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ANALYTIC INFERENCE AND THE INFORMATIONAL MEANING OF THE LOGICAL OPERATORS

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ABSTRACT

This paper elaborates on some ideas put forward in [20] and provides a more detailed exposition of an "informational semantics" for the logical operators. In this semantics the meaning of a logical operator is specified solely in terms of the information that is *actually possessed* by an agent. We show that the informational meaning of the logical operators that arises from this semantics is consistent with a *strong manifestability requirement*: any agent who grasps the (informational) meaning of the logical operators should be able to tell, in practice and not only in principle, whether or not s(he) holds the information that a given complex sentence is true, or the information that it is false, or neither of the two. This informational semantics also allows us to draw a sharp demarcation between "analytic" and "synthetic" (i.e. non-tautological) inferences in propositional logic, which defies the empiricist dogma that all logical inferences should belong to the first class. It also provides the means for defining degrees of syntheticity of logical inferences that may be related, on the one hand, to the "cognitive effort" required by an agent to recognize their validity and, on the other, to the computational resources that need to be consumed for this task.

1. Introduction

Despite all doubts cast upon the analytic-synthetic distinction, purely deductive reasoning is still usually described as being "analytic" or "tautological": its validity depends exclusively on the meaning of certain words (the "logical words") and so the information carried by the conclusion of an inference is already contained in the information carried by its premises.

This view appears to provide the strongest possible justification of deductive inference — which explains part of its enduring success in philosophical circles[1] — and has found in Bar-Hillel and Carnap's notion of "semantic

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[1] As Michael Dummett puts it: "Once the justification of deductive inference is perceived as philosophically problematic at all, the temptation to which most philosophers succumb is to offer too strong a justification: to say, for instance, that when we recognize

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information" [12, 7] its final theoretical settlement. Roughly speaking, the "semantic information" carried by a sentence is the set of all relevant possible worlds that it *excludes* (i.e., in which it is false), an idea first put forward by Popper [34] to argue for his thesis that only theories that are (empirically) falsifiable are (empirically) informative, and explicitly related by Bar-Hillel and Carnap to Shannon and Weawer's theory of information [42]. According to this notion, any classical inference is informationally trivial in that the set of possible worlds ruled out by its conclusion is included in the set of possible worlds ruled out by (the conjunction of) its premises.

As pointed out by Cohen and Nagel, however, this trivialization of logic sounds paradoxical:

> If in an inference the conclusion is not contained in the premises, it cannot be valid; and if the conclusion is not different from the premises, it is useless; but the conclusion cannot be contained in the premises and also possess novelty; hence inferences cannot be both valid and useful.[2]

In his influential book *Logic, Language and Information* Jaakko Hintikka labeled this paradox as a genuine "scandal of deduction": if deductive reasoning is "analytic" and logical truths carry no semantic information, "in what other sense, then, does deductive reasoning give us new information? Is it not perfectly obvious there is some such sense, for what point would there otherwise be to logic and mathematics?" [30, p. 222]. Hintikka explicitly relates the scandal to the undecidability of first order logic, which implies some inevitable uncertainty about the validity of inferences involving quantifiers. If one takes seriously the "old important idea" that information consists in reducing uncertainty, then "relief from this sort of uncertainty ought to be reflected by any realistic measure of the information which we *actually possess* (as distinguished from the information we in some sense have *potentially available* to us) and with which we can in fact operate" ([30, p. 229], our emphasis).

Without going into details, we observe that Hintikka's proposal for solving the paradox classifies as non-analytic only some inferences of the non-monadic predicate calculus and so leaves the Cohen-Nagel paradox unsettled in the domain of propositional logic:

> The truths of propositional logic are […] tautologies, they do not carry any new information. Similarly, it is easily seen that in the logically valid inferences of propositional logic the information carried by the conclusion is smaller or

the premises of a valid inference as true, we have thereby already recognized the truth of the conclusion"[25, p. 195].

[2] [13, p. 173]. See also [37] and [26] for a discussion and interesting positive proposals to solve the paradox. The difficulty is related to the so-called "problem of logical omniscience". See [18] for a discussion of the of the latter that is closely related to the ideas put forward in [20] and in this paper.

> at most equal to the information carried by the premisses. The term "tautology" thus characterizes very aptly the truths and inferences of propositional logic. One reason for its one-time appeal to philosophers was undoubtedly its success in this limited area" ([30, p. 154]).

This is highly unsatisfactory, especially in the light of the discovery that classical propositional logic is NP-hard and, therefore, most likely to be intractable. Computational intractability can be described as "undecidability in practice" and so it is highly plausible that there will ever be, in practice, a certain amount of uncertainty about the validity of some inferences even in the domain of propositional logic. Now, if information is to be related also to relief from this kind of practical uncertainty, then knowing the solution of a hard problem in propositional logic certainly gives us information that is new and practically useful.

This problem has been recently addressed in [20] on the basis of an "informational semantics" for Boolean Logic. In this approach the meaning of the Boolean operators is fixed exclusively in terms of the information that we *actually possess*. This semantics can be used to draw a sharp demarcation between "analytic" and "synthetic" (i.e. non-tautological) inferences in propositional logic, which defies the persistent dogma of empiricism according to which all logical inferences should belong to the first class.[3] In this view, the synthetic inferences are exactly those that cannot be justified on the grounds of the informational meaning of the logical operators, in that they make *essential* use of "virtual information", namely information that we do not actually possess and is by no means contained in the data, but is temporarily assumed "for the sake of the argument".

The main aim of the present contribution is to offer a more comprehensive treatment of informational semantics and clarify its connection with a "strong manifestability requirement" that relates the meaning of a sentence to the information that we actually possess. Indeed, we propose a direct, "positive", account of the informational meaning of the logical operators that is quite different from, and perhaps more natural than, the purely negative constraint-based approach put forward in [20]. Moreover, we show how it may be interestingly related to some ideas that can be traced in the works of W.V.O. Quine and M. Dummett. On the more technical side, we provide a detailed proof of a fundamental result (the subsentence property of strictly tautological arguments, Proposition 4) that was only sketched in [20] and show the connection between the informational meaning of the logical operators and a kind of non-deterministic valuation system that was essentially anticipated by Quine (Proposition 1).

[3] As explained in Section 3 this dogma has survived Quine's criticism against the analytic/synthetic distinction.

We believe that the approach outlined here[4] may have a practical and scientific interest that goes beyond its possible philosophical merits, and may be of use for researchers in the areas of cognitive science, artificial intelligence and behavioural economics. We suggest that the depth at which "virtual information" must be invoked in order to recognize the validity of an inference can be taken as a measure of the "cognitive effort" required to perform this task. From an AI perspective, this cognitive effort is reflected by the computational complexity of the corresponding decision problem, i.e. the problem of deciding whether or not a certain conclusion follows from the premises when virtual information can be used only up to a given fixed depth.

The rest of the paper is organized as follows: in Section 2 we critically discuss the traditional view that all logical inferences are "analytic" and suggest that a way out of the Cohen-Nagel paradox may consist in replacing classical semantics with a suitable "informational semantics" in which the meaning of the logical operators is specified solely in terms of the information that we *actually possess*. We also propose a "strong manifestability" requirement according to which any agent should be able to tell, in practice and not only in principle, whether or not s(he) actually possesses a certain piece of information. Next, in Section 3 we briefly reconsider Quine's views about the analytic/synthetic distinction with regard to logical laws and argue that an anticipation of informational semantics can be found in some remarks of his on the "primitive" meaning of the logical operators. In Sections 4 and 5 we discuss the informational meaning of the logical operators in detail and show how it justifies suitable introduction and elimination rules of a "natural deduction" system. The consequence relation defined by this system aims to capture the valid inferences of classical propositional logic that are genuinely "tautological". Next, in Section 6, we show that deducibility and refutability in this natural deduction system enjoy the "subsentence" (or "subformula") property and admit of a (straightforward) feasible decision procedure, which implies that they are consistent with our "strong manifestability" requirement. Finally, in Section 7 we discuss a way of classifying the valid inferences of classical propositional logic according to their degree of "syntheticity", which corresponds to the depth at which they make essential use of "virtual information".

2. Are analytic inferences really "tautological"?

According to one sense of the word "analytic", analytic inferences are those whose validity relies exclusively on the meaning of the logical operators.

[4] See also the recent [19] for a computationally oriented treatment and further technical results.

Given this meaning, the conclusion must necessarily be true whenever the premises are true. In most logical systems valid deductive inferences are defined in such a way that they turn out to be analytic in this sense. For example, in classical propositional logic, valid inferences are exactly those that are faithful to, and licensed by, the truth-tables for the logical operators, which are usually taken as definitions of their meaning. Let us call this the *semantic sense* of "analytic":

(1) An inference is analytic (in the semantic sense) if, and only if, whenever the premises are true, the conclusion must also be true by virtue only of the accepted meaning of the logical operators.

This is, admittedly, a somewhat vague notion of analytic inference that may be made more precise by specifying how the meaning of the logical operators is to be explicated. Observe that (1) makes plausible sense only when this meaning is explicated in terms of some notion of truth[5] and that this is the same notion of truth that is referred to in the righthand side of the equivalence. For example, if the meaning of $\neg$ and $\vee$ is specified by the classical truth-conditions, then the truth of B follows from the truth of $A \to B$ and $\neg A \to B$ by virtue only of the meaning stipulations, provided that (i) the truth-conditions are expressed in terms of the classical (alethic) notion of truth, satisfying the traditional principles of Bivalence and Non-Contradiction, and (ii) this is the same kind of truth that is preserved by the inference.

This semantic sense of the word "analytic", when applied to an inference, is usually associated with another sense, that we may call its *informational sense*, according to which an analytic inference is one that is "tautological", i.e. does not increase information. This sense can be made more explicit as follows:

(2) An inference is analytic (in the informational sense, or "tautological") if, and only if, whenever we possess the information that the premises are true, we thereby possess the information that the conclusion is true.

Going back to the above example, the inference from $A \to B$ and $\neg A \to B$ is analytic in the informational sense if the information that the premises are true somehow contains the information that the conclusion B is true.

This is a definition whose content much depends on our understanding of the crucial notion of "possessing the information" that a certain sentence is true. We shall not attempt an in-depth analysis of this primitive intuitive notion except for three general remarks.

[5] In this respect, one can agree that "*every* semantic theory has its goal in an account of the way in which a sentence is determined as true, when it is true, in accordance with its composition". [25, p. 31].

First, we assume that this notion is understood in its (ordinary) weak sense, according to which it may well be that an agent holds the information that A is true even if A is actually false. So, we do not assume a strong notion of information that complies with Floridi's "veridicality thesis" according to which information must be truthful ([27] and [28, Chapters 4-5]). In our loose sense, "possessing the information that A is true" is compatible with being misinformed about A.

Second, observe that the notion of possessing the information that a sentence is true is naturally associated with the symmetrical notion of possessing the information that a sentence is false, and the two notions are by no means complementary, even when the underlying notions of truth and falsity are. Not possessing the information that A is true is obviously not equivalent to possessing the information that A is false, even if we accept that falsity should be identified with the lack of truth: we certainly cannot assume that either we possess the information that A is true or we possess the information that A is false.

Third, we follow Hintikka in claiming that an important, if not the most important, sense of each of these symmetrical epistemic notions is that of a relation between an agent a and a sentence A that obtains if and only if *a actually*, and not only *potentially*, possesses the information that A is true (respectively, false) and can operate with it (see quotation on p. 2). We can make this more precise by requiring that the two notions satisfy the following condition:[6]

(3) *Strong Manifestability*. If an agent a grasps the informational meaning of a sentence A, then a should be able to tell, in practice and not only in principle, whether or not (s)he actually possesses the information that A is true, or the information that A is false or neither of them.

Here "in practice" can be interpreted in the sense that a has a *feasible* procedure to establish whether or not s(he) possesses the information that A is true (respectively false), on the sole basis of the information that s(he) *explicitly* holds and of the meaning of the logical operators occurring in A. In the sequel we shall use the expression "to hold the information that A is true (false)" as synonymous with "to actually possess the information that A is true (false)" in a sense that satisfies the strong manifestability requirement.

[6] This condition can be compared with an analogous requirement put forward by Michael Dummett in [24]: "… whenever the condition for the truth of a sentence is one that we have no way of bringing ourselves to recognize as obtaining whenever it obtains, it seems plain that there is no content to an ascription of an implicit knowledge of what that condition is, since there is no practical ability by means of which such knowledge may be manifested" [24, p. 82]. For a clear discussion of manifestability and its connection with the realism/anti-realism debate, see [33].

We are then led to what we may call the *strict informational sense* of the word "analytic":

(4) An inference is analytic (in the *strict* informational sense, or "strictly tautological") if, and only if, whenever we *hold* the information that the premises are true, we thereby *hold* the information that the conclusion is true.

This strict informational sense of "analytic" seems to be the main sense that underlies the Cohen-Nagel paradox, which can be construed as the following argument:

1. Classically valid inferences are analytic in the semantic sense
2. If an inference is analytic in the semantic sense, then it is informationally trivial

$\therefore$ Classically valid inferences are informationally trivial

where the conclusion clashes with the fact that, owing to the lack of a feasible decision procedure, deductive inference does reduce our practical uncertainty even in the domain of propositional logic. This is a genuine conflict only if "informationally trivial" is construed as "analytic in the strict informational sense": any agent who holds the information that the premises are true, thereby holds the information that the conclusion is true, where "holding the information" is intended in the practical, operational sense that satisfies our strong manifestability requirement. Then, since classical propositional logic is NP-hard, it is highly implausible that a feasible decision procedure will ever be found.[7] Hence, there is no guarantee that, for any agent a who grasps the classical meaning of the logical operators, a holds the information that A is true, whenever a holds the information that the sentences in Δ are true and A is a classical consequence of Δ.

Where is the catch? The first premise of the "paradox" is usually shown to be true by defining the meaning of the logical operators via the standard truth-conditions, which hinge on the classical information-transcending notions of truth and falsity as primary semantic notions. However, there is a patent *mismatch* between these notions and the central epistemic notions underlying the strict informational sense of "analytic", namely *holding (actually possessing) the information* that a sentence is true, respectively false, which is the only sense that makes the argument sound paradoxical. Hence, if the first premise is recognized as true on the grounds of the classical semantics for the logical operators, the second premise is far from being compelling, or even plausible, and so the whole argument is not much of a paradox, after all. Moreover, *any* way of fixing the meaning of the

[7] An NP-hard problem is tractable if and only if P = NP, against the generally accepted conjecture that P $\neq$ NP.

logical operators that makes the first premise true — that is, such that all classically valid inferences turn out to be analytic in the semantic sense — would make the second premise highly implausible because of the lack of a feasible decision procecure for classical validity. Similar considerations hold if "classically valid" is replaced by "intuitionistically valid", for intuitionistic logic is PSPACE-complete and so the existence of a feasible decision procedure for intuitionistic propositional logic is even less likely than for classical propositional logic.[8]

Can we draw a demarcation between the classical inferences that are analytic in the strict informational sense — i.e., truly informationally trivial or "tautological" — and those that are not? Our discussion suggests that this question may be fruitfully addressed by specifying a suitable *informational semantics*, whose primary notions are not classical truth (and falsity), but their epistemic counterparts of "holding the information" that a given sentence is true, respectively false, construed in a practical, operational sense that satisfies the strong manifestability requirement (3). As explained in the sequel, the main task of such informational semantics is that of specifying the conditions for, and the consequences of, holding (actually possessing) the information that a complex sentence is true, respectively false, in terms of the information that we hold about its component sentences (and nothing else). This will provide the means for drawing the required demarcation between the classical inferences that may be regarded as "analytic" — namely those that can be justified by this weaker informational semantics — and those that may not and can therefore be called "synthetic", in the twofold sense that their validity does not depend *only* on the informational meaning of the logical operators and their conclusion provides information that is practically *new*, that is, not contained in the information carried by the premises in an objective and non-psychological sense.

In the next section we shall show how an anticipation of this informational semantics can be found in some remarks made by W.V.O. Quine in his *The Roots of Reference* concerning the analytic/synthetic distinction and its relation with the "primitive" meaning of the logical operators.

3. Quine and the "primitive" meaning of the logical operators

The conception of logical deduction as "analytic", and therefore "tautological", is a persistent dogma of (logical) empiricism which seems to be somewhat independent of Quine's two dogmas [38] as well as from Davidson's

[8] Hence, shifiting from classical to intuitionistic logic is no solution. Although the intuitionistic semantics for the logical opeators can be expressed in terms of information states, the meaning of the logical operators is not defined solely in terms of the information that an agent actually possesses in a given state. On this point see also Section 4 below and [18].

"third dogma" [21]. After all, Quine's well-known arguments against the analytic-synthetic distinction spared the claim that the notion of analyticity had been sufficiently clarified in the restricted domain of logic. According to [38], statements that are analytic "by general philosophical acclaim" fall into two classes: those that may be called *logically true*, such as "no unmarried man is married" and those that may be turned into logical truths by replacing synonyms with synonyms, such as "no bachelor is married". Admittedly, Quine's problem was that "we lack a proper characterization of this second class of analytic statements" for, in his view, "the major difficulty lies not in the first class of analytic statements, the logical truths, but rather in the second class, which depends on the notion of synonymy" ([38], pp. 22-32 of the 1961 edition). Four decades later, while his reservations over the notion of analyticity remained the "the same as ever", Quine clarified that they concerned only "the tracing of any demarcation, even a vague and approximate one, across the domain of sentences in general" [40, p. 270]. But the impossibility of tracing a sharp demarcation does not imply that there may not be undebatable cases of analytic sentences. Indeed, "It is intelligible and often useful in discussion to point out that some disagreement is purely a matter of words rather than of fact" [40, p. 270]. The so-called "logical laws" are the most natural candidates for such paradigmatic examples of analytic sentences: it seems plausible that a disagreement about a logical truth can always be reduced to a disagreement about the meaning of some logical word that occurs in it.

In fact, in *The Roots of Reference* Quine had already suggested that, in order to fit the undisputed cases of analytic sentences, one may provide a rough theoretical definition of analyticity by saying that (i) a sentence is analytic for the native speakers of a language if they learn its truth in the very process of learning how to use the words that occur in it, and (ii) "recondite" sentences should still count as analytic if they can be obtained by "a chain of inferences each of which individually is assured by the learning of the words" [39, pp. 79-80]. In this perspective, logical truths may qualify as analytic in the traditional sense, although the very existence of enduring disagreement on some logical laws — e.g. on the law of excluded middle on the part of intuitionists — may suggest that such laws are not similarly bound up with the learning of the logical words and "should perhaps be seen as synthetic" [39, p. 80].

In his latest work Quine appears to leave aside this idea that some logical laws may be synthetic. For example, in his *Two dogmas in retrospect*, he argues that by the above criterion "all logical truths […] — that is, the logic of truth functions, quantification, and identity — would then perhaps qualify as analytic, in view of Gödel's completeness proof" [40, p. 270] and later on, in a 1993 interview, he seems to abandon any hesitation and make his position crystal-clear:

> Yes so, on this score I think of the truths of logic as analytic in the traditional sense of the word, that is to say true by virtue of the meaning of the words. Or as I would prefer to put it: they are learned or can be learned in the process of learning to use the words themselves, and involve nothing more. [11, p. 199].[9]

However, Quine's work contains also the germ of a different approach. In *The Roots of Reference* he outlined a dispositional theory of what he called "the primitive meaning" of the logical operators and observed that this semantics fails to be truth-functional. As mentioned above, Quine conceded that a sentence can be regarded as "analytic" when "*everybody* learns its truth by learning its words" or follows from such basic analytic truths by means of inference rules whose validity is also learned in the same process of learning the words. In particular, we learn the meaning of the logical words by "finding connections of dispositions" [39, p. 78]. For example, we find that "people are disposed to assent to an alternation [disjunction] when they are disposed to assent to a component" and so "the law that an alternation is implied by its components is thus learned, we might say, with the word 'or' itself; and similarly for the other laws" (*ibid.*). Can we conjecture that all basic laws and inference rules of Boolean logic are learned in a similar way, so that every validly derived law or inference can, at least tentatively, qualify as "analytic" in the sense outlined above? Not quite. While the governing circumstances that fix the meaning of negation are "strangely simple" (p. 75) — we learn to assent (respectively dissent) to $\neg A$ exactly when we dissent (respectively assent) to A — conjunctions and disjunctions immediately appear to be far more problematic. As for conjunction:

> A governing circumstance that goes far towards fixing its meaning is that a conjunction commands assent when and only when each component does. […] If we were content always to affirm conjunctions or leave them alone, then what has just been said would be the whole story. […] It is in dissent that the rub comes. […] The circumstances of dissent from a conjunction have to be mastered independently of the excessively simple rule of assent. Still, one of the rules of dissent is simple enough: the conjunction commands dissent whenever a component does. […] Conjunction has its blind spot, however, when neither component commands assent or dissent. There is no direct way of mastering this quarter. In some such cases the conjunction commands dissent and in others it commands nothing. This sector is mastered only later, in theory-laden ways. Where the components are "it is a mouse" and "it is a chipmunk", and neither is affirmed nor denied, the conjunction will still be denied. But where the components are "it is a mouse" and "it is in the kitchen", and neither is affirmed nor denied, the conjunction will perhaps be left in abeyance. [39, pp. 76-77].

This passage is perfectly in tune with our programmatic informational approach to the meaning of the logical operators, we just need to reword

[9] Quoted in [22].

p \ q	assent	abstain	dissent
assent	assent	abstain	dissent
abstain	abstain	?	dissent
dissent	dissent	dissent	dissent

p \ q	assent	abstain	dissent
assent	assent	assent	dissent
abstain	assent	?	abstain
dissent	assent	abstain	dissent

TABLE 1. Quine's incomplete 3-valued tables for conjunction (left) and disjunction (right).

"*a* actually holds the information that *A* is true (false)" as "*a* is disposed to assent (dissent) to *A*". Quine stresses that the case of disjunction is quite similar and dual to that of conjunction. Disjunction, like conjunction, has its blind quarters where neither component commands assent or dissent. So, "we might assent to the alternation of 'it is mouse' and 'it is a chipmunk' or we might abstain" (p. 77). The situation calls for a 3-valued logic that fails to be fully truth-functional, in that the truth-tables (or "verdict tables" as Quine calls them, the "verdicts" being assent, dissent and abstention) for the logical operators are incomplete. The incomplete truth-tables for conjunction and disjunction given by Quine, are reproduced in Table 1 [39, p. 77].

In Quine's view, the conjunction and disjunction operators that emerge from these incomplete 3-valued tables are "more primitive than the genuine truth-functional conjunction and disjunction, in that they can be learned by induction from observation of verdictive behaviour" [39, p. 78]. Moreover, they are "independent of our parochial two-valued logic, and independent of other truth-value logics" (*ibid.*). In fact:

> Truth-values represent a more advanced, more theory-laden level of linguistic development; and it is in terms of a theory, different theories for different subject-matters, that we eventually learn (if at all) what verdict to give to the cases of conjunction and alternation that are indeterminate at the centre of the verdict table. Two-valued logic is a theoretical development that is learned, like other theory, in indirect ways upon which we can only speculate. [39, p. 78].

But if Quine's incomplete 3-valued tables are all we can safely say about connections of dispositions to assent or dissent to a given compound sentence, why not take Quine's suggestion seriously and qualify as analytic, in a particularly strict sense of the word, all logical laws and inferences that can be fully justified with reference only to these incomplete, but admittedly more primitive, specifications? Strictly speaking, we should limit our attention to inferences, since these tables, like other many-valued tables, do not qualify any sentence as logically valid (i.e. true under all interpretations). For example, as Quine recognizes, the law of excluded middle is not bound up with the very learning of "or" and "not", as described by the verdict tables, and indeed cannot be derived from them. So, "it lies rather in the blind quarter of alternation" and perhaps "though true by our lights, should be seen as synthetic." But, given this proviso, why not go all the way and qualify as analytic, in a primitive sense, all and only the basic inference rules that can be immediately justified by the incomplete verdict tables and, in a derived sense, all the inferences that can be justified by taking the transitive closure of these basic inference rules? The immediate connection between Quine's views and our sought informational semantics should be apparent. As remarked above, it is sufficient to assume that "holding the information" that a certain sentence is true, respectively false, for an agent a is tantamount to being in the disposition to assent (dissent) to A.

4. Informational semantics for the logical operators

By "informational semantics" for the logical operators we mean an explication of their meaning that takes the epistemic notions "the agent a holds (i.e. actually possesses) the information that A is true" and "a holds the information that A is false" as *primary* semantic notions. Such an explication should consist in specifying the conditions for, and the consequences of, a's holding the information that a complex sentence A is true, respectively false, in terms of the information that a holds concerning the immediate subsentences of A.

Our discussion in the previous section strongly suggests that these conditions and consequences should agree with Quine's incomplete 3-valued tables for the logical operators. Accordingly, in Table 2, they are renamed as "informational 3-valued matrices", rewritten in a more convenient format and extended so as to include material conditional. Here the value of a sentence is 1 when we *hold the information* that it is true ("informational truth", corresponding to Quine's "assent"), 0 when we *hold the information* that it is false ("informational falsity", corresponding to Quine's "dissent") and $\bot$, or "undefined", when it is neither 1 nor 0 ("informational indeterminacy", corresponding to Quine's "abstain"); the "blind spots" correspond to the entries where two alternative possible values are given, indicating that the value of the compound sentence is not uniquely determined by the values of its component sentences, but can be either of the two values shown, in accordance with Quine's remarks quoted in the previous section. Such *non-deterministic matrices* where independently rediscovered by Crawford and Etherington [14] and used to provide a semantic characterization of unit-resolution.[10]

[10] The general theory of non-deterministic valuations has been developed by Arnon Avron and coauthors (see [3, 4, 1, 5, 2, 6] among others) and used as a tool for investigating proof-theoretical properties of Gentzen-style sequent calculi.

$\wedge$	1	0	$\bot$
1	1	0	$\bot$
0	0	0	0
$\bot$	$\bot$	0	$0/\bot$

$\vee$	1	0	$\bot$
1	1	1	1
0	1	0	$\bot$
$\bot$	1	$\bot$	$1/\bot$

$\to$	1	0	$\bot$
1	1	0	$\bot$
0	1	1	1
$\bot$	1	$\bot$	$1/\bot$

$\neg$	
1	0
0	1
$\bot$	$\bot$

TABLE 2. Informational 3-valued matrices.

By *signed sentence* we mean any expression of the form $T\ A$ or $F\ A$, where A is a sentence of a standard propositional language. In the sequel we shall use X, Y, etc. to denote arbitrary signed sentences and the boldface capital greek letters $\mathbf{\Gamma}$, $\mathbf{\Delta}$, etc. to denote sets of signed sentences.

We consider each agent a as equipped with a database Δ_a (a's "working memory") consisting of a *finite* set of signed sentences representing the information that *a explicitly* holds (at a given time). Let us write $a \Vdash T\ A$ for "a holds the information that A is true" and $a \Vdash F\ A$ for "a holds the information that A is false", so that $\Vdash$ is a relation between an agent a and a signed sentence X.

A minimal requirement on $\Vdash$ is that, for every signed sentence X:

(Reflexivity) $$X \in \Delta_a \Longrightarrow a \Vdash X.$$

The problem is that of extending the image of a under $\Vdash$ beyond the signed sentences contained in Δ_a, in order to characterize the information that *a implicitly* holds by the sole means of the information in Δ_a and of her grasping of the (informational) meaning of the logical operators.

For this purpose, observe that the informational 3-valued matrices (Table 2) that have been discussed and justified in the previous section, immediately imply the following sufficient conditions:

S1. $v(A) = 1$ and $v(B) = 1 \Longrightarrow v(A \wedge B) = 1$,
S2. $v(A) = 0$ or $v(B) = 0 \Longrightarrow v(A \wedge B) = 0$,
S3. $v(A) = 1$ or $v(B) = 1 \Longrightarrow v(A \vee B) = 1$,
S4. $v(A) = 0$ and $v(B) = 0 \Longrightarrow v(A \vee B) = 0$,
S5. $v(A) = 0$ or $v(B) = 1 \Longrightarrow v(A \to B) = 1$,
S6. $v(A) = 1$ and $v(B) = 0 \Longrightarrow v(A \to B) = 0$,
S7. $v(A) = 0 \Longrightarrow v(\neg A) = 1$,
S8. $v(A) = 1 \Longrightarrow v(\neg A) = 0$,

where $v(A)$ denotes the value of the sentence A. This is all there is to say about the *conditions* for holding the information that a complex sentence is true or false. Notice that these are formally the same as the classical truth-conditions, and so shifting from the classical to the informational

$$\frac{F\,A}{T\,\neg A}\;T\neg\text{-}\mathscr{I} \qquad \frac{T\,A}{F\,\neg A}\;F\neg\text{-}\mathscr{I}$$

$$\frac{T\,A}{T\,A\vee B}\;T\vee\text{-}\mathscr{I}1 \qquad \frac{T\,B}{T\,A\vee B}\;T\vee\text{-}\mathscr{I}2 \qquad \frac{\begin{array}{c}F\,A\\ F\,B\end{array}}{F\,A\vee B}\;F\vee\text{-}\mathscr{I}$$

$$\frac{F\,A}{F\,A\wedge B}\;F\wedge\text{-}\mathscr{I}1 \qquad \frac{F\,B}{F\,A\wedge B}\;F\wedge\text{-}\mathscr{I}2 \qquad \frac{\begin{array}{c}T\,A\\ T\,B\end{array}}{T\,A\wedge B}\;T\wedge\text{-}\mathscr{I}$$

$$\frac{F\,A}{T\,A\to B}\;T\to\text{-}\mathscr{I}1 \qquad \frac{T\,B}{T\,A\to B}\;T\to\text{-}\mathscr{I}2 \qquad \frac{\begin{array}{c}T\,A\\ F\,B\end{array}}{F\,A\to B}\;F\to\text{-}\mathscr{I}$$

TABLE 3. Sufficient conditions (introduction rules)
for the standard Boolean operators.

interpretation makes no difference as far as these sufficient conditions are concerned.

It follows that the image of a under $\Vdash$ must be closed under all the inference rules in Table 3, in the following sense: for all sentences A and B, any agent a who *holds the information* that the signed sentences above the line obtain, thereby *holds the information* that the signed sentence below the line obtains.[11]

Let us now turn our attention to the *consequences* of holding the information that a complex sentence is true or false. This is quite unproblematic for the negation operator as well as for true conjunctions, false disjunctions and false conditionals, for suitable necessary conditions can be immediately read off the informational matrices and are given by N1-N5 below:

N1. $v(A \wedge B) = 1 \Longrightarrow v(A) = 1$ and $v(B) = 1$,
N2. $v(A \vee B) = 0 \Longrightarrow v(A) = 0$ and $v(B) = 0$,
N3. $v(A \to B) = 0 \Longrightarrow v(A) = 1$ and $v(B) = 0$,
N4. $v(\neg A) = 1 \Longrightarrow v(A) = 0$,
N5. $v(\neg A) = 0 \Longrightarrow v(A) = 1$,

which are, again, formally identical to the classical necessary conditions.

[11] By saying that a signed sentence $T\,A$ or $F\,A$ "obtains" we simply mean that A is true or, respectively, false. Alternatively, we might use the signs T and F to mean assent or dissent, based on the information the agent actually possesses, along the lines of [43] and [41]. On the use of signed sentences in classical natural deduction see also [10].

On the other hand, holding the information that a conjunction is false, or that a disjunction is true, or that a material conditional is true has no *direct* consequences that can be expressed exclusively in terms of the information that we hold concerning their immediate subsentences. In particular, the following classical-like conditions *cannot* be justified by the informational matrices in Table 2 and are, indeed, *intuitively wrong* according to the ordinary sense of "information":

N*6. $v(A \wedge B) = 0 \Longrightarrow v(A) = 0$ or $v(B) = 0$,
N*7. $v(A \vee B) = 1 \Longrightarrow v(A) = 0$ or $v(B) = 1$,
N*8. $v(A \rightarrow B) = 1 \Longrightarrow v(A) = 0$ or $v(B) = 1$.

For example, if I suffer from low vision, I may still hold the information that the digit at which the optometrist is pointing is either a "2" or a "7" — as well as the information that it is not both a "2" and a "7", and the information that if it is a "2", then the next one is also a "2" — without holding any clear information about the component sentences.[12]

In Gentzen's Natural Deduction — a proof-theoretic presentation of logical consequence that mirrors the intuitionistic explanations of the logical operators[13] — this problem is bypassed by means of a technical device, known as "discharging of assumptions". For example, in the $\vee$-elimination rule

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A \vee B\end{matrix} \qquad \begin{matrix}\Delta, [A] \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\Lambda, [B] \\ \vdots \\ C\end{matrix}}{C}$$

the sentences in square brackets represent information that is by no means contained in the information that is actually "given" to the agent who is performing this inference step (except for special cases in which the rule application is redundant), represented by the "undischarged" assumptions Γ, Δ and Λ. In [20] we called "virtual information" this kind of information that plays an essential role in an inference step without being actually possessed by the agent who performs it.

[12] Compare this example with Quine's remarks on the blind quarters of conjunction and disjunction quoted in the previous section.

[13] See [29, 35]; see also [46] for an excellent exposition. Gentzen suggested that the rules of his system of Natural Deduction could be taken as definitions of the logical operators. Indeed, he proposed that the introduction rules would be sufficient for this purpose and that the elimination rules could be ultimately "justified" in terms of the introduction rules. This idea was later refined into criteria of admissibility for putative definitions of the logical operators that culminated in Prawitz's *inversion principle*: no information can be obtained from applying an elimination rule to a sentence *A* that would not have already been available if *A* had been obtained by means of an introduction rule.

This use of virtual information in Gentzen's $\vee$-elimination rule complies with the intuitionistic view, according to which holding the information that a sentence is true ultimately means possessing a *canonical proof* for it, that is, a proof obtained "by the most direct means". In Natural Deduction this is a proof whose last step is the application of an introduction rule.[14] Indeed, in intuitionistic terms, we have a canonical proof of $A \vee B$ if and only if we have either a proof of A or a proof of B. So any application of the above rule is fully justified and would be indeed redundant had we obtained the information that the premise $A \vee B$ is true by means of a canonical proof. However, this does not seem to be a compelling feature of our understanding of $\vee$ in relation to a more ordinary notion of "information". As Michael Dummett puts it:

> I may be entitled to assert "A or B" because I was reliably so informed by someone in a position to know, but if he did not choose to tell me which alternative held good, I could not apply an or-introduction rule to arrive at that conclusion. […] Hardy may simply not have been able to hear whether Nelson said "Kismet hardy" or "Kiss me Hardy", though he heard him say one or the other: once we have the concept of disjunction, our perceptions themselves may assume an irremediably disjunctive form ([25], pp. 266-267). […]
>
> Unlike mathematical information, empirical information decays at two stages: in the process of acquisition, and in the course of retention and transmission. An attendant directing theatre-goers to different entrances according to the colours of their tickets might even register that a ticket was yellow or green, without registering which it was, if holders of tickets of either colours were to use the same entrance; even our observations are incomplete, in the sense that we do not and cannot take in every detail of what is in our sensory fields. That information decays yet further in memory and in the process of being communicated is evident. In mathematics, any effective procedure remains eternally available to be executed; in the world of our experience, the opportunity for inspection and verification is fleeting ([25], pp. 277-278).

In the light of this discussion, it can be argued that the inference steps involving the introduction and subsequent "discharge" of virtual information — like the elimination of disjunction in Gentzen's natural deduction — should be regarded as "synthetic" in a sense that appears to be close to Kant's sense, in that they force an agent to consider information that is not included in the information "given" to her.[15] Such "synthetic" inference steps are part of our common reasoning practice and are not problematic in

[14] See [25] (Chapter 11) and [36] for a thorough discussion.

[15] "In an analytical judgement I do not go beyond the given conception, in order to arrive at some decision respecting it. […] But in synthetic judgements, I must go beyond the given conception, in order to cogitate, in relation with it, something quite different from what was cogitated in it […]" (I. Kant, *Critique of Pure Reason* [1781], Book II, Chapter II, Section II. Quoted from the english translation by J.M.D. Meiklejohn, ebooks Adelaide, 2009, http://ebooks.adelaide.edu.au/k/kant/immanuel/k16p/index.html).

simplest cases. However, when recognizing that a conclusion validly follows from the premises requires weaving one's way in and out of a complex recursive pattern of virtual information, the situation may soon get out of control.[16]

It should be now apparent that the informational semantics we are investigating in this paper, which hinges upon the primary semantic notion of "actually possessing" information concerning the truth or falsity of a sentence, cannot fix the meaning of a logical operator by means of conditions that make essential use of what we have called "virtual information". It follows, that the information that an agent holds may assume, to use Dummett's words, an "irremediably disjunctive form". That a disjunction is true may be piece of information that an agent holds without having epistemically cogent grounds for either of the two disjuncts or having lost them, and with which s(he) is anyway ready to operate. This involves *reacting* in a certain way when exposed to information concerning A or B. Suppose that an agent a, who holds the information that $A \vee B$ is true, is exposed to the information that A is false. Then a's reaction would be that of holding the information that B is true. Similar considerations can be made for the case in which a holds the information that a conjunction $A \wedge B$ is false or the information that a conditional $A \rightarrow B$ is true. This kind of "dispositional attitude" that is associated with holding the information that a signed sentence of the form $T\,A \vee B$, $F\,A \wedge B$ or $T\,A \rightarrow B$ obtains, is reflected by time honoured inference principles, such as Disjunctive Syllogism, Modus Ponens or Modus Tollens. These considerations lead to replacing the intuitively wrong classical-like conditions N*6-N*8, with the following weaker ones:

N6. $v(A \wedge B) = 0$ and $v(A) = 1 \Longrightarrow v(B) = 0$,

N7. $v(A \wedge B) = 0$ and $v(B) = 1 \Longrightarrow v(A) = 0$,

N8. $v(A \vee B) = 1$ and $v(A) = 0 \Longrightarrow v(B) = 1$,

N9. $v(A \vee B) = 1$ and $v(B) = 0 \Longrightarrow v(A) = 1$,

N10. $v(A \rightarrow B) = 1$ and $v(A) = 1 \Longrightarrow v(B) = 1$,

N11. $v(A \rightarrow B) = 1$ and $v(B) = 0 \Longrightarrow v(A) = 0$,

These are, indeed, the strongest necessary conditions that can be read off the informational matrices in Table 2. It follows that the image of a under $\Vdash$ must be closed also under all the inference rules in Table 4 which specify the immediate consequences of our holding the information that a certain

[16] Similar considerations apply to the treatment of disjunction in Beth's semantics for intuitionistic logic or to the treatment of conditional in both Kripke's and Beth's semantics. In order to recognize that a sentence is true at a state s a reasoning agent may have to travel from his actual information state s to a "virtual" state s^* containing information that is not contained in s. It is not surprising, then, that the decision problem for intuitionistic logic, even for the pure $\{\rightarrow\}$-fragment, is PSPACE complete, that is, among the hardest problems in PSPACE ([44, 45]).

$$\frac{T\neg A}{FA}\,T\neg\text{-}\mathscr{E} \qquad \frac{F\neg A}{TA}\,F\neg\text{-}\mathscr{E}$$

$$\frac{\begin{array}{c}TA\vee B\\ FA\end{array}}{TB}\,T\vee\text{-}\mathscr{E}1 \qquad \frac{\begin{array}{c}TA\vee B\\ FB\end{array}}{TA}\,T\vee\text{-}\mathscr{E}2 \qquad \frac{FA\vee B}{FA}\,F\vee\text{-}\mathscr{E}1 \qquad \frac{FA\vee B}{FB}\,F\vee\text{-}\mathscr{E}2$$

$$\frac{\begin{array}{c}FA\wedge B\\ TA\end{array}}{FB}\,F\wedge\text{-}\mathscr{E}1 \qquad \frac{\begin{array}{c}FA\wedge B\\ TB\end{array}}{FA}\,F\wedge\text{-}\mathscr{E}2 \qquad \frac{TA\wedge B}{TA}\,T\wedge\text{-}\mathscr{E}1 \qquad \frac{TA\wedge B}{TB}\,T\wedge\text{-}\mathscr{E}2$$

$$\frac{\begin{array}{c}TA\rightarrow B\\ TA\end{array}}{TB}\,T\rightarrow\text{-}\mathscr{E}1 \qquad \frac{\begin{array}{c}TA\rightarrow B\\ FB\end{array}}{FA}\,T\rightarrow\text{-}\mathscr{E}2 \qquad \frac{FA\rightarrow B}{TA}\,F\rightarrow\text{-}\mathscr{E}1 \qquad \frac{FA\rightarrow B}{FB}\,F\rightarrow\text{-}\mathscr{E}2$$

TABLE 4. Necessary conditions (elimination rules)
for the four standard Boolean operators.

complex sentence is true or false making *no use of virtual information*. The rules in Tables 3 and 4, taken together, can be seen as the *introduction* and *elimination* rules of a "natural deduction" system that reflects the informational meaning of the logical operators in much the same way as Gentzen's *LJ* reflects their intuitionistic meaning.[17]

The shift from classical to informational semantics is driven by the shift from the classical notions of truth and falsity to their informational counterparts as primary semantic notions in the explication of the logical operators. The classical notions are governed by two fundamental Aristotelian assumptions:

Principle of Non-Contradiction: no sentence can be both true and false;
Principle of Bivalence: every sentence is either true or false.

These assumptions can be called *structural* in that they do not pertain to the explication of the meaning of the logical operators, which they do not mention at all, but are assumptions on the underlying notion of truth that is used in such explications (e.g., via the truth-tables).

[17] We observe, in passing, that these rules also satisfy Prawitz's inversion principle (See fn. 13): each elimination rule allows us to extract *no more* information from a complex sentence than the information contained in its *direct* informational grounds. They do not satisfy stricter versions of this principle, because we admit of the possibility that an agent holds the information that some sentences (a disjunction, a conditional) are true and some sentences (a conjunction) are false without having direct grounds for holding it. For an overview on inversion principles see [32].

What structural properties should we assume concerning the new primary semantic notions? As for the Principle of Non-Contradiction, it may be reasonably assumed that this principle lifts to the informational level: we cannot sensibly claim that we hold *both* the information that a certain sentence *A* is true and the information that *A* is false, since this situation would be more naturally represented by saying that we hold *no* information about *A*.[18] Thus, even if one is not willing to endorse the somewhat controversial view that "information encapsulates truth" ([27] and [28, Chapters 4-5]), one may still maintain that a minimal interpretation of "holding information" is one that satisfies the *informational version* of the Principle of Non-Contradiction:

(IPNC) *For no agent a and sentence A, it is the case that a holds both the information that A is true and the information that A is false.*

Notice that (IPNC) does not require that actual information "encapsulates truth", nor does it even require that it "encapsulates consistency", but only that information "encapsulates *surface* consistency", namely that if we cannot *actually possess* both the information that *A* is true and the information that *A* is false, for under these circumstances we had better say that we do not actually possess any information about *A*.

Nothing of the kind, on the other hand, can be said of the Principle of Bivalence, which cannot lift to the informational level without turning into a counterintuitive *Principle of Omniscience*:

(PO) *For every agent a and sentence A, either a holds the information that A is true or a holds the information that A is false.*

In what follows we shall therefore assume that our primary notions of "holding the information that *A* is true (respectively false)" satisfies (IPNC), but *not* (PO).

[18] Interestingly enough, this seems to be consistent with Kant's view in his *Critique of Pure Reason*, where he regarded the Principle of Non-Contradiction as "the supreme principle of all analytical judgements", and claimed that: "[This principle] is a universal but purely negative criterion of all truth. But it belongs to logic alone, because it is valid of all cognitions, merely as cognitions and without respect to their content, and declares that the contradiction entirely nullifies them."(I. Kant, *Critique of Pure Reason* [1781], Book II, Chapter II, Section I. Quoted from the english translation by J.M.D. Meiklejohn, ebooks Adelaide, 2009, http://ebooks.adelaide.edu.au/k/kant/immanuel/k16p/index.html). We realize, however, that this point may be controversial and that one may be interested in a notion of information that allows for the possibility of holding inconsistent information. On the other hand, in the light of the strong manifestability requirement, one should wonder under what circumstances it really makes sense to distinguish a situation in which we hold inconsistent information about a sentence from one in which we hold no information at all about it. Allowing for inconsistent information becomes a more urgent issue when "information" is intended in a broader sense that does not satisfy strong manifestabiity, so that such inconsistencies may remain hidden.

We conclude this section by observing that closure under the introduction and elimination rules is equivalent to closure under the non-deterministic valuation rules specified by the informational matrices in Table 2. Consider a *non-deterministic valuation* v such that (i) for every sentence p, $v(p) \in \{1, 0, \bot\}$,[19] and (ii) for every complex A, $v(A)$ is chosen in agreement with the informational 3-valued tables in Table 2 (by this we mean that, whenever the value is not uniquely determined, a value is chosen among the two allowed ones). We call such a valuation a *3ND-valuation*. Say that a 3ND-valuation v *satisfies* a set $\mathbf{\Gamma}$ of signed formulas, if for every $T\,A \in \mathbf{\Gamma}$, $v(A) = 1$ and for every $F\,A \in \mathbf{\Gamma}$, $v(A) = 0$. Then it can be shown that:

Proposition 1. *Let $\mathbf{\Gamma}$ be a set of signed sentences such that: (i) for no atomic sentence p, $T\,p$ and $F\,p$ are both in $\mathbf{\Gamma}$, and (ii) $\mathbf{\Gamma}$ is closed under the introduction and elimination rules in Tables 3 and 4. Then, there exists a 3ND-valuation v that satisfies $\mathbf{\Gamma}$.*

5. Informational models and intelim sequences

An *informational model* is a triple $M = (S, \phi, \Vdash)$ where

- S is a non-empty set;
- ϕ is a function mapping each element a of S to a finite set Δ_a of signed sentences;
- $\Vdash$ is a relation between elements of S and signed sentences such that
  - $\Vdash$ satisfies (Reflexivity) and (IPNC),
  - for every $a \in S$, the image of a under $\Vdash$ is closed under all the rules in Tables 3 and 4.

Intuitively, S is a set of *agents* and for each agent a, $\phi(a) = \Delta_a$ represents the information explicitly stored by a (or a's "working memory"). All the agents in S share the same language, that is, the same mechanism for expanding a priori this explicit information, represented by the *informational forcing* relation $\Vdash$, and this mechanism includes, but may not be restricted to, the semantic stipulations witnessing that each agent grasps the same (informational) meaning of the logical operators, and that the primary semantic notion $\Vdash$ satisfies the structural properties (Reflexivity) and (IPNC). In other words, a model consists in a set of agents who share the same meaning stipulations, including those that are taken as definitions of

[19] Recall that 1 is informational truth (holding the information that the sentence is true), 0 is informational falsity (holding the information that it is false) and $\bot$ is informational indeterminateness (holding no information about its truth or falsity).

the logical operators, and intend the notion of "holding the information" that a sentence is true or false as satisfying (Reflexivity) and (IPNC).[20]

Let us now say that a set $\mathbf{\Gamma}$ of signed sentences *informationally implies* a signed sentence X, and write $\mathbf{\Gamma} \vDash X$, if for every informational model M and every $a \in S$, $a \Vdash X$ whenever $a \Vdash \mathbf{\Gamma}$. (Henceforth we use "$a \Vdash \mathbf{\Gamma}$" as an abbreviation of "for all $Y \in \mathbf{\Gamma}$, $a \Vdash Y$"). Let us also say that $\mathbf{\Gamma}$ is *informationally inconsistent* if for all models M and all $a \in S$, $a \nVdash Y$ for some $Y \in \mathbf{\Gamma}$. Observe that, according to our definition of $\vDash$, if $\mathbf{\Gamma}$ is informationally inconsistent, then $\mathbf{\Gamma} \vDash X$ for every signed sentence X.

Readers more in tune with a valuation-based approach, and who are not put-off by unusual objects such as non-deterministic valuations, may switch, by virtue of Proposition 1 (see also footnote 20) to the following simpler characterization:

Corollary 1. $\mathbf{\Gamma}$ informationally implies *X if and only if every 3ND-valuation that satisfies $\mathbf{\Gamma}$ satisfies also X.*

This simpler non-deterministic semantics shows that there is nothing essential in the "multi-agent" approach outlined above. On the other hand, the exposition in terms of informational models, that can be intuitively interpreted as a set of agents sharing the same meaning stipulations, paves the way for further developments, e.g., assigning different inferential powers to different agents (see Section 7).

We claim that this semantics provides an adequate explication of the "informational meaning" of the classical operators, that is, of that part of their meaning that can be fully specified solely in terms of the primary semantic notions of an agent's holding ("actually possessing") the information that a sentence is true, respectively false. We also claim that the relation $\vDash$ models a notion of logical inference that is analytic *both* in the semantic and in the strict informational sense explained in Section 2: any agent a who grasps the informational meaning of the logical operators *does* hold the information that the conclusion A is true whenever s(he) holds the information that all the premises in Γ are true.

To substantiate these claims, however, we cannot rely only on our intuition, as we unwarrantedly did before when saying that the rules in Tables 3 and 4 "appear to be intuitively sound". We have to show also that taking the *closure* under these rules as a characterization of the information that is

[20] Observe that, by Proposition 1, for every informational model M and every $a \in S$, there exists a 3ND-valuation that satisfies $\{X \mid a \Vdash X\}$, for the latter is a set of signed sentences that satisfies the sufficient condition. Moreover, for every 3ND-valuation v, the set $\{X \mid v \text{ satisfies } X\}$ is closed under the inference rules (that is, the latter are sound with respect to 3ND-valuations) and trivially satisfies (IPNC). So, the information held by an agent could be equivalently described by a 3ND-valuation.

actually possessed by the agent is consistent with our "manifestability" requirement (3). In other words we have to make sure that our explications of the informational meaning of the logical operators and of the relation of "informational implication" never put an agent a who grasps the informational meaning of the logical operators in the uncomfortable position of being supposed to hold the information that a certain sentence A is true (or false) — because this follows "analytically" from the information that a *explicitly* holds and from the "intuitively sound" meaning stipulations — and yet possess no feasible procedure to decide whether or not this is the case. In other words, we have to make sure that for no agent a, $\Delta_a \vDash X$ and, yet, a is unable to tell, in practice and not only in principle, whether or not s(he) holds the information that X obtains. This amounts to showing that the relation $\vDash$ is *feasible*.

At first sight, this is far from being obvious, for our sufficient conditions in Table 3 imply that any agent a who holds some information at all, always holds information concerning sentences of arbitrary complexity. So, we have to make sure that establishing whether or not $\mathbf{\Gamma} \vDash X$ does not require any consideration of sentences whose complexity is not comparable to the total complexity of $\mathbf{\Gamma}$ and X.

6. Intelim sequences, subsentence property and tractability

We have already observed that the closure conditions on $\Vdash$ expressed in Tables 3 and 4 can be seen, respectively, as the introduction and elimination rules of a "natural deduction" system tailored to the informational meaning of the logical operators. We now show that this natural deduction system provides each agent a feasible means for satisfying the manifestability requirement (3). All proofs of the propositions stated in this section are given in the Appendix.

Given a finite set $\mathbf{\Gamma}$ of signed sentences, let us call *intelim sequence for* $\mathbf{\Gamma}$ any finite sequence $X_1, \ldots, X_n$ of signed formulas such that, for $i = 1, \ldots, n$, either (i) X_i is in $\mathbf{\Gamma}$ or (ii) X_i results from applying one of the rules in Tables 3 and 4 to preceding elements of the sequence. An intelim sequence for $\mathbf{\Gamma}$ is *closed* if it contains both $T\,A$ and $F\,A$ for some sentence A. Otherwise it is *open*. A *proof* of X from $\mathbf{\Gamma}$ is an intelim sequence for $\mathbf{\Gamma}$ that contains X. A *refutation* of $\mathbf{\Gamma}$ is a closed intelim sequence for $\mathbf{\Gamma}$. We say that X is *intelim deducible* from $\mathbf{\Gamma}$, and write $\mathbf{\Gamma} \vdash X$ if there is an intelim proof of X from $\mathbf{\Gamma}$. We also say that $\mathbf{\Gamma}$ is *intelim inconsistent*, and write $\mathbf{\Gamma} \vdash$, if there is an intelim refutation of $\mathbf{\Gamma}$. Otherwise, we say that $\mathbf{\Gamma}$ is *intelim consistent*.

Proposition 2. *If $\mathbf{\Gamma}$ is intelim inconsistent, then $\mathbf{\Gamma} \vdash X$ for every signed sentence X.*

Given that the introduction and elimination rules of our natural deduction system, taken as definitions of the logical operators, are part of the closure conditions on the informational forcing relation $\Vdash$, the following proposition is far from being surprising and its proof is entirely routine:

Proposition 3. *For every finite* $\mathbf{\Gamma}$ *and every* X, $\mathbf{\Gamma} \vDash X$ *if and only if* $\mathbf{\Gamma} \vdash X$.

From the above proposition it immediately follows that:

Corollary 2. *For every finite* $\mathbf{\Gamma}$, $\mathbf{\Gamma}$ *is informationally inconsistent if and only if* $\mathbf{\Gamma} \vdash$.

Taking for granted the notion of *immediate subsentence* of a given sentence, let us say that A is a *subsentence* of B if there is a sequence $C_1, \ldots, C_n$, such that $C_1 = A$, $C_n = B$ and, for all $i = 1, \ldots, n-1$, C_i is an immediate subsentence of C_{i+1}. Let us now say that a signed sentence X is a *signed subsentence* of a signed sentence Y if, for $S_1, S_2 \in \{T, F\}$, $X = S_1\,A$ and $Y = S_2\,B$, for some A, B such that A is a subsentence of B. Finally, let us say that (i) a *refutation* π of $\mathbf{\Gamma}$ has the *subsentence property* if every element of π is a signed subsentence of some signed sentence in $\mathbf{\Gamma}$; (ii) a *proof* π of X from $\mathbf{\Gamma}$ has the *subsentence property* if every element of π is a signed subsentence of some signed sentence in $\mathbf{\Gamma} \cup \{X\}$. We can show that:

Proposition 4 (Subsentence Property)**.** *If* $\mathbf{\Gamma} \vdash X$, *then for every intelim proof* π *of* X *from* $\mathbf{\Gamma}$*:*

(1) *if* π *is an open intelim sequence, there is an intelim proof* π' *of* X *from* $\mathbf{\Gamma}$ *such that* π' *has the subsentence property and the length of* π' *is less than or equal to the length of* π*;*
(2) *if* π *is a closed intelim sequence, there is an intelim refutation* π' *of* $\mathbf{\Gamma}$ *such that* π' *has the subsentence property and the length of* π' *is less than or equal to the length of* π.

Given the structure of informational sequences — that unlike typical justification procedures for classical or intuitionistic logic involve no branching or "case reasoning" — the subsentence property ensures that the most straightforward decision procedure for $\vDash$ is feasible and so, in accordance with our manifestability requirement (3), every agent a has a feasible procedure to decide whether or not s(he) holds the information that A is true, or the information that A is false, or neither of them.

This straightforward decision procedure can be informally described as follows ($\|\mathbf{\Gamma}\|$ is the total number of symbols occurring in a suitable encoding of $\mathbf{\Gamma}$):

1. form a list S_1 of all the signed sentences in Δ_a (the order is immaterial); this step requires time $O(n)$, where $n = \|\Delta_a\|$;

2. form a list S_2 of all the signed subsentences of all signed sentences in $\Delta_a \cup \{TA\}$ ($\Delta_a \cup \{FA\}$); this step requires time $O(n)$, where $n = \|\Delta_a \cup \{TA\}\|$ ($\|\Delta_a \cup \{FA\}\|$) and the size of the list is $O(n)$;
3. for all X in S_2,
   3.1. check whether or not X follows from the current elements of S_1 by one of the rules in Tables 3 and 4;
   3.2. if yes, add X to S_1, remove X from S_2 and restart the loop; each run of this loop (instructions 3.1 and 3.2) requires time $O(n)$.

An analysis of this algorithm shows that it exits from the loop described in instruction 3 within $O(n^2)$ runs. So the procedure terminates in time $O(n^3)$ and is, therefore, feasible.

In fact, this is a very unsophisticated procedure that can be easily improved on and it is not difficult to show that:

Proposition 5. *Whether or not* $\Gamma \vDash X$ *can be decided in time* $O(n^2)$, *where* n *is the total number of occurrences of symbols in* $\Gamma \cup \{X\}$.

It is worth noticing that $\vDash$ is a consequence relation in Tarski's sense, that is, it satisfies the usual conditions of Reflexivity, Monotonicity, Transitivity and Substitution Invariance. We may call it (classical) *informational logic*.

Notice also that, although it delivers a tractable notion of tautological inference, informational logic has *no tautologies*: there are no sentences A such that $\emptyset \vDash TA$.

The empty information state cannot license the truth or falsity of any sentence at all, a property that informational logic has in common with Belnap's four-valued logic [8, 9] and Kleene's 3-valued logic [31].[21]

7. Synthetic inferences in classical propositional logic

What about the inferences that are valid in classical logic but are not valid under the informational meaning of the logical operators? A very simple example is the inference that concludes B from premises $A \vee B$ and $\neg A \vee B$. If we hold the information that $A \vee B$ and $\neg A \vee B$ are both true, the closure conditions (or equivalently the informational matrices) do not allow us to conclude that we hold the information that B is true. Indeed, any non-deterministic valuation v such that $v(A) = v(B) = \bot$, $v(\neg A) = \bot$ and $v(A \vee B) =$

[21] Space prevents us from making a proper comparison between informational logic and these multi-valued logics that share part of its motivations. However, it is perhaps useful to stress here that informational logic, unlike the other two logics in question, cannot be characterized by a finite valuation system (see [19], Proposition 2.49). For instance, in both these logics the disjunction of two undefined (neither-true-nor-false) sentences is always undefined, while in informational logic it may be either undefined or true.

$v(\neg A \vee B) = 1$ is faithful to the informational matrices and provides a counterexample. Is there a plausible sense in which these inferences can be construed as being "synthetic", albeit a priori, so as to vindicate Kant's tenet in the realm of propositional logic?

Given a signed sentence X, let X^* denote the unsigned sentence A such that $X = SA$ for $S \in \{T, F\}$. Given a set of signed sentences $\mathbf{\Gamma}$, let $\mathbf{\Gamma}^*$ be the set $\{X^* \mid X \in \mathbf{\Gamma}\}$ and let Sub($\mathbf{\Gamma}^*$) be the set of all subsentences of the sentences in $\mathbf{\Gamma}^*$. The relation $\vDash_k$, for $k \in \mathbb{N}$, is defined as follows:

(1) $\mathbf{\Gamma} \vDash_0 X$ if and only if $\mathbf{\Gamma} \vDash X$;
(2) for every $k > 1$, $\mathbf{\Gamma} \vDash_k X$ if and only if there is an $A \in \mathrm{Sub}(\mathbf{\Gamma}^* \cup \{X^*\})$ such that $\mathbf{\Gamma} \cup \{TA\} \vDash_{k-1} X$ and $\mathbf{\Gamma} \cup \{FA\} \vDash_{k-1} X$.

Observe that, according to the above definition, $\vDash_j \subseteq \vDash_k$ whenever $j \leq k$. It is not difficult to show that:

Proposition 6. *The relation $\vDash_\infty = \bigcup_{k \in \mathbb{N}} \vDash_k$ is the consequence relation of classical propositional logic.*

The transition from $\vDash_k$ to $\vDash_{k+1}$ can be represented in terms of a *proof rule* that licenses the validity of a *higher-depth* inference whenever certain other inferences of immediately lower depth are recognized as valid. Let $\vdash_0 = \vdash$. For every $k \geq 0$, the proof rule for $k + 1$-depth inferences is the following:[22]

(PCB) For all $A \in \mathrm{Sub}(\mathbf{\Gamma}^* \cup \{X\}^*)$,

$$\frac{\Gamma \cup \{TA\} \vdash_k X \quad \Gamma \cup \{FA\} \vdash_k X}{\Gamma \vdash_{k+1} X}$$

Notice that the deducibility relation $\vdash_k$ includes all the deducibility relations $\vdash_j$ with $j \leq k$.

Consider the case in which $k = 0$. An inference step involving an application of the proof rule in (PCB) with $k = 0$ invites to restrict our attention to informational models $M = (S, \phi, \Vdash)$ and agents $a \in S$ that, besides holding the information expressed by $\mathbf{\Gamma}$, also hold either the information that A is true or the information that A is false, for some $A \in \mathrm{Sub}(\mathbf{\Gamma}^* \cup \{X\}^*)$. Except for the special cases in which $\mathbf{\Gamma} \vdash_0 TA$ or $\mathbf{\Gamma} \vdash_0 FA$, in which the application of the rule is redundant, this information is by no means necessarily held by any agent who holds the information expressed by $\mathbf{\Gamma}$. So, TA and FA in the premises of (PCB) express "virtual information" (see Section 4 above), namely information that goes *beyond* the information actually possessed by an arbitrary agent who actually possesses the information expressed by $\mathbf{\Gamma}$. Logicians are familiar with virtual information from

[22] The name "PCB" stands for "Principle of Controlled Bivalence".

the so-called discharge rules of "natural deduction" systems. However, in Gentzen-style natural deduction the use of virtual information is intertwined with some of the inference rules that fix the (classical or intuitionistic) meaning of the logical operators. In our approach, on the other hand, virtual information plays no role in the explication of their informational meaning and so analytic inferences are exactly those that make no use of it.

For each given $k > 0$, the proof rule (PCB) allows for the nested use of virtual information up to a certain fixed limit, and so k can be taken as a "degree of syntheticity" of the inferences that are valid in $\vDash_k$ and invalid in $\vDash_{k-1}$. For unbounded k, this is a proof system for full classical propositional logic that enjoys the subsentence property. However, this presentation of classical logic allows also for representing proofs that do not have the subsentence property simply by removing the restriction to subsentences in the rule (PCB).[23]

Given Proposition 5 it is not difficult to show that, for each fixed k, $\vDash_k$ admits of a feasible decision procedure:

Proposition 7. *Whether or not $\Gamma \vDash_k X$, for each $k \in \mathbb{N}$, can be decided in time $O(n^{k+2})$, where n is the total number of occurrences of symbols in $\Gamma \cup \{X\}$.*

We conclude with the suggestion that the *degree of syntheticity* of a classical inference — defined as the smallest $k > 0$ such that the inference is valid in $\vDash_k$ and invalid in $\vDash_{k-1}$ — may be regarded as a plausible and natural measure of the "cognitive effort" required by a reasoning agent to recognize the inference in question as classically valid, in that it is related to the *depth* at which virtual information concerning appropriate subsentences of the premises or of the conclusion must be assumed and manipulated. Moreover, such increasing "cognitive effort" is related to the computational complexity of the most straightforward decision procedure that can be based on the informational meaning of the logical operators and on the depth-increasing rule (PCB). In this perspective, it would be quite interesting to test this hypothesis empirically, in the spirit of recent work in experimental logic (see, for instance, [23]).

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[23] On the connection between the rule (PCB) and the cut rule of Gentzen's sequent calculus, as well as on the advantages of cut-based formalizations of classical logic, see [15, 16].

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APPENDIX

Proof of Proposition 1. Hint: First show that if a set $\mathbf{\Gamma}$ of signed formula is closed under the introduction and elimination rules and $\mathbf{\Gamma}$ contains both TA and FA for some sentence A, then $\mathbf{\Gamma}$ contains Tp and Fp for some atomic sentence p. Next consider the function v such that, for every sentence A, $v(A) = 1$ if $TA \in \mathbf{\Gamma}$, $v(A) = 0$ if $FA \in \mathbf{\Gamma}$ and $v(A) = \bot$ otherwise. Show that such a v agrees with the informational 3-valued tables. □

Proof of Proposition 2. If $\mathbf{\Gamma}$ is intelim inconsistent, there is a closed intelim sequence for $\mathbf{\Gamma}$, call it π, that contains both TA and FA for some sentence A. Now, if $X = TB$, π can be extended to $\pi, TA \vee B$, by applying the introduction rule $T\vee\text{-}\mathscr{I}1$ using TA as premise and, then, to $\pi, TA \vee B, TB$ by applying the elimination rule $T\vee\text{-}\mathscr{E}1$ using $TA \vee B$ and FA as premises. The result is an intelim sequence for $\mathbf{\Gamma}$ that contains X and, therefore, $\mathbf{\Gamma} \vdash X$. If $X = FB$, π can be extended to $\pi, FA \wedge B$, by applying the introduction rule $F\wedge\text{-}\mathscr{I}1$ using FA as premise and, then, to $\pi, FA \wedge B, FB$ by applying the elimination rule $F\wedge\text{-}\mathscr{E}1$ using $FA \wedge B$ and TA as premises. The result, again, is an intelim sequence for $\mathbf{\Gamma}$ that contains X and, therefore, $\mathbf{\Gamma} \vdash X$. □

Proof of Proposition 3. The if direction is trivial. For the only-if direction, suppose $\mathbf{\Gamma} \nvdash X$. By Proposition 2, $\mathbf{\Gamma}$ must be intelim consistent. Let S be the set of all finite intelim consistent sets of signed sentences, ϕ be the identical mapping of S onto itself, and $\Vdash$ the relation that holds between a set $\Delta \in S$ and a signed formula X if and only $\Delta \vdash X$. First, observe that, by definition, $\Vdash$ satisfies (IPNC). Otherwise, for some $\Delta \in S$ there should be a sentence A such that $\Delta \vdash TA$ and $\Delta \vdash FA$ and this implies that there should also be closed intelim sequence for Δ — obtained by concatenating any intelim proof of TA from Δ with any intelim proof of FA from Δ — against the assumption that Δ is intelim consistent. Second, observe that, for every $\Delta \in S$: (i) the image of Δ under $\Vdash$ is closed under the conditions in Tables 3 and 4; (ii) $\Delta \Vdash Y$ for all $Y \in \Delta$. Hence, $M = (S, \phi, \Vdash)$ is an informational model. However, since $\mathbf{\Gamma}$ is intelim consistent, $\mathbf{\Gamma} \in S$, but $\mathbf{\Gamma} \nVdash X$, and so $\mathbf{\Gamma} \nvDash X$. □

Proof of Corollary 2. $\mathbf{\Gamma} \vdash$ if and only if $\mathbf{\Gamma} \vdash TA$ and $\mathbf{\Gamma} \vdash FA$, for some sentence A. So, by Proposition 3, $\mathbf{\Gamma} \vdash$ if and only if $\mathbf{\Gamma}$ is informationally inconsistent. □

Proof of Proposition 4. If $\mathbf{\Gamma} \vdash X$, then there is an intelim proof of X from $\mathbf{\Gamma}$. If π is closed, then there is an intelim refutation of $\mathbf{\Gamma}$ whose length is less than or equal to the length of the intelim proof. Let π be an intelim proof of X from $\mathbf{\Gamma}$ or an intelim refutation of $\mathbf{\Gamma}$ and let $\mathbf{\Phi}\pi$ be defined as follows:

- if π is a proof of X from $\mathbf{\Gamma}$, $\mathbf{\Phi}_\pi$ is the set of all signed sentences occurring in π that are not signed subsentences of any signed sentence in $\mathbf{\Gamma} \cup \{X\}$;
- if π is a refutation of $\mathbf{\Gamma}$, $\mathbf{\Phi}_\pi$ is the set of all signed sentences occurring in π that are not signed subsentences of any signed sentence in $\mathbf{\Gamma}$.

Notice that, by definition of intelim sequence, $\mathbf{\Phi}_\pi$ never contains atomic sentences. The proof is by induction on $|\mathbf{\Phi}_\pi|$, i.e. the number of elements of $\mathbf{\Phi}_\pi$.

Base: $|\mathbf{\Phi}_\pi| = 0$. In this case it is obvious that π has the subsentences property.

Step: $|\mathbf{\Phi}_\pi| > 0$. We show that π can be transformed into an intelim sequence π' for $\mathbf{\Gamma}$ with the following properties:

(i) If π is a proof of X from $\mathbf{\Gamma}$, π' is either a proof of X from $\mathbf{\Gamma}$ or a refutation of $\mathbf{\Gamma}$;
(ii) if π is a refutation of $\mathbf{\Gamma}$, π' is a refutation of $\mathbf{\Gamma}$;
(iii) $|\mathbf{\Phi}_{\pi'}| < |\mathbf{\Phi}_\pi|$;
(iv) the length of π' is less than or equal to the length of π.

Let Y be a signed sentence in $\mathbf{\Phi}_\pi$ of maximal logical degree (where by *logical degree* of a signed sentence we mean the number of occurrences of logical operators in it). There are several cases depending on the logical form of Y. We discuss only the case in which $Y = TA \vee B$, since it is fully representative of all the others. Since $Y = TA \vee B$ has maximal logical degree in $\mathbf{\Phi}_\pi$, then it occurs in π as the result of an introduction. Therefore, either TA or TB must also occur in π before Y.

Let us say that $Y = TA \vee B$ is *used* in π if there is a signed sentence Z in π such that either $Z = FA \vee B$, or Z occurs after Y as the result of an application of a rule with Y as (one of the) premise(s). Now, either Y is used in π or it is not. If not, then Y can be simply removed from π and the resulting sequence π' satisfies (i)-(iv) above. Otherwise, there are two cases: (i) if $Z = FA \vee B$ occurs in π, since Y has maximal logical degree in $\mathbf{\Phi}_\pi$, $FA \vee B$ also has maximal logical degree in $\mathbf{\Phi}_\pi$ and therefore it occurs in π as the result of an introduction. Thus, both FA and FB occur in π before $FA \vee B$. Moreover, if Z is used in π as premise of a rule application, it can only be used as major premise[24] of a (redundant) application of $F \vee$-$\mathscr{E}1$ or $F \vee$-$\mathscr{E}2$ with FA or FB as conclusion. It follows that the sequence π' obtained from π by removing $Z = FA \vee B$ is a closed intelim sequence and

[24] By "major premise" we mean just "premise" when the elimination rule has only one premise and "more complex premise" when it has two premises.

satisfies (i)-(iv); (ii) if Z occurs after Y as the result of a rule application with Y as premise, since Y has maximal logical degree in $\mathbf{\Phi}_\pi$, Y can be used only as major premise of an elimination. So, either π contains FA, and TB is obtained from Y and FA by an application of $T\vee-\mathscr{E}1$, or π contains FB, and TA is obtained from Y and FB by an application of $T\vee-\mathscr{E}2$. Since either TA or TB occurs in π before Y, it follows that either the conclusion obtained by the elimination was already present in π or π is closed independent of this rule application to Y. Hence, in either case the sequence π' obtained from π by removing Y satisfies properties (i)-(iv). □

Proof of Proposition 5. A proof can be adapted from [17], [20]. □

Proof of Proposition 6 (Sketch). Consider a finite set Γ of unsigned sentences and an unsigned sentence A such that Γ classically implies A. Let Λ be the set of all the atomic sentences that occur in the sentences of $\Gamma \cup \{A\}$ and suppose that $\Lambda = \{p_1, \ldots, p_n\}$. Consider all the possible sets $\{X_1, \ldots, X_n\}$ of signed sentences such that every X_i, $i = 1, \ldots, n$, is equal to T_{p_i} or F_{p_i}. There are 2^n such sets. For each of these sets consider the set $\{TB \mid B \in \Gamma\} \cup \{X_1, \ldots, X_n\}$. Since $\vDash_0$ can simulate each line of the classical truth-tables, this set is either informationally inconsistent, or such that $\{X_1, \ldots, X_n\}$ informationally implies TA. In either case, by definition of $\vDash_0$,

$$\{TB \mid B \in \Gamma\} \cup \{X_1, \ldots, X_n\} \vDash_0 TA.$$

It follows, by definition of $\vDash_k$, that $\{TB \mid B \in \Gamma\} \vDash_{2^n} TA$. Hence, for sufficiently large k, $\vDash_k$ can justify the validity of any classical inference. □

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