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INVARIANCE PRINCIPLES IN POLYADIC INDUCTIVE LOGIC

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ABSTRACT

We show that the Permutation Invariance Principle can be equivalently stated to involve invariance under *finitely many* permutations, specified by their action on a particular finite set of formulae. We argue that these formulae define the polyadic equivalents of unary atoms. Using this we investigate the properties of probability functions satisfying this principle, in particular, we examine the idea that the Permutation Invariance Principle provides a natural generalisation of (unary) Atom Exchangeability. We also clarify the status of the Principle of Super Regularity in relation to invariance principles.

KEYWORDS

Invariance Principle, Symmetry, Inductive Logic, Probability Logic, Rationality, Super Regularity.

1. Introduction

Principles based on symmetry have provided a powerful source of rational principles in Pure Inductive Logic since the early explorations of the subject [4, 1, 2]. This has reasons both in the significant role symmetry plays in everyday life decisions and in its mathematical appeal, and so considerable effort has gone into formalising what exactly we mean by a 'symmetry'. This paper shall identify a symmetry with an automorphism (first introduced in [11] and explicated below), focusing on the class of automorphisms that permute state formulae. Identifying degrees of belief with (subjective) probability, we argue that a rational choice of probability function should respect such symmetries.

We shall investigate properties of permutations of state formulae that determine automorphisms and suggest a method to generate probability functions that are invariant under these symmetries. We shall then propose what we think is a straightforward polyadic generalisation for the unary

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notion of atoms and use this to argue that the Permutation Invariance Principle is perhaps a more natural generalisation of the extensively studied Atom Exchangeability than the previously proposed Spectrum Exchangeability [6].

Throughout this exposition we shall be concerned with first order languages L with countably many constants $a_1, a_2, a_3, \ldots$, a finite number of relation symbols $R_1, R_2, \ldots, R_q$ of (finite) arities $r_1, r_2, \ldots, r_q$ respectively and no function symbols nor equality. $r = \max\{r_1, r_2, \ldots, r_q\}$ will denote the highest arity of any relation symbol in our language. We shall also intend that the constants a_i exhaust the universe, in the sense that every individual can be represented by a constant from the a_i. We will use variables $x_1, x_2, x_3, \ldots$ to construct formulae in L and in attempt to simplify notation, shall identify formulae which are logically equivalent; hence in particular we shall often use $=$ rather than $\equiv$ between logically equivalent formulae. We denote the set of sentences of L by SL and the set of quantifier-free sentences by $QFSL$.

The *state descriptions* of L are sentences of the form

$$\Theta(b_1, \ldots, b_n) = \bigwedge_{j=1}^{q} \bigwedge_{i_1, \ldots, i_{r_j} \in \{1, \ldots, n\}} \pm R_j(b_{i_1}, \ldots, b_{i_{r_j}}),$$

where $\pm R_j$ stands for R_j or $\neg R_j$ and $b_1, \ldots, b_n$ are distinct constants from the a_i. Similarly, for $\Theta(b_1, \ldots, b_n)$ a state description and $z_1, \ldots, z_n$ a (distinct) choice of variables from the x_i, $\Theta(z_1, \ldots, z_n)$ is a *state formula* of L.

We will henceforth use upper case Θ, Φ, Ψ to denote state descriptions (or state formulae), so whenever an expression such as $\Phi(a_1, \ldots, a_n)$ appears in this account it should be taken to mean a state description on $a_1, \ldots, a_n$.

We define a function w to be a *probability function* on SL if $w : SL \to [0,1] \subset \mathbb{R}$ satisfies the following three conditions:

For $\theta, \phi, \exists x_j\, \psi(x_j) \in SL$,

(P1) If $\vDash \theta$ then $w(\theta) = 1$.

(P2) If $\vDash \neg(\theta \wedge \phi)$ then $w(\theta \vee \phi) = w(\theta) + w(\phi)$.

(P3) $w(\exists x_j\, \psi(x_j)) = \lim_{n\to\infty} w\left(\bigvee_{i=1}^{n} \psi(a_i)\right)$.

This definition ensures that w has properties traditionally expected of rational numerical beliefs, so that for instance, logically equivalent sentences are given the same values by w. State descriptions are especially useful when working in this context since any probability function is completely determined by its values on state descriptions, see for example Theorem 11.2 of [7] based on work by Gaifman in [3].

Moreover by a result in [8], any function w defined merely on state descriptions $\Theta(a_1, a_2, \ldots, a_n)$, $n \in \mathbb{N}$ of L which satisfies[1]

(P1′) $w(\Theta(a_1, a_2, \ldots, a_n)) \geq 0$,

(P2′) $w(\top) = 1$,

(P3′) $w(\Theta(a_1, a_2, \ldots, a_n)) = \sum_{\Phi(a_1, a_2, \ldots, a_{n+1}) \models \Theta(a_1, a_2, \ldots, a_n)} w(\Phi(a_1, a_2, \ldots, a_{n+1}))$

extends uniquely to a probability function on *QFSL* and hence on *SL*.

Following the framework set out in [11], let $\mathcal{T}L$ denote the set of structures for the language L with universe $\{a_i \,|\, i \in \mathbb{N}^+\}$, where each constant symbol a_i is interpreted as a_i. Let *BL* be the two-sorted structure with universe $\mathcal{T}L$, the sets

$$[\theta] = \{\mathcal{M} \in \mathcal{T}L \,|\, \mathcal{M} \models \theta\}$$

for $\theta \in SL$ and the membership relation between elements of $\mathcal{T}L$ and these sets.

An *automorphism* η of *BL* is a bijective mapping from $\mathcal{T}L$ onto $\mathcal{T}L$ such that for each $\theta \in SL$,

$$\eta[\theta] = \{\eta\mathcal{M} \,|\, \mathcal{M} \in \mathcal{T}L,\, \mathcal{M} \models \theta\} = [\phi]$$

for some $\phi \in SL$ and conversely, for each $\phi \in SL$,

$$\eta^{-1}[\phi] = \{\eta^{-1}\mathcal{M} \,|\, \mathcal{M} \in \mathcal{T}L,\, \mathcal{M} \models \phi\} = [\theta]$$

for some $\theta \in SL$.

We will henceforth write[2] $\eta(\theta)$, or just $\eta\theta$, for the sentence $\phi \in SL$ such that $\eta[\theta] = [\phi]$. Note that $\eta\theta$ is determined up to logical equivalence only.

As is customary in investigations of this nature, we assume a rational agent is aware of the structure *BL*, inhabits one of the structures $\mathcal{M} \in \mathcal{T}L$ but is unaware of which particular $\mathcal{M}$ it is. When the agent chooses his/her rational probability function w, it would therefore be reasonable to assume that justification for the probability $w(\theta)$ for $\theta \in SL$ (equivalently $[\theta] \in BL$) should apply also to $w(\eta\theta)$ for any automorphism η of *BL*. This gives us the following symmetry principle for a probability function w on *SL*:

[1] For $n = 0$, $\Theta(a_1, a_2, \ldots, a_n)$ stands for any tautology and is denoted by $\top$.

[2] Thus avoiding overuse of square brackets, which also denote restrictions of formulae, see page 545. The notation is now established so we keep to it; it should be obvious from the context what is meant.

The Invariance Principle, INV

For any automorphism η of BL and any $\theta \in SL$

$$w(\theta) = w(\eta\theta).$$

2. The Permutation Invariance Principle and Super Regularity

Previous investigations into INV for probability functions on unary languages have proved INV to be too strong a principle [10], leaving only one (somewhat unsatisfactory) function that satisfies it. On the other hand, it is not yet clear what its full effect for general languages is. The reason for INV eliminating nearly all probability functions in the unary context is that some automorphisms can force state descriptions with different numbers of constants to have the same probabilities, which – combined with all other conditions INV imposes – is almost never satisfied.

This raises the question of what happens if we require our automorphisms to map state descriptions to state descriptions respecting the number of constants, and what the probability functions that satisfy this weaker version of INV would be, where we only demand that $w(\theta) = w(\eta\theta)$ for $\theta \in SL$ and for such automorphisms η. It turns out [11] that such automorphisms must be in a certain sense uniform and that they must be (up to a permutation of all constants) of the type described below.

We say that a function F *permutes state formulae* if for each n and (distinct) variables $z_1, \dots, z_n$, F permutes the state formulae $\Theta(z_1, \dots, z_n)$ in these variables.

An automorphism η of *BL permutes state formulae* if there is a function $\overline{\eta}$ that permutes state formulae such that for any $b_1, \dots, b_n$ and state formulae $\Theta(z_1, \dots, z_n)$

$$\eta(\Theta(b_1, \dots, b_n)) = \overline{\eta}(\Theta(z_1, \dots, z_n))(b_1, \dots, b_n),$$

where $\overline{\eta}(\Theta(z_1, \dots, z_n))(b_1, \dots, b_n)$ is the state description arrived at by applying $\overline{\eta}$ to $\Theta(z_1, \dots, z_n)$ and then substituting $b_1, \dots, b_n$ into the resulting state formula.

Let F be a function permuting state formulae and satisfying the following conditions from [11]:

(A) For each state formula $\Theta(z_1, \dots, z_m)$ and surjective mapping $\sigma : \{y_1, \dots, y_n\} \twoheadrightarrow \{z_1, \dots, z_m\}$,

$$(F(\Theta))_\sigma = F(\Theta_\sigma),$$

where Θ_σ is the unique state formula $\Psi(y_1, \dots, y_n)$ such that

$$\Psi(\sigma(y_1), \dots, \sigma(y_n)) = \Theta(z_1, \dots, z_m).$$

(B) For each state formula $\Theta(z_1,\ldots,z_m)$ and distinct $i_1,\ldots,i_k \in \{1,\ldots,m\}$

$$(F(\Theta))[z_{i_1},\ldots,z_{i_k}] = F(\Theta[z_{i_1},\ldots,z_{i_k}]),$$

where $\Theta[z_{i_1},\ldots,z_{i_k}]$ is the *restriction* of $\Theta(z_1,\ldots,z_m)$ to these variables, that is, the state formula on $z_{i_1},\ldots,z_{i_k}$ implied by Θ.

Note that where no confusion may arise, we write Θ in place of $\Theta(z_1,\ldots,z_n)$ and $F(\Theta)$ for $F(\Theta(z_1,\ldots,z_n))$ in favour of clarity of notation.

By Theorems 1 and 2 of [11], every function F that permutes state formulae and satisfies the conditions (A) and (B) describes a function $\overline{\eta}$ that extends to an automorphism η of BL permuting state formulae and conversely, every such $\overline{\eta}$ is a function F that permutes state formulae and satisfies (A) and (B).

Restricting the Invariance Principle to include only the automorphisms of BL that permute state formulae gives us

The Permutation Invariance Principle, PIP

For any permutation of state formulae F that satisfies (A) and (B) and a state description $\Theta(b_1,\ldots,b_n)$

$$w(\Theta(b_1,\ldots,b_n)) = w((F(\Theta))(b_1,\ldots,b_n)).$$

Lemma 1. *Let F be a function that permutes state formulae and satisfies (A) and (B). Then F is uniquely determined by its action on state formulae of r variables, where r is the highest arity of an L-relation symbol.*

Proof. Consider a state formula $\Psi(z_1,\ldots,z_s)$ where $s < r$ and let $\Theta(z_1,\ldots,z_r)$ be such that $\Theta \vDash \Psi$. By the condition (B)

$$F(\Psi) = F(\Theta[z_1,\ldots,z_s]) = (F(\Theta))[z_1,\ldots,z_s]$$

so the values of F on state formulae of less than r variables are determined by its values on state formulae of r variables.

Now let $\Psi(z_1,\ldots,z_r,\ldots,z_n)$ be a state formula with $n > r$ and suppose there is a function F_1 that permutes state formulae and satisfies (A) and (B) such that $F(\Theta) = F_1(\Theta)$ for all state formulae Θ on r variables or fewer, but $F(\Psi) \neq F_1(\Psi)$. Then there must be a relation symbol R_j of L and (not necessarily distinct) $z_{i_1},\ldots,z_{i_{r_j}}$ from $\{z_1,\ldots,z_n\}$ such that

$$F(\Psi) \vDash R_j(z_{i_1},\ldots,z_{i_{r_j}}) \text{ and } F_1(\Psi) \vDash \neg R_j(z_{i_1},\ldots,z_{i_{r_j}}) \tag{1}$$

(or vice versa). Let $z_{k_1},\ldots,z_{k_r}$ be distinct variables from $\{z_1,\ldots,z_n\}$ such that all of $z_{i_1},\ldots,z_{i_{r_j}}$ are included amongst them.

By the condition (B) and since F, F_1 agree on state formulae of r variables, we have

$$(F(\Psi))[z_{k_1},\dots,z_{k_r}] = F(\Psi[z_{k_1},\dots,z_{k_r}])$$
$$= F_1(\Psi[z_{k_1},\dots,z_{k_r}]) = (F_1(\Psi))[z_{k_1},\dots,z_{k_r}],$$

contradicting (1). This shows the claim holds also for state formulae with more than r variables, as required. □

As an immediate consequence of the above lemma, the set

$$\mathcal{F} = \{F \mid F \text{ permutes state formulae and satisfies (A) and (B)}\}$$

is finite and we can therefore generate a probability function w' that satisfies PIP from an arbitrary probability function w by averaging over 'permuted versions' of w. Furthermore, as the next proposition shall show, w' will preserve some characteristic properties of w and thus bear witness to their compatibility with PIP.

One of the most widely accepted principles in Pure Inductive Logic is that a rational probability function should treat the individual constants a_i equally. This is formally stated as:

Constant Exchangeability, Ex

For any formula $\theta(x_1,\dots,x_n)$ of L and distinct constants $b_1,\dots,b_n$, $b_1',\dots,b_n'$ from the a_i,

$$w(\theta(b_1,\dots,b_n)) = w(\theta(b_1',\dots,b_n')).$$

Ex is sometimes imposed at the start of investigations in Inductive Logic as the first requirement a rational probability function should obey. It is implied by INV but not by PIP. We do not assume it here but we will explain the role it has in what follows.

For the remainder of this section, we will focus on the rather elusive Principle of Super Regularity and clarify its status with respect to INV and PIP.

Super Regularity, SReg

For any consistent $\theta \in SL$,

$$w(\theta) > 0.$$

Note that the condition is imposed on *all* consistent sentences so this principle is stronger than that of Regularity (see for example [8]), where only quantifier free sentences are involved.

Let w be an arbitrary probability function on SL and define $w' : SL \to [0,1]$ for state descriptions $\Theta(a_1, \dots, a_n)$, $n \in \mathbb{N}$, by

$$w'(\Theta(a_1,\dots,a_n)) = \frac{1}{|\mathcal{F}|} \sum_{F \in \mathcal{F}} w_F(\Theta(a_1,\dots,a_n)) \tag{2}$$

where $w_F(\Theta(a_1,\dots,a_n)) = w\big((F(\Theta))(a_1,\dots,a_n)\big)$.

Lemma 2. *The function w' defined in* (2) *extends uniquely to a probability function on SL. Moreover,* (2) *holds even when the constants $a_1,\dots,a_n$ are replaced by any other distinct constants $b_1,\dots,b_n$.*

Proof. Let $F \in \mathcal{F}$. Clearly w_F satisfies (P1′) and (P2′) on page 543. To check (P3′) note that by the condition (B) on F, for state descriptions $\Phi(a_1,\dots,a_n,a_{n+1})$, $\Theta(a_1,\dots,a_n)$ we have

$$\Phi(a_1,\dots,a_{n+1}) \vDash \Theta(a_1,\dots,a_n) \Leftrightarrow (F(\Phi))(a_1,\dots,a_{n+1}) \vDash (F(\Theta))(a_1,\dots,a_n).$$

Consequently, since (P3′) holds for w and since $(F(\Phi))(a_1,\dots,a_{n+1})$ run through the state descriptions for $a_1,\dots,a_{n+1}$ when $\Phi(a_1,\dots,a_{n+1})$ do so, we have

$$\begin{aligned} w_F(\Theta(a_1,\dots,a_n)) &= w((F(\Theta))(a_1,\dots,a_n)) \\ &= \sum_{(F(\Phi))(a_1,\dots,a_{n+1}) \vDash (F(\Theta))(a_1,\dots,a_n)} w((F(\Phi))(a_1,\dots,a_{n+1})) \\ &= \sum_{\Phi(a_1,\dots,a_{n+1}) \vDash \Theta(a_1,\dots,a_n)} w_F(\Phi(a_1,\dots,a_{n+1})) \end{aligned}$$

so (P3′) holds for w_F and hence w_F extends uniquely to a probability function on SL. w' is therefore a convex combination of probability functions on SL and thus defines a probability function on SL.

The rest of the lemma follows upon noting that any probability function v on SL satisfies

$$v(\Theta(b_1,b_2,\dots,b_n)) = \sum_{\Phi(a_1,a_2,\dots,a_k) \vDash \Theta(b_1,b_2,\dots,b_n)} v(\Phi(a_1,a_2,\dots,a_k)),$$

where k is large enough for the $b_1,\dots,b_n$ to be included amongst $a_1,\dots,a_k$. □

Proposition 3. *The probability function w' defined in* (2) *satisfies PIP. If, in addition, the original probability function w satisfies Ex + SReg then so does w'.*

Proof. To see that w' satisfies PIP, let $\Theta(z_1,\ldots,z_n)$, $\Phi(z_1,\ldots,z_n)$ be state formulae of L with $G(\Theta)=\Phi$ for some $G\in\mathcal{F}$. Consider the set

$$\mathcal{F}'=\{FG^{-1}|\,F\in\mathcal{F}\}.$$

$\mathcal{F}$ is closed under composition and inverse of functions [11], so $FG^{-1}\in\mathcal{F}$ and $\mathcal{F}'\subseteq\mathcal{F}$. Conversely, every $F\in\mathcal{F}$ can be written as the composition of $FG\in\mathcal{F}$ and G^{-1}, so $\mathcal{F}\subseteq\mathcal{F}'$. Therefore,

$$\begin{aligned}
w'(\Theta(b_1,\ldots,b_n)) &= \frac{1}{|\mathcal{F}|}\sum_{F\in\mathcal{F}} w((F(\Theta))(b_1,\ldots,b_n))\\
&= \frac{1}{|\mathcal{F}|}\sum_{F\in\mathcal{F}} w((FG^{-1}(G(\Theta)))(b_1,\ldots,b_n))\\
&= \frac{1}{|\mathcal{F}'|}\sum_{FG^{-1}\in\mathcal{F}'} w((FG^{-1}(\Phi))(b_1,\ldots,b_n))\\
&= \frac{1}{|\mathcal{F}|}\sum_{F\in\mathcal{F}} w((F(\Phi))(b_1,\ldots,b_n))\\
&= w'(\Phi(b_1,\ldots,b_n)).
\end{aligned}$$

Suppose w satisfies Ex. Then for every $F\in\mathcal{F}$ and distinct $b_1,\ldots,b_n$, $b_1',\ldots,b_n'$ from the a_i

$$w((F(\Theta))(b_1,\ldots,b_n))=w((F(\Theta))(b_1',\ldots,b_n'))$$

so w_F satisfies Ex on state descriptions and hence[3] on SL. Consequently, so does w'.

Now suppose w is super regular. Recall that the extension of each w_F to a probability function on SL is unique and that w' is defined as the weighted sum of these extensions. Notice that the permutation that maps each state formula to itself trivially satisfies (A) and (B), so $\mathrm{Id}\in\mathcal{F}$. It follows that w must be the extension to SL of w_{Id} defined on state descriptions of L and w is therefore one of the summands in the calculation of w'. So $w'(\theta)\geq\frac{1}{|\mathcal{F}|}w(\theta)>0$ for every consistent $\theta\in SL$ and so w' is super regular.

□

[3] Note that it suffices to check that a probability function satisfies Ex on state descriptions of L, since then its extension to SL would satisfy Ex (for details see the forthcoming [12]).

The existence of a probability function w' that satisfies PIP, SReg and Ex follows, since a w satisfying Ex and SReg exists (see Chapter 25 of the forthcoming [12] for details). Note also that since PIP implies the principles of Predicate Exchangeability (Px), Strong Negation (SN) and Variable Exchangeability (Vx) (see for example [11]), w' will also satisfy these principles.

The consistency of Super Regularity with PIP is interesting due to the restrictive nature of SReg. Yet this consistency becomes perhaps even more noteworthy in view of the fact that INV contradicts SReg, as we shall now show. The case for languages containing only unary relation symbols follows from the results in [10] that we have already mentioned and we will extend it to polyadic languages. For simplicity, we shall construct the argument for a binary language; however the result generalises similarly to languages of higher arities.

For this purpose, let L_1 denote the language with a single unary predicate symbol P and let L be the language with a single binary relation symbol R. Let $\phi \in SL$ be the sentence

$$\forall x(\forall y R(x,y) \vee \forall y \neg R(x,y)).$$

For $\mathcal{M} \in \mathcal{T}L$ such that $\mathcal{M} \vDash \phi$, define $\beta(\mathcal{M}) \in \mathcal{T}L_1$ by

$$\mathcal{M} \vDash R(a_i, a_1) \Leftrightarrow \beta(\mathcal{M}) \vDash P(a_i),$$

so β is a bijection between $\{\mathcal{M} \in \mathcal{T}L \mid \mathcal{M} \vDash \phi\}$ and $\mathcal{T}L_1$.

For $\psi \in SL$, define ψ^* to be the result of replacing each occurrence of $R(t_1, t_2)$ in ψ, where t_1, t_2 are any terms of L, by $P(t_1)$. Then for $\mathcal{M} \vDash \phi$ it follows easily by induction on complexity of L-formulae that

$$\mathcal{M} \vDash \psi \Leftrightarrow \beta(\mathcal{M}) \vDash \psi^*.$$

Similarly, for $\xi \in SL_1$ we define ξ^+ to be the result of replacing each occurrence of $P(t_1)$ in ξ by $R(t_1, a_1)$. Then for $\mathcal{M} \vDash \phi$

$$\mathcal{M} \vDash \xi^+ \Leftrightarrow \beta(\mathcal{M}) \vDash \xi.$$

In [10] an automorphism[4] δ of BL_1 is specified, with the property

$$\delta[P(a_1) \wedge P(a_2)] = [P(a_1) \wedge P(a_2) \wedge P(a_3)].$$

Using this automorphism δ, define a bijection $\tau : \mathcal{T}L \to \mathcal{T}L$ in the following way:

[4] Referred to as γ in [10].

$$\tau(\mathcal{M}) = \begin{cases} \beta^{-1}(\delta(\beta(\mathcal{M}))) & \text{if } \mathcal{M} \vDash \phi, \\ \mathcal{M} & \text{otherwise.} \end{cases}$$

Then τ is an automorphism of BL since for $\psi \in SL$,

$$\tau[\psi] = \tau[(\psi \wedge \neg\phi) \vee (\psi \wedge \phi)] = [\psi \wedge \neg\phi] \cup [(\delta(\psi^*))^+ \wedge \phi],$$

and $[\psi \wedge \neg\phi] \cup [(\delta(\psi^*))^+ \wedge \phi]$ is $[(\psi \wedge \neg\phi) \vee ((\delta(\psi^*))^+ \wedge \phi)]$. Similarly for $\tau^{-1}[\psi]$.

Let $\psi \in SL$ be the sentence $R(a_1, a_1) \wedge R(a_2, a_1)$. Then ψ^* is $P(a_1) \wedge P(a_2)$, $\delta(\psi^*) = P(a_1) \wedge P(a_2) \wedge P(a_3)$, so $(\delta(\psi^*))^+ = R(a_1, a_1) \wedge R(a_2, a_1) \wedge R(a_3, a_1)$.

Consequently, for any probability function w satisfying INV, we require

$$w(R(a_1, a_1) \wedge R(a_2, a_1) \wedge \phi) = w(R(a_1, a_1) \wedge R(a_2, a_1) \wedge R(a_3, a_1) \wedge \phi).$$

On the other hand,

$$\begin{aligned} w(R(a_1, a_1) \wedge R(a_2, a_1) \wedge \phi) = {} & w(R(a_1, a_1) \wedge R(a_2, a_1) \wedge R(a_3, a_1) \wedge \phi) + \\ & w(R(a_1, a_1) \wedge R(a_2, a_1) \wedge \neg R(a_3, a_1) \wedge \phi) \end{aligned}$$

However then

$$w(R(a_1, a_1) \wedge R(a_2, a_1) \wedge \neg R(a_3, a_1) \wedge \phi) = 0$$

and this sentence is satisfiable. Therefore w cannot satisfy Super Regularity.

3. PIP and Polyadic Atom Exchangeability

Lemma 1 exemplified the unique role state formulae on r variables[5] play in determining automorphisms of BL that permute state formulae. Results in this section will demonstrate a second application of these state formulae, allowing us to show that PIP is in fact a natural generalisation of the thoroughly studied unary principle of Atom Exchangeability.

To emphasise the unary context where appropriate, we use symbols $P_1, P_2, \ldots, P_q$ for unary predicates and L_q for the language containing just these predicate symbols.

[5] Recall that r is the highest arity of a relation symbol in the language L with which we are concerned.

Atom Exchangeability, Ax

For any permutation τ of $\{1,2,\dots,2^q\}$ and constants $b_1,\dots,b_n$,

$$w\left(\bigwedge_{i=1}^{n}\alpha_{h_i}(b_i)\right)=w\left(\bigwedge_{i=1}^{n}\alpha_{\tau(h_i)}(b_i)\right),$$

where $\alpha_1(x),\alpha_2(x),\dots,\alpha_{2^q}(x)$, the *atoms* of L_q, are formulae of the form

$$\pm P_1(x)\wedge\pm P_2(x)\wedge\ldots\wedge\pm P_q(x).$$

Note that a state formula of L_q on n variables, $\bigwedge_{i=1}^{n}\bigwedge_{j=1}^{q}\pm P_j(x_i)$, is then just a conjunction of atoms $\bigwedge_{i=1}^{n}\alpha_{h_i}(x_i)$, $h_i\in\{1,2,\dots,2^q\}$, and a state formula for one variable is an atom. So Ax above is the statement that two state descriptions that are mapped one to the other by a permutation of atoms, should get the same probability. In the unary case, permutations of atoms are in an obvious bijection with permutations of state formulae satisfying (A) and (B) and so *in the unary context* PIP is equivalent to Ax.

We extend the notion of atoms to any polyadic language L, by defining a *polyadic atom* to be a state formula on r variables. We label the polyadic atoms $\gamma_1(x_1,\dots,x_r),\gamma_2(x_1,\dots,x_r),\dots,\gamma_N(x_1,\dots,x_r)$ in a fixed order, where the total number of atoms N is $2^{\sum_{j=1}^{q}r^{r_j}}$, since each state formula of r variables contains r^{r_j} conjuncts for each $j=1,2,\dots,q$. Unless indicated otherwise, γ_k will stand for $\gamma_k(x_1,\dots,x_r)$, with these variables. Note that for purely unary languages, this definition exactly describes the atoms of the language in the original sense.

In a manner corresponding to the case for unary languages, every state formula of the polyadic language L may be written as a conjunction of polyadic atoms. Namely,

$$\Theta(z_1,\dots,z_n)=\bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,n\}^r}\gamma_{h_{i_1,\dots,i_r}}(z_{i_1},\dots,z_{i_r}).\tag{3}$$

In contrast however, not every such conjunction describes a state formula of L, since some of these will be inconsistent. For instance, for L containing a single binary relation symbol R and a state formula $\Theta(z_1,z_2,z_3,z_4)$, in order for the conjunction to be consistent we must have $\gamma_{h_{3,4}}(z_3,z_4)=\gamma_{h_{4,3}}(z_4,z_3)$ and when $i=j$ the $\gamma_{h_{i,j}}$ need to be those atoms where R occurs either only positively or only negatively.

Note that when $i_1,\dots,i_r$ in (3) are distinct,

$$\gamma_{h_{i_1,\dots,i_r}}(z_{i_1},\dots,z_{i_r})=\Theta[z_{i_1},\dots,z_{i_r}].$$

On the other hand, when $i_1,\ldots,i_r$ are not all distinct we have

$$\gamma_{h_{i_1,\ldots,i_r}}(x_1,\ldots,x_r) = (\Theta[z_{i_{m_1}},\ldots,z_{i_{m_s}}])_\sigma$$

where $i_{m_1},\ldots,i_{m_s}$ are the distinct numbers among $i_1,\ldots,i_r$, and $\sigma : \{x_1, \ldots, x_r\} \twoheadrightarrow \{z_{i_{m_1}},\ldots,z_{i_{m_s}}\}$ is defined by $\sigma(x_j) = z_{i_{m_k}} \Leftrightarrow i_j = i_{m_k}$, so

$$\gamma_{h_{i_1,\ldots,i_r}}(z_{i_1},\ldots,z_{i_r}) = \Theta[z_{i_{m_1}},\ldots,z_{i_{m_s}}].$$

By Lemma 1 every permutation of state formulae that satisfies the conditions (A) and (B), equivalently a permutation that extends to an automorphism permuting state formulae, is determined by its restriction to the *atoms* of L. Let Γ denote the set of permutations τ of $\{1,\ldots,N\}$ such that the permutation ξ of atoms defined by $\xi(\gamma_k(x_1,\ldots,x_r)) = \gamma_{\tau(k)}(x_1,\ldots,x_r)$ is a permutation of state formulae satisfying (A) and (B). With these definitions, PIP is clearly equivalent to what may be termed

Polyadic Atom Exchangeability – Permutation Version

For any state description

$$\bigwedge_{\langle i_1,\ldots,i_r\rangle \in \{1,\ldots,n\}^r} \gamma_{h_{i_1,\ldots,i_r}}(b_{i_1},\ldots,b_{i_r})$$

and $\tau \in \Gamma$,

$$w\left(\bigwedge_{\langle i_1,\ldots,i_r\rangle \in \{1,\ldots,n\}^r} \gamma_{h_{i_1,\ldots,i_r}}\left(b_{i_1},\ldots,b_{i_r}\right)\right) = w\left(\bigwedge_{\langle i_1,\ldots,i_r\rangle \in \{1,\ldots,n\}^r} \gamma_{\tau(h_{i_1,\ldots,i_r})}\left(b_{i_1},\ldots,b_{i_r}\right)\right).$$

This represents PIP as a generalisation of Ax as stated above, except that we limit the ‘allowed’ permutations of polyadic atoms to those in Γ. The next result will determine exactly which[6] permutations of atoms define a permutation of state formulae that satisfies (A) and (B).

Lemma 4. *A permutation τ of $\{1,\ldots,N\}$ is in Γ if and only if for each $m \le r$, distinct $1 \le i_1,\ldots,i_m \le r$, $\sigma : \{x_1,\ldots,x_r\} \twoheadrightarrow \{x_{i_1},\ldots,x_{i_m}\}$ and $k, s \in \{1,\ldots,N\}$*

$$(\gamma_k[x_{i_1},\ldots,x_{i_m}])_\sigma = \gamma_s(x_1,\ldots,x_r) \Leftrightarrow (\gamma_{\tau(k)}[x_{i_1},\ldots,x_{i_m}])_\sigma = \gamma_{\tau(s)}(x_1,\ldots,x_r). \quad (4)$$

[6] Note that the condition in the following lemma is trivial when L is purely unary in accordance with the aforementioned fact that any permutation of unary atoms extends to a permutation of state formulae satisfying (A) and (B).

Proof. Suppose first that $\tau \in \Gamma$ and let η be the associated automorphism of BL. Then $\overline{\eta}$ satisfies (A) and (B). Also, $\overline{\eta}(\gamma_k) = \gamma_{\tau(k)}$ and by (B)

$$\overline{\eta}(\gamma_k[x_{i_1},\ldots,x_{i_m}]) = \overline{\eta}(\gamma_k)[x_{i_1},\ldots,x_{i_m}] = \gamma_{\tau(k)}[x_{i_1},\ldots,x_{i_m}]$$

so by (A)

$$(\gamma_{\tau(k)}[x_{i_1},\ldots,x_{i_m}])_\sigma = (\overline{\eta}(\gamma_k[x_{i_1},\ldots,x_{i_m}]))_\sigma = \overline{\eta}((\gamma_k[x_{i_1},\ldots,x_{i_m}])_\sigma).$$

Hence (4) follows from $\gamma_{\tau(s)} = \overline{\eta}(\gamma_s)$.

To prove the opposite direction, assume that τ satisfies (4). First observe that for $i_1,\ldots,i_r$ *not necessarily distinct*, $\gamma_k(z_{i_1},\ldots,z_{i_r})$ is consistent just when $\gamma_{\tau(k)}(z_{i_1},\ldots,z_{i_r})$ is consistent. This is the case since for a polyadic atom γ_v, $\gamma_v(z_{i_1},\ldots,z_{i_r})$ is consistent just when $\gamma_v(x_1,\ldots,x_r)$ is $(\gamma_v[x_{m_1},\ldots,x_{m_t}])_\sigma$ where $i_{m_1},\ldots,i_{m_t}$ are the distinct numbers amongst $i_1,\ldots,i_r$ and σ is defined by $\sigma(x_j) = x_{m_u} \Leftrightarrow i_j = i_{m_u}$.

Another observation we need is that if two atoms γ_k, γ_h have the property that restricting one to some m variables and the other to some (other) m variables produces the same state formula up to renaming the variables then the same holds for $\gamma_{\tau(k)}, \gamma_{\tau(h)}$. Expressed more formally, for (distinct) $x_{i_1},\ldots,x_{i_m}$ and $x_{j_1},\ldots,x_{j_m}$ from $\{x_1,\ldots,x_r\}$ we have

$$\begin{aligned}&\gamma_k[x_{i_1},\ldots,x_{i_m}] = (\gamma_h[x_{j_1},\ldots,x_{j_m}])(x_{i_1}/x_{j_1},\ldots,x_{i_m}/x_{j_m})\\ &\quad\Leftrightarrow \gamma_{\tau(k)}[x_{i_1},\ldots,x_{i_m}] = (\gamma_{\tau(h)}[x_{j_1},\ldots,x_{j_m}])(x_{i_1}/x_{j_1},\ldots,x_{i_m}/x_{j_m}) \qquad (5)\end{aligned}$$

where $(\gamma_h[x_{j_1},\ldots,x_{j_m}])(x_{i_1}/x_{j_1},\ldots,x_{i_m}/x_{j_m})$ is the result of replacing every occurrence in $\gamma_h[x_{j_1},\ldots,x_{j_m}]$ of x_{j_v} by x_{i_v}, $v = 1,\ldots,m$.

To see this, consider for example $\sigma_1 : \{x_1,\ldots,x_r\} \twoheadrightarrow \{x_{i_1},\ldots,x_{i_m}\}$ and $\sigma_2 : \{x_1,\ldots,x_r\} \twoheadrightarrow \{x_{j_1},\ldots,x_{j_m}\}$ defined by

$$\sigma_1(x_i) = \begin{cases} x_i & \text{if } i \in \{i_1,\ldots,i_m\},\\ x_{i_1} & \text{otherwise,}\end{cases} \qquad \sigma_2(x_i) = \begin{cases} x_{j_v} & \text{if } i = i_v \in \{i_1,\ldots,i_m\},\\ x_{j_1} & \text{otherwise.}\end{cases}$$

Then the left hand side of (5) holds just if for these σ_1, σ_2 we have

$$(\gamma_k[x_{i_1},\ldots,x_{i_m}])_{\sigma_1} = \gamma_s(x_1,\ldots,x_r) = (\gamma_h[x_{j_1},\ldots,x_{j_m}])_{\sigma_2}$$

for some $\gamma_s(x_1,\ldots,x_r)$, in which case

$$(\gamma_{\tau(k)}[x_{i_1},\ldots,x_{i_m}])_{\sigma_1} = \gamma_{\tau(s)}(x_1,\ldots,x_r) = (\gamma_{\tau(h)}[x_{j_1},\ldots,x_{j_m}])_{\sigma_2}$$

follows by (4), implying the right hand side of the equivalence. The other direction follows similarly.

Let the function F be defined as follows. For a state formula $\Theta(z_1,...,z_n) = \bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,n\}^r} \gamma_{h_{i_1,\dots,i_r}}(z_{i_1},\dots,z_{i_r})$,

$$F(\Theta(z_1,\dots,z_n)) = \bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,n\}^r} \gamma_{\tau(h_{i_1,\dots,i_r})}(z_{i_1},\dots,z_{i_r}). \tag{6}$$

By the first of the above observations each conjunct in (6) is consistent. Moreover, the whole conjunction must be consistent, since otherwise there would be $\langle i_1,\dots,i_r\rangle$ and $\langle j_1,\dots,j_r\rangle$ from $\{1,\dots,n\}^r$ and distinct $k_1,\dots,k_t$ occurring both amongst $\{i_1,\dots,i_r\}$ and $\{j_1,\dots,j_r\}$ such that for some relation symbol R_j of L of arity r_j and some $m_1,\dots,m_{r_j}$ from $\{1,\dots,t\}$,

$$\gamma_{\tau(h_{i_1,\dots,i_r})}(z_{i_1},\dots,z_{i_r}) \vDash R_j(z_{k_{m_1}},\dots,z_{k_{m_{r_j}}}),$$

$$\gamma_{\tau(h_{j_1,\dots,j_r})}(z_{j_1},\dots,z_{j_r}) \vDash \neg R_j(z_{k_{m_1}},\dots,z_{k_{m_{r_j}}}).$$

This would mean that

$$\gamma_{\tau(h_{i_1,\dots,i_r})}(z_{i_1},\dots,z_{i_r})[z_{k_1},\dots,z_{k_t}] \neq \gamma_{\tau(h_{j_1,\dots,j_r})}(z_{j_1},\dots,z_{j_r})[z_{k_1},\dots,z_{k_t}],$$

so by the second observation

$$\gamma_{h_{i_1,\dots,i_r}}(z_{i_1},\dots,z_{i_r})[z_{k_1},\dots,z_{k_t}] \neq \gamma_{h_{j_1,\dots,j_r}}(z_{j_1},\dots,z_{j_r})[z_{k_1},\dots,z_{k_t}].$$

However this is impossible since both are $\Theta(z_1,\dots,z_n)[z_{k_1},\dots,z_{k_t}]$. Therefore F defined by (6) permutes state formulae and clearly $F(\gamma_k(x_1,\dots,x_r)) = \gamma_{\tau(k)}(x_1,\dots,x_r)$.

It remains to check that F satisfies conditions (A) and (B). The condition (B) is obvious and for (A), let

$$\Theta(z_1,\dots,z_m) = \bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,m\}^r} \gamma_{h_{i_1,\dots,i_r}}(z_{i_1},\dots,z_{i_r})$$

and let $\sigma:\{y_1,\dots,y_n\} \twoheadrightarrow \{z_1,\dots,z_m\}$. Writing σ also for the mapping from $\{1,\dots,n\}$ to $\{1,\dots,m\}$ that sends j to i iff $\sigma(y_j) = z_i$, we have

$$(\Theta(z_1,\dots,z_m))_\sigma = \bigwedge_{\langle j_1,\dots,j_r\rangle\in\{1,\dots,n\}^r} \gamma_{h_{\sigma(j_1),\dots,\sigma(j_r)}}(y_{j_1},\dots,y_{j_r}),$$

$$F(\Theta(z_1,\dots,z_m)) = \bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,m\}^r} \gamma_{\tau(h_{i_1,\dots,i_r})}(z_{i_1},\dots,z_{i_r})$$

so both $(F(\Theta(z_1,\dots,z_m)))_\sigma$ and $F(\Theta(z_1,\dots,z_m)_\sigma)$ are

$$\bigwedge_{\langle j_1,\dots,j_r\rangle\in\{1,\dots,n\}^r} \gamma_{\tau(h_{\sigma(j_1),\dots,\sigma(j_r)})}(y_{j_1},\dots,y_{j_r}).$$

□

We shall now show that another formulation of Ax which while in the unary case is easily seen to be equivalent to the one given above, in the polyadic context becomes a principle that is *not obviously* equivalent to PIP but somewhat surprisingly turns out to be so nevertheless.

Atom Exchangeability (II)

Let

$$\Theta(b_1,\dots,b_n) = \bigwedge_{i=1}^{n}\alpha_{h_i}(b_i), \qquad \Phi(b_1,\dots,b_n) = \bigwedge_{i=1}^{n}\alpha_{k_i}(b_i),$$

be state descriptions of a unary language. If for all $0 \le i,\ j \le n$ we have $h_i = h_j \Leftrightarrow k_i = k_j$ then $w(\Theta) = w(\Phi)$.

The immediate polyadic counterpart of this is

Polyadic Atom Exchangeability - Spectral-Equivalence Version, PAx

Let

$$\Theta(b_1,\dots,b_n) = \bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,n\}^r} \gamma_{h_{i_1,\dots,i_r}}(b_{i_1},\dots,b_{i_r})$$

and

$$\Phi(b_1,\dots,b_n) = \bigwedge_{\langle i_1,\dots,i_r\rangle\in\{1,\dots,n\}^r} \gamma_{k_{i_1,\dots,i_r}}(b_{i_1},\dots,b_{i_r})$$

be state descriptions of L such that for all $\langle i_1,\dots,i_r\rangle, \langle j_1,\dots,j_r\rangle \in \{1,\dots,n\}^r$

$$h_{i_1,\dots,i_r} = h_{j_1,\dots,j_r} \Leftrightarrow k_{i_1,\dots,i_r} = k_{j_1,\dots,j_r}. \tag{7}$$

Then $w(\Theta(b_1,\dots,b_n)) = w(\Phi(b_1,\dots,b_n))$.

We shall show that PIP is equivalent to PAx and in order to do so we will use an equivalent characterisation of when two state formulae can be mapped one to another by a permutation satisfying (A) and (B).

State formulae $\Theta(z_1,\dots,z_n)$, $\Phi(z_1,\dots,z_n)$ are *similar* if for all (distinct) $i_1,\dots,i_t$ and $j_1,\dots,j_s$ from $\{1,\dots,n\}$ and $\sigma:\{z_{i_1},\dots,z_{i_t}\}\twoheadrightarrow\{z_{j_1},\dots,z_{j_s}\}$

$$\Theta[z_{i_1},\dots,z_{i_t}] = (\Theta[z_{j_1},\dots,z_{j_s}])_\sigma \Leftrightarrow \Phi[z_{i_1},\dots,z_{i_t}] = (\Phi[z_{j_1},\dots,z_{j_s}])_\sigma.$$

Theorem 5. *[11] State formulae Θ and Φ are similar if and only if there is a permutation of state formulae that satisfies (A) and (B) and maps Θ to Φ.*

Theorem 6. *The principle PAx is equivalent to PIP.*

Proof. First assume that w satisfies PAx. Suppose that F is a permutation of state formulae that satisfies (A) and (B), Θ is a state formula and $\Phi = F(\Theta)$. Assuming $\Theta(b_1,\dots,b_n)$ and $\Phi(b_1,\dots,b_n)$ are written as in the statement of PAx, by the condition (B) we have that $F(\gamma_{h_{i_1,\dots,i_r}}) = \gamma_{k_{i_1,\dots,i_r}}$ so (7) holds. Hence

$$w(\Theta(b_1,\dots,b_n)) = w(\Phi(b_1,\dots,b_n)),$$

showing PIP for w.

Now suppose that w satisfies PIP. Let Θ, Φ be as in the statement of PAx and such that (7) holds. It suffices to show that Θ and Φ are similar since then it will follow by Theorem 5 and PIP that $w(\Theta(b_1,\dots,b_n)) = w(\Phi(b_1,\dots,b_n))$, proving PAx for w.

So suppose that for distinct $i_1,\dots,i_t$ and $j_1,\dots,j_s$ from $\{1,\dots,n\}$ and $\sigma : \{z_{i_1},\dots,z_{i_t}\} \twoheadrightarrow \{z_{j_1},\dots,z_{j_s}\}$ we have

$$\Theta[z_{i_1},\dots,z_{i_t}] = (\Theta[z_{j_1},\dots,z_{j_s}])_\sigma.$$

Then for every choice of $m_1,\dots,m_{r_j}$ (with possible repeats) from $\{i_1,\dots,i_t\}$ and each relation symbol R_j of arity r_j,

$$\Theta \vDash R_j(z_{m_1},\dots,z_{m_{r_j}}) \Leftrightarrow \Theta \vDash R_j(\sigma(z_{m_1}),\dots,\sigma(z_{m_{r_j}}))$$

since

$$\Theta[z_{i_1},\dots,z_{i_t}] \vDash R_j(z_{m_1},\dots,z_{m_{r_j}}) \Leftrightarrow \Theta[z_{j_1},\dots,z_{j_s}] \vDash R_j(\sigma(z_{m_1}),\dots,\sigma(z_{m_{r_j}})).$$

With a slight abuse of notation, writing $\sigma(i_d) = j_e$ instead of $\sigma(z_{i_d}) = z_{j_e}$, this means that for any $m_1,\dots,m_r$ (with possible repeats) from $\{i_1,\dots,i_t\}$ we have $h_{m_1,\dots,m_r} = h_{\sigma(m_1),\dots,\sigma(m_r)}$, as $\gamma_{h_{m_1,\dots,m_r}}$ describes every relation involving variables from $\{z_{m_1},\dots,z_{m_r}\}$ and similarly for $\gamma_{h_{\sigma(m_1),\dots,\sigma(m_r)}}$.

If we had

$$\Phi[z_{i_1},\dots,z_{i_t}] \neq (\Phi[z_{j_1},\dots,z_{j_s}])_\sigma$$

then reasoning as above, this would mean that for some $m_1,\dots,m_r$ from $\{i_1,\dots,i_t\}$, $k_{m_1,\dots,m_r} \neq k_{\sigma(m_1),\dots,\sigma(m_r)}$. However this would contradict (7), so $\Phi[z_{i_1},\dots,z_{i_t}] = (\Phi[z_{j_1},\dots,z_{j_s}])_\sigma$ and since the same argument can be repeated with Θ and Φ interchanged, we conclude that Θ and Φ are similar as required. □

We have thus far shown that two versions of Atom Exchangeability on unary languages result in the principle PIP on polyadic languages when formulated using polyadic atoms. The third remaining formulation of Ax in the unary context utilises the idea of *spectrum* of a state description, as we shall shortly explain. This version can easily be seen to be equivalent to the previous statements of unary Ax *if we assume that Ex holds*. It would be natural to ask therefore, whether a polyadic formulation of this version of Ax would be equivalent to PIP + Ex. We shall show that for the most immediate polyadic counterpart of this principle the answer would be no. Whether other possible definitions of polyadic spectrum do indeed provide an equivalence with PIP + Ex remains a topic for further research.

Atom Exchangeability (III)

For $\Theta(b_1,\ldots,b_n)$ a state description of a unary language L_q, the probability

$$w(\Theta(b_1,\ldots,b_n)) = w\left(\bigwedge_{i=1}^{n}\alpha_{h_i}(b_i)\right)$$

depends only on the *spectrum* of this state description, that is on the multiset $\{m_1,\ldots,m_{2^q}\}$ where $m_j = |\{i \mid h_i = j\}|$, the number of times the atom α_j appears in $\Theta(b_1,\ldots,b_n)$.

By analogy, in the polyadic case this gives rise to defining the *p-spectrum* (polyadic, atom-based spectrum) of a state description

$$\Theta(b_1,\ldots,b_n) = \bigwedge_{\langle i_1,\ldots,i_r\rangle\in\{1,\ldots,n\}^r}\gamma_{h_{i_1,\ldots,i_r}}(b_{i_1},\ldots,b_{i_r})$$

of a polyadic language L as the multiset $\{m_1,\ldots,m_N\}$ where

$$m_j = |\{\langle i_1,\ldots,i_r\rangle\in\{1,\ldots,n\}^r \mid h_{i_1,\ldots,i_r} = j\}|.$$

For ease of notation, we usually omit zero entries from our multisets.

We remark that current use of the term *spectrum* in polyadic languages, as in the statement of Spectrum Exchangeability (Sx) [6, 5] for instance, involves the strong notion of *indistinguishability* of constants in a particular state description, where b_i and b_l are indistinguishable in $\Theta(b_1,\ldots,b_n)$ if for any relation R_j of L and $b_{k_1},\ldots,b_{k_u},b_{k_{u+2}},\ldots,b_{k_{r_j}}$ from $\{b_1,\ldots,b_n\}$

$$\Theta(b_1,\ldots,b_n)\vDash R_j(b_{k_1},\ldots,b_{k_u},b_i,b_{k_{u+2}},\ldots,b_{k_{r_j}})$$
$$\Leftrightarrow \Theta(b_1,\ldots,b_n)\vDash R_j(b_{k_1},\ldots,b_{k_u},b_l,b_{k_{u+2}},\ldots,b_{k_{r_j}}).$$

The spectrum of Θ is then defined as the multiset of the sizes of the classes of indistinguishable constants. This is clearly different from the notion of p-spectrum. Unless the language is unary, this type of indistinguishability is *not preserved* when the state description is extended, that is when we consider another state description with additional constants that implies the given one.

On the other hand, in the definition of p-spectrum of a state description we consider ordered r-tuples of constants (possibly with repeats), classifying them purely by the way these r constants relate to each other in the state description and disregarding their connections to the other constants. If we choose to define p-indistinguishability of two r-tuples in a state description to mean satisfying the same atom within it, we find that this notion of p-indistinguishability is 'forever': extending the state description to more constants does not change it.

With this in mind, we arrive at the following new polyadic symmetry principle:

Atom-Based Spectrum Exchangeability, p-Sx

The probability of a state description of a polyadic language L depends only on its p-spectrum.

Examining this new principle, we can see[7] that p-Sx implies Ex; it also implies PAx (and hence PIP), since any two state descriptions that satisfy (7) necessarily have the same p-spectrum.

We now show the converse does not hold in general by pointing out a probability function that satisfies PIP + Ex but gives different probabilities to state descriptions with the same p-spectrum. For this purpose, we employ one of the probability functions $u_{\overline{E}}^{\overline{p},L}$. A general construction of these functions is presented in [9], where it is also proved that the $u_{\overline{E}}^{\overline{p},L}$ satisfy PIP and Ex. Here we explain the definition only in the special case used in what follows.

Let L be the language with a single binary relation symbol R.

Imagine having an urn containing balls of 4 different colours (referred to as colours 1, 2, 3, 4), in equal proportions. Let $\equiv_2$ be the equivalence on the set of ordered pairs of these colours which has the following equivalence classes:

$$\{\langle 1,1\rangle,\langle 3,3\rangle\} \;\; \{\langle 2,2\rangle,\langle 4,4\rangle\} \;\; \{\langle 1,2\rangle,\langle 3,4\rangle\} \;\; \{\langle 2,1\rangle,\langle 4,3\rangle\}$$

$$\{\langle 1,3\rangle\} \;\; \{\langle 3,1\rangle\} \;\; \{\langle 1,4\rangle\} \;\; \{\langle 4,1\rangle\} \;\; \{\langle 2,3\rangle\} \;\; \{\langle 3,2\rangle\} \;\; \{\langle 2,4\rangle\} \;\; \{\langle 4,2\rangle\}.$$

We pick balls from the urn repeatedly, with replacement, to obtain a sequence of colours $\langle c_1,\ldots,c_n\rangle$. Each such sequence thus has a probability $\left(\frac{1}{4}\right)^n$

[7] If w satisfies p-Sx then obviously $w(\Theta(b_1,\ldots,b_n)) = w(\Theta(b_{1'},\ldots,b_{n'}))$ when $b_1,\ldots,b_n$ and $b'_1,\ldots,b'_n$ are distinct constants and $\Theta(b_1,\ldots,b_n)$ is a state description, so Ex follows.

of being chosen. Having picked the sequence $\langle c_1,\dots,c_n\rangle$, we pick uniformly at random a state description *consistent* with this sequence, where a state description

$$\Theta(b_1,\dots,b_n) = \bigwedge_{\langle i_1,i_2\rangle\in\{1,\dots,n\}^2} \gamma_{h_{i_1,i_2}}(b_{i_1},b_{i_2})$$

of L is *consistent with* $\langle c_1,\dots,c_n\rangle$ if for any $\langle i_1,i_2\rangle$, $\langle j_1,j_2\rangle\in\{1,\dots,n\}^2$,

$$\langle c_{i_1},c_{i_2}\rangle \equiv_2 \langle c_{j_1},c_{j_2}\rangle \Rightarrow h_{i_1,i_2} = h_{j_1,j_2}. \tag{8}$$

For a state description $\Psi(b_1,\dots,b_n)$, we define $w(\Psi(b_1,\dots,b_n))$ to be the probability that $\Psi(b_1,\dots,b_n)$ is picked by the above process. It can be checked that w is one[8] of the functions $u_{\overline{E}}^{\overline{p},L}$ defined in [9].

Any state description of L on n constants may be represented by an $n\times n$ $\{0,1\}$-matrix where 1 or 0 at the (i,j)th entry means this state description implies $R(b_i,b_j)$ or $\neg R(b_i,b_j)$ respectively. Let $\Theta(b_1,b_2,b_3,b_4)$ and $\Phi(b_1,b_2,b_3,b_4)$ be the following state descriptions:

$$\Theta:\ \begin{matrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\1&1&1&1\end{matrix} \qquad \Phi:\ \begin{matrix}1&0&1&1\\0&1&0&1\\1&0&1&0\\1&1&0&1\end{matrix}.$$

The p-spectrum of both is $\{10,6\}$, so it remains to show that $w(\Theta)\neq w(\Phi)$.

To see this, note that neither Θ nor Φ are consistent with any sequence of colours $\langle c_1,c_2,c_3,c_4\rangle$ in which a colour appears more than once, since if $c_{j_1}=c_{j_2}$ ($j_1,j_2\in\{1,2,3,4\}$), then $\langle c_i,c_{j_1}\rangle\equiv_2\langle c_i,c_{j_2}\rangle$ and so it would have to be the case that for each $i\in\{1,2,3,4\}$, as matrices,

$$\Theta[b_i,b_{j_1}] = \Theta[b_i,b_{j_2}]$$

(so in particular in the matrix for Θ there would be two identical columns) and similarly for Φ, but there is no pair j_1,j_2 where $j_1\neq j_2$ for either Θ or Φ with this property. So consider a sequence $\langle c_1,c_2,c_3,c_4\rangle$ where each

[8] Where $\overline{p}=\langle 0,\frac14,\frac14,\frac14,\frac14,0,0,\dots\rangle$ and $\overline{E}$ is defined by $\langle c_1,\dots,c_k\rangle\equiv_k\langle d_1,\dots,d_k\rangle$ if and only if one of the following holds:

- $\langle c_1,\dots,c_k\rangle=\langle d_1,\dots,d_k\rangle$,
- For each $j\in\{1,\dots,k\}$, either $c_j=1$ and $d_j=3$, or $c_j=2$ and $d_j=4$,
- For each $j\in\{1,\dots,k\}$, either $c_j=3$ and $d_j=1$, or $c_j=4$ and $d_j=2$.

colour appears exactly once. For some permutation ν of $\{1,2,3\}$ we must have $\langle c_{\nu(1)}, c_{\nu(2)}\rangle \equiv_2 \langle c_{\nu(3)}, c_4\rangle$ but

$$\Theta[b_{\nu(1)}, b_{\nu(2)}] = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \qquad \Theta[b_{\nu(3)}, b_4] = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$$

for *every* ν, so Θ is consistent with no sequence $\langle c_1, c_2, c_3, c_4\rangle$ and hence $w(\Theta) = 0$.

On the other hand, Φ is consistent for example with the sequence $\langle 1,2,3,4\rangle$ and hence $w(\Phi) \neq 0$. Thus w is a function that satisfies PIP and Ex (see [9]) without satisfying p-Sx as required.

4. Conclusion

In this paper we have investigated the special role state formulae on r variables, where r is the highest arity of a relation symbol in our language, play in the context of polyadic symmetry. We have shown that the automorphisms of our structure that permute state formulae are determined by their action on these particular state formulae. We have proposed that state formulae on r variables form the natural extension into the polyadic context of unary atoms and using this idea we have shown that PIP results as a natural extension of two formulations of the unary Ax, whilst a third possible formulation of Ax (equivalent to the others only as long as Ex is assumed) does not yield PIP when extended in the most obvious way to the polyadic. It is our hope that the definition of polyadic atoms will enable further research into polyadic Inductive Logic, making it possible to formulate new rational principles and to find plausible extensions of unary principles currently without a polyadic counterpart. For example, the long standing Johnson's Sufficientness Principle [4] is potentially a fruitful direction for future investigation.

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