘this proposition is untrue’ but also in the little-studied asserted and referential propositions ‘it is (un)true that this proposition is untrue’.

These versions of the liar paradox, and some others, were formulated in GGan. We do not discuss any solutions or accommodations of the paradoxes either there or here; but their presence in L2 and ML2 is one of the reasons given in sub-section 2.1 for making DN explicit.

2.7. *The relation of identity*

The relation of identity is often assigned to logic, and much philosophy is po(u)red over it. Highly influential is the *salva veritate* criterion of identity based upon indiscernibility (hereafter ‘II’), where the substitution of a component in a proposition by an identical component leaves unchanged its truth-value. But its broad popularity is questionable (Wessel, H. (1994); for surely it is an absurd convention to regard, say, the tri-equation $4 = 2 + 2 = 2 \times 2$ as containing any identities when, for instance, 2×2 possesses the property of multiplication while 4 and 2 + 2 do not (compare Tarski, A. (1941, ch. 3)). A better tactic is to invoke *closure*, where a collection of distinguishable objects a, b, ... is closed with respect to a means of combination '·' when for all a and b a·b (including a·a) is of the same kind as a and b (for example, negative integers under addition).

Identity of (compound) symbols is recognised by mathematicians in algebraic identities, such as axioms or theorems satisfied by all members of an algebra; model theorists correctly insist on including the quantifiers when stating them. Examples of good practice include the law of commutativity ('for all a and b, if a = b or a ≠ b, then a·b = b·a'), and the law of reflexivity ('for all a, a ≥ a') in some logics and mathematical theories, or theorems such as completing the square.

A legitimate use of the identity relation is that of co-identification of referents such as 'the morning star' and 'the evening star', or of 'the trio of points of intersection of pairs of perpendicular bisectors of any planar triangle' thanks to the theorem that they coincide. Identity can also hold between moments, such as of the particular direction shared by a collection of parallel lines. The category of self-referring self-reference exemplifies non-identity (sub-section 5.4).

A related issue is the status of universals in a logic, especially their dependence upon, or independence of, the domains of references (if any) of its predicates and relations *f* to the physical world. Landini, G. and Foster, T. R. (1991) provide a difficult but valuable discussion of three different stances over 'realism' in connection with II. In the 'logical' stance independence is upheld, and only the implication 'for all *f* if *fx* then *fy*' is needed to establish II. The 'attribute' stance also affirms independence but admits only universals with physical reference and so requires equivalence in its rendering of II in terms of 'for all *f*, *fx* iff *f*y').

The 'natural' kind requires that universals depend upon reference, and therefore also uses 'iff' in its formulation of II. Only the first kind belongs to logical knowledge as understood here; the other two draw also upon the settings.

3. Alongside logic

3.1. *Neighbours*

'Alongside' refers to foundational topics of various kinds that both apply to mathematics and to logics and are settings of momental logic; like logics, some of them are also self-referential. Several of them also apply to each other; none reduces to another one except perhaps for special cases. This plurality of topics is maintained, and no efforts are made to reduce some of them to the others; the network of relationships between them all is *extremely* complicated, and has never been elaborated in detail.[9]

In this section we note briefly theories of collections, metamathematics, axiomatisation, model theory and definitions, ending with a lament over the excessive use of 'existence'. Some branches of mathematics are also nearby, especially finite and transfinite arithmetic, recursion theory, common algebra and several abstract algebras. Other topics include the part of semiotics (theories of signs and sign families) that deals with notations; and diagrams and graphical representation of knowledge, which has a rich history in logic but rather a low status in its philosophy (Allwein, G. and Barwise, J. (1996). Linguistics involves logic to a notable extent (McCawley, J. D. 1981); so does semantics, including the paradoxes of naming. Computer science also relates to these neighbours as well as logics and several branches of mathematics, for it examines the theoretical foundations of information and computation and of techniques for their setting in computer systems.

3.2. *Theories of collections*

Theories of collections possess uniquely close links to logic, especially via an (implicit or explicit) comprehension principle that associates a propositional function with the collection of values of the argument variable(s) satisfying it (for example, 'x is a man' vis-à-vis the collection of men). More or less in chronological order from the 1840s to the 1910s, there developed

1) part-whole theory, where membership is not distinguished from inclusion: the traditional theory, elaborated especially by Bernard Bolzano, Hermann and Robert Grassmann, and the algebraic logicians (1830s+);
2) Cantorian (-Dedekindian) set theory, including point set topology, transfinite arithmetic and order-types (1870s+): to become central in the 'logistic' programme led by Giuseppe Peano of axiomatising both mathematical theories and the 'mathematical logic' involved in them (1890s+), and formally encased in the 'logicistic' programmes of Gottlob Frege (1870s+) and especially of A. N. Whitehead and Russell (1900s+) that reduced (some) mathematics to mathematical logic;
3) the germs of multiset theory with A. B. Kempe (in the context of part-whole theory): the most general theory of all in allowing for multiple

[9] A nervous attempt to cope with some of them was made in Grattan-Guinness, I. (2011b). The methods of developing mathematical theories given in Pólya, G. (1954) and Grattan-Guinness, I. (2008) – analogising, importation, reduction, convolution, and so on – are also relevant to logics.

membership (1886) but overlooked by almost everyone and still little known even after its post-War rebirth;
4) axiomatic Cantorian set theory (1900s+), initially expounded by Ernst Zermelo; and
5) several variants of the above in connection with other programmes, such as Husserl's theory of manifolds (of which the parts/moments distinction of sub-section 2.2 is an offshoot, 1880s+) and intuitionist mathematics and logic (1900s+).[10]

These various theories, and now also fuzzy set theory (1960s+), also played and still play important roles in the study of paradoxes, axiomatisation and model theory. It seems better to abandon the aspiration of having collections of things as "stuff" inside from logical knowledge and assign them to mathematics, or maybe treat them as a body of knowledge of their own.[11] The title of Quine's survey (Quine 1969), 'set theory *and* its logic', has it right: sets are not propositional functions; even a finite collection of words or of propositions is not a linguistic object in the way that a property can be construed.

3.3. *Metamathematics*

In the programme of 'metamathematics' led by David Hilbert, especially the second phase from 1917 onward, the aim was to axiomatise a logic (for him, L2) and a mathematical theory and study its (lack of) consistency, completeness and independence of the axioms.[12] Hilbert used the word 'formalism' *only* to refer to the axiomatised version of the theory; L. E. J. Brouwer named metamathematics 'formalism' as a criticism, and unfortunately it caught on from around 1930.

Further, Gödel's incompletability theorems (Godel, K. 1931) showed that Hilbert's conception of metamathematiocs could not be upheld. However, it has continued in revised versions, and is applied to other logics, finite and transfinite arithmetic, recursion theory, set theory, geometries, abstract algebras, mechanics and probability theory. Perhaps less success has been found elsewhere, such as axiomatising branches of physics (for example, thermodynamics), economics and biology. Hilbert also used the term 'proof theory'

[10] Quite a lot of these various histories and their historical developments can be garnered from van Heijenoort, J. (1967b), Moore, G. H. (1982), Hallett, M. (1984), Simons, P. (1987), Ferreirós, J. (1999), Grattan-Guinness, I. (2000) and Hartimo, M. (2010), and their own further references.

[11] For example, a recent proposed definition of numbers as moments of multisets Grattan-Guinness, I. (2011a) uses L2 but is neither a part nor a moment of it.

[12] Zermelo's axiomatisation of set theory, recently mentioned, is a well-known early example of a follower of the first phase in the late 1900s; an even better one is Georg Hamel (1909) on mechanics.

in these contexts; we take it to encompass, for example, the study in GGan of the relationship between direct and indirect proof-methods.[13]

3.4. *Model theory*

Model theory is another subject that is practiced both in logic and in mathematics. It examines links between the syntax and the semantics of a formal language F that contains constants and variables, connectives and quantifiers, operation symbols for building up terms from them, and relation symbols to express relations between terms (for instance, equality). An interpretation of F is a syntactically determined 'structure'[14] S that includes objects corresponding to these notions (for example in arithmetic, the successor relation and positive integers) in which set-theoretic relations such as union and improper inclusion play roles. A well-formed formula in F lacking free variables (but maybe containing quantified ones) is a 'sentence'; if it is true in a structure S, then conversely S is a 'model' of F. Examples include models of axiomatic set theory (to be distinguished from the use of sets in the basic formulation of model theory); and the Peano axioms for the positive integers, which however are satisfied by other systems such as the odd positive integers and model only progressions.

The connections between model theory and metamathematics and some logics (for example, modal and combinatory) have been close. A recent example is F. W. Lawvere's categorical logic, which was partly inspired by analogies between category theory and aspects of (modern) algebraic logic (Bell, J. L. 2005). However, links have not always been followed. For example, a natural manner of formulating the property that proposition R is a logical consequence of the collection of propositions C is that there is no model in which each member of C is true and R is untrue; but even Tarski, a leading model theorist, did not use it in his textbook on logic (Tarski, A. 1941, esp. pp. 29-32, 119-120).

3.5. *Definitions*

Definitions, much understudied in theories of knowledge in general, are especially significant in mathematics and logics (Dubislav, W. 1931). The

[13] The historical literature on metamathematics is considerable. Mancosu, P. (1998) is a useful source book on both Hilbert and his dispute over logic with Brouwer. Still important on the history of axiomatics is Cavaillès, J. (1938).

[14] This use of the word 'structure' is singularly unfortunate in referring to an "object" rather than a moment; the older word 'system' was much preferable ((Corcoran, J. 1980, 188-190), a paper on the important property of categoricity). On links between model theory and set theory see, for example, Hodges, W. (2009); on its development within algebras and logic see Sinaceur, H. (1991, pts. 2-4).

most common kind is nominal definition, where a term within a theory is defined by a well-formed defining expression. When the term plays a major role in the theory, it has the status of a stipulation. Typical of an important case in a branch of mathematics is that of the continuity of a mathematical function; it also exemplifies competition between different definitions of a term, as several versions are available.

Another significant kind is contextual definition. In logic a connective will be defined this way in terms of others (and all of them from ‘nor’ or from ‘not-and’). Each definition is framed within a propositional moment, such as of material implication:

(If X, then Y) := (not-X or Y),

where X and Y themselves are propositional moments. It also played a major role in the Whitehead-Russell programme because their comprehension principle and theory of definite descriptions were formulated contextually.

A further kind arises when an axiom system, maybe along with some or all of its models, is regarded as furnishing implicit definitions of its concepts and notions (Gabriel, G. 1978). Other kinds include analytic and synthetic definitions, especially in (neo-)Kantian philosophy; creative definitions relative to an axiom system; distinctions of sense, when one type of a term is distinguished from another type (such as continuity from semi-continuity of a mathematical function in the context of sub-section 2.1); definitions under hypothesis; definition by mathematical induction, and more generally by recursion; and so-called essentialist definitions, when it is (controversially) claimed that the essence of some term is captured. Each kind of definition can be formulated in terms of moments, whether or not definitions are regarded as part of logic.

3.6. *Existence*

Existence lurks both inside and outside logic and mathematics, sometimes tied to kinds of definition (Krasner, M. 1957-1958). The word has been used rarely here, for in a very untidy situation one sense of it often contradicts another one; but in many contexts it is a significant moment. For example, $+\sqrt{13}$ exists as a real number in arithmetic, as an irrational number defined from rational numbers by (say) cuts, as a referent of the indefinite description ‘fourth root of 169’, and not as the referent of the definite description ‘the negative square root of 13’. Sets can also exist: for example, thanks to some ontological claim about collections, <u>or</u> as definable in finite terms by a propositional function (so that a set created by an axiom of choice does not exist in this sense), <u>or</u> by being not empty (so that Russell’s paradoxical set does exist in this sense). Constructions can exist, as in many existence theorems; <u>or</u> not, including impossibility theorems such as the insolubility of the quintic equation by radicals. Optimisation is involved in instances such as the existence in a geometry of the shortest distance between two points, or of the derivation of a proposition in logic in some minimal number of or symbols.

3.7. *Effects of our proposals*

The proposals concerning negations and assertion made in the previous article, and the momental logic formulated in this one, have some consequences for these neighbouring disciplines, although none of a revolutionary or comprehensive character. For example, negations encroach upon syntax, assertion affects model theory and semantics and could use several operator algebras, and logical moments involve new kinds of definitions. The impact of these proposals would increase when extended from propositions to sentences and statements. The encouragement to pay more attention to assertion and to the specification of ranges of significance and universes of discourse, applies also to some of these neighbours. But the main potential impact of these proposals lies in the use of the distinction between parts and moments of a totality, which deserves to be far better known in mathematics, logical knowledge and philosophy.

4. On structuralism: logic(s) ≠ mathematics

The emphasis on moments of various kinds (propositional, functional, assertional, deductional) as dependent parts, and the detachment from L2 of theories of collections, suggest that philosophically this characterisation of logical knowledge is *structuralist*, with the word 'structure' taken as a synonym for both 'form' and 'moment'. This kind of position has been advocated for mathematics (for example, in Mac Lane, S. (1986)), and indeed structures are very significant in mathematics (Vercelloni, L. 1989), especially in algebras: commutativity, associativity, distributivity, idempotence, duality, symmetry, and many others not necessarily rooted in algebra, such as linear combination and sum of squares. However, the *reduction* of mathematical theories to structures is surely excessive precisely *because* structures are moments *of* the mathematical totalities that may possess them, such as sets or groups or dodecahedra (compare Krömer, R. (2007, esp. ch. 7) on this point in the context of category theory). By contrast, the central role given to moments lets structuralism fit logic rather nicely in the algebra of assertion described in sub-section 2.3, even technically so.

The same view holds for non-bivalent logics. They have more or other truth-values from those in L2; for example, modal logics use 'necessary' and 'possible' as well as 'true' and 'untrue', and some constructivist logics deny a truth-value to propositions that use double negations.[15] But each logic can be taken to be a structuralist theory, with its own momental structure, use of internal as well as external negations, and an articulable theory of assertion. Forceful analogies hold between the bivalent 'it is (un)true that' and, for example, the epistemic 'It is (not) known that', the convinced 'It is (not) believed that', and the modal 'It is (not) necessary *or* possible that'. They are also much involved with self-reference, which is our last topic.

5. The status of self-reference

A feature of our discussion of both L2 and the neighbouring topics is self-reference. Extremely ubiquitous, it manifests itself not only in logics but also words, signs (including families of them), properties, propositions, whole theories including some philosophical positions.[16] A wide variety of settings lies in the life and social sciences, many of them named in the form 'self-'; and in literature a wide range of delightful whimsies have been created.[17] We distinguish between two categories of setting.

5.1. *Linguistic self-reference*

Linguistic self-reference is tied to languages and logics, and involves the formation of clauses and propositions. An attractive example from methodology is that in theory we develop a theory in order to put it into practice, but in practice we do not! Again, we use quotation marks when naming signs, but then what signs can we use to name quotation marks? Surely not the self-referential "' and '". The last sentence of this paragraph is also an example.

Certain philosophical positions cannot be self-referring: for example, the requirement of logical positivism that meaningful asserted propositions must be empirically verifiable *or* tautological does not itself exhibit these properties. Especially interesting is Karl Popper's requirement that a scientific

[15] Some logics invoke real numbers for their truth-values; for example, probability logic and fuzzy logic take all real numbers in the interval [0,1] (where the latter works better with sub-intervals. But these resorts are question-begging, as they assume that the theory of real numbers is itself consistent.

[16] Champlin, T. (1988) is a fine survey of settings, which are well captured also in Bolander, T. (2009). For Thomas-Fogiel, I. (2011, ch. 6) self-reference plays a role in avoiding the 'death of philosophy' that some philosophers of various kinds have predicted over the ages.

[17] Notable exploiters include Lawrence Sterne's *Tristram Shandy* (Sterne, L. 1781, Book 9, esp. the empty chs. 18-19), Lewis Carroll, much more in his *Alice* tales and poetry than in his logic books (Jourdain, P. E. B. 1918), Joseph Heller's *Catch 22* (Heller, J. 1961, esp. ch. 5), and various scenarios due to Stanislaw Lem (Swirski, P. 1997) and Douglas Hofstadter (1979).

proposition must be falsifiable, for it is not itself falsifiable; then he widened it to criticisability, which is criticisable and led Bartley, W. W. (1964) to identify 'comprehensively critical rationalism', an important self-referring alternative to both justificationism and relativism. His main target was theology, where several examples involve maximality (Does Jesus worship Jesus? Can God destroy God? ...).[18]

When applied to logic, self-reference is recognised as a cause of paradoxes, and guides the development of several solutions of them (as was noted in section 5 of GGan). But it is also the *source* of paradox in the unasserted proposition 'This proposition is not self-referential'. In the late 1900s Kurt Grelling imitated this kind of paradox when he defined 'heterological' words in a language as those that do not exhibit any of their properties *as* words, unlike 'autological' words such as 'word' or 'four-syllabled', which do; therefore 'heterological' is heterological if and only if it is not (Peckhaus 1995). This kind of paradox is also formulable for phrases about/in a language like 'not a noun', 'contains three words' and 'does not contain three words', and mnemonic acronyms such as 'MAD' for 'mutually assured destruction'. Propositions such as the liar paradox can be stated in heterological terms.

Autological words and phrases are examples of linguistically self-referential propositions that are not paradoxical. Consider these self-referring instructions written in imperative mood on signboards:

DO NOT WALK BEYOND THIS SIGN	DO NOT DEFACE THIS SIGN	DO NOT READ THIS SIGN

The first may be an important safety warning while the other two are useless, but only the third is paradoxical.

5.2. *Self-reference elsewhere*

Our second category, may not even be stated in linguistic form at all, or at least not involve linguistic self-reference. An example from semiotics is the flag of the United Kingdom, which is a superposition of the flags of England, Ireland and Scotland (poor Wales was left out).

A rich source of cases is technology, where self-reference controls safety devices mounted on equipment and activated; the word 'feedback' is often used in these contexts. For example, a servomechanism such as a governor mounted on the drive shaft of a steam engine or waterwheel shuts down the machine when the angular velocity of the shaft exceeds the value that has

[18] Note also the excellent line in Leonard Bernstein's *Mass* (1971): 'I'll believe in thirty religions if they'll believe in me'.

been assigned to be dangerous; but then the governor itself shuts down. Similar devices are used in electrical engineering, such as a fuse set inside a plug, or a sensor set into the floor of a lift that prevents the lift doors from closing when an excessive load is detected; and in heating technology with thermostats built around a multi-metallic strip. The operation of computers shows many examples: my computer has an icon on the screen called 'My computer'; I cannot copy a programme onto itself; and so on harmlessly if on occasion frustratingly. Some testing procedures are self-referring, such as ringing the bell on my bicycle to check that it is working properly rather than to warn other traffic of my presence.

5.3. *Cross-reference: extensions to pluralities and interactions*

Many of the examples of self-reference can be extended to situations involving more than one entity. In logic, for example, a well-known version of the liar paradox is the pair of propositions

A := 'Proposition B is true' and B := 'Proposition A is untrue',

with further versions extended to $n > 2$ propositions. Paradoxes of sets can be extended to paradoxes of multi-sets where multiple membership is involved (subsection 3.2). Semiotics has many properties concerning families of signs, such as the pair '$\wedge$' and '$\vee$' in logics reflecting the duality of conjunction and (either mode of) disjunction. The physical sciences have many theories centered upon interactions: a major instance is the perturbations of the orbits of the planets around the Sun in our solar system that are held to be caused by influences upon each other. The life and social sciences are full of cases: MAD was already mentioned in sub-section 5.1. A particularly interesting case is the so-called but mis-named 'suicide pacts', which are actually mutually agreed murders (Champlin, T. 1988, ch. 10).

5.4. *Specification*

It is tempting to formulate self-reference as the property of a proposition of including itself in its domain of reference, and to apply to instances of it the characterisation of logic outlined in this article: momental logic in its four sectors distinguished from the setting, locating paradoxes in the appropriate metatheory, and so on. But even the small collection of examples used here shows that self- and cross-reference are far more ubiquitous than propositions in languages and logics; they are *not* confined to formation in language but are also manifest, for example, semiotic recognition. So we take a setting to be a totality as described in sub-section 2.1, whether linguistic or not, and then specify each instance of self-reference as a moment of that totality. Then self-referentially self-referring propositions are not identical with self-referring ones (that is, self-reference is not idempotent), because they are moments of moments. They launch a metahierarchy of hierarchies; and so on ...

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