Logique & Analyse **225** (2014), 83–112

MATHEMATICS, LOGICS, AND PHILOSOPHY: THE ANALYTIC/SYNTHETIC DISTINCTION IN KANT, BOLZANO AND PEIRCE

MICHAEL OTTE

ABSTRACT

The analytic/synthetic distinction lies at the heart of Kant's *Critique of Pure Reason* and Bolzano employed difficulties in Kant's presentation to re-conceptualize the whole relationship between science or mathematics and logics. Bolzano was not any more concerned with epistemology, but with science and mathematics as a cultural and logical phenomenon and he made analyticity (syntheticity) a characteristic of the form of a proposition. His semantic reformulation of Kant's problem trivialized it in a sense (WL §305). On the other hand, the problem, by being transferred to the wider socio-cultural context, showed its greater complexity. The analytic/synthetic distinction since became accepted as a fundamental "Dogma" (Quine) of scientific positivism and analytical philosophy.

Peirce agrees with Bolzano that the first question that Kant should have asked was, how synthetic propositions are possible at all. His criticism of Kant's views on the analytic/synthetic distinction departs, however, into a quite different direction. Peirce, being primarily interested in the growth of knowledge, rather than in the construction of a logical system, stresses the universal importance of semiotics. Peirce considers the distinction between signs and objects, between general and particular, as only relative and thus the analytic/synthetic distinction became recognized as relative too.

What seems fascinating is, that a rather specific question — the meaning and reference of the analytic/synthetic distinction — can be turned into a probe to investigate the wider philosophical contexts. The differences between the three philosophers amount to fundamental philosophical differences, rather than reflecting different ideas of the analytic/synthetic distinction itself. Rather than assuming that we all "live" (mentally, that is, as philosophers) in the same world, believing that there is the "right" answer, we could think about, how the same idea or problem comes to look like within different worlds. In the philosophy of mathematics it is absolutely useless, for instance, to ask out of context, "what numbers really are?"

1. Introduction

Kant, Bolzano, and Peirce, were all involved in a logical project of some sort. By its originator Aristotle's logic had been intimately connected to rhetoric (Aristotle distinguishes, for instance between dialectical syllogisms

doi: 10.2143/LEA.225.0.3011357.

and apodictic ones). During the 19th century this connection led to two different conceptions of logic and of mathematics as well: “Logic as calculus and logic as language”, (Heijenoort, 1967). Hintikka, who had stressed repeatedly the importance of this distinction, comments on it as follows:

> “An initial reference-point … is provided by Leibniz’ distinction between two components of his ambitious project in mathematical logic…. On the one hand, Leibniz proposed to develop a *characteristica universalis* … whose symbolic structure would reflect directly the structure of the world of our concepts. On the other hand, Leibniz’ ambition included the creation of a *calculus ratiocinator* which was conceived of by him as a method of symbolic calculation which would mirror the processes of human reasoning” (Hintikka 1997, p. I).

Now, Kant’s logical project rests on the conviction that by language and conceptual thinking alone one cannot reach objectivity. One cannot get existential assertions out of definitions and linguistic maneuvers alone. “Being is evidently not a real predicate”, Kant has famously stated refuting the proofs of God’s existence, which were so essential to classical Rationalism (Critique, B 626). Kant believed that reference to objects can be addressed only from an active or constructive perspective and that a different logic, transcendental logic, was required for that (Kant, B 81). The notion of transcendental logic distinguishes Kant’s philosophy from straightforward empiricism. Kant, however, did maintain to a certain degree the epistemological framework of the “Classical Age” (Foucault), with its oscillations between rationalism and empirism.

Bolzano and Peirce, both gave up the relation of logic to human thinking and endorsed each one of the two conceptions of logic presented by Heijenoort and Hintikka. Bolzano defines logic as universal and as a language, while, according to Peirce logic is a theory of diagrammatic reasoning.

Bolzano called his logic a “doctrine of science” (*Wissenschaftslehre* (WL)) which considers the relationships between “propositions in themselves”. Bolzano himself indicates essentially two reasons why his approach to logic seems rather novel. Firstly logic had traditionally been a subject matter to be taught only to novices and secondly the sciences had to have reached a certain extension and maturity, before one might be able to even think about a doctrine of science (WL §5).

Bolzano critizised those, who have considered logic to be the science of how humans think. “One is inclined to feel that it is too little for a doctrine of science (Wissenschaftslehre) not to want to rise oneself beyond the consideration of the laws which bind only our thinking, rather than proceed to the propositions and truths in themselves, which would be the next higher thing” (Bolzano, WL §16). Bolzano’s philosophy is not concerned with individual cognition and knowledge and he criticized Kant’s restrictions. Bolzano concentrates on science as a reality *sui generis* and tries to outline the requirements of a theory of knowledge representation and conceptual reasoning.

The first thing a sign or representation must fulfill, in order to function as a sign at all, is consistency and Bolzano bases the ontology of his doctrine of science exactly on the principle of consistency. The difference between Kant and Bolzano comes therefore clearly out in their respective treatment of the logical law of non-contradiction. The principle of consistency, according to Kant, only applies if there is an object given. The statement that "a triangle has three angles", says Kant, "does not enounce that three angles necessary exist, but upon the condition that a triangle exists three angles must necessarily exist in it" (Kant, B 622). With Bolzano things are exactly the opposite way, that is, the law of non-contradiction comes first and the whole universe of possible "representations in themselves" becomes established thereby. Sometimes this is expressed by saying that "the entire ontology is based on the principle of non-contradiction" (Neemann 1972, p. 49). However, such statements might be misleading, because Bolzano does not believe that the logically possible must exist and he does not claim a "pre-established harmony" between concepts and objects in the sense of Leibniz (Bolzano, WL §280, Annot.1).

On the contrary, he draws, like Kant a principal and categorical distinction between theory and objective reality. Bolzano makes accordingly the consideration of the truth or falsehood of "propositions in themselves" (Sätze an sich) the focus and starting point of his logic or doctrine of science (Wissenschaftslehre).

Thus both, Kant as well as Bolzano, having refuted the knowledge foundations of Classical Rationalism had to find new contexts to their philosophies of science. Both had the advantage over their predecessors that objective knowledge was an indubitable reality in their times (with respect to the "dilemma" of classical rationalism see: Hacking 1980, pp 176f.). Kant assumes something like the "objectivity" of the subjective and bases his "Copernican Revolution" of epistemology on it. Kant writes:

> "Hitherto it has been assumed that all our knowledge must conform to objects. But all attempts to extend our knowledge of objects by establishing something in regard to them a priori, by means of concepts, have, on this assumption, ended in failure. We must therefore make trial whether we may not have more success in the tasks of metaphysics, if we suppose that objects must conform to our knowledge. This would agree better with what is desired, namely, that it should be possible to have knowledge of objects *a priori*, determining something in regard to them prior to their being given" (Critique, Introduction).

Starting from the <u>fact</u> of knowledge, Kant sets out to explain how knowledge is possible by identifying its conditions and possibilities in the reality and objectivity of human activity itself. And the logic he had in mind was "transcendental logic", which "would examine the origin of our cognitions of objects, so far as that origin cannot be ascribed to the objects themselves" (Kant, B 81). Transcendental logic therefore is "a doctrine of mind and of rational knowledge, by which we think of objects completely a priori".

Bolzano starts, half a century after Kant, from the reality of science as a socio-cultural phenomenon and framed his logic, as was said above already, as a "doctrine of science" So what is a science then? Bolzano answers: "It is the conjunction of truths which is of a kind that deserves to be presented in a particular book" (WL, §1). A doctrine of science or logic accordingly is the aggregate of

> "(…) all rules which we must follow when we divide the total sum of truths into individual sciences and represent them in their respective books. There can also be no doubt that the sum of these rules itself deserves to be called a science, (…) I allow myself to call it the theory of science (*Wissenschaftslehre)*" (WL §1).

Bolzano consequently constitutes a kind of linguistic ontology by hypostatizing linguistic sense and thereby conceiving of a "third world", beyond the world of physical objects or physical states and the world of states of consciousness. Popper, who had coined this term, "third world", had acknowledged inspiration from Bolzano in his search for an "objective epistemology" (Popper 1972, p. 106). This third realm of the "in-itself" is brilliantly wielded by Bolzano, as has repeatedly been observed, to define and explain truth and falsity, logical consequence, compatibility, derivability and the analytic/synthetic distinction.

Few things, Coffa writes, "have proved more difficult to achieve in the development of semantics than recognition of the fact that between our subjective representations and the world of things we talk about, there is a third element: what we say" (Coffa, 1993, p. 76). And this third element, that is sense or meaning "cannot be constituted either from psychological content or from the real world correlates of our representations" (Coffa, 1993, p. 77).

Peirce shares the semantic view. "Kant regarded mathematical propositions as synthetical judgments a priori; wherein there is this much truth, that they are not, for the most part, what he called analytical judgments; that is, the predicate is not, in the sense he intended, contained in the definition of the subject. But if the propositions of arithmetic for example are true cognitions, or even forms of cognitions, this circumstance is quite aside from their mathematical truth" (Peirce, CP 4.232).

Semantics can, however, be understood in two different ways, namely as the branch of linguistics that deals with the study of meaning and communication, or, secondly, as the study of the relationships between signs or symbols and what they represent. Bolzano adheres to the first view, Peirce endorses the second.

Bolzano was particularly interested in mathematics or arithmetic as a language, while Peirce did emphasize mathematics as an activity and mathematical logic as a theory of that activity.

Logic, he writes, "is only another name for semiotic, the … formal doctrine of signs. By describing the doctrine as … formal, I mean that we observe the characters of such signs as we know, and from such an observation, by a process which I will not object to naming Abstraction we are led to statements eminently fallible … as to what must be the characters of all signs used by … an intelligence capable of learning by experience. As to that process of abstraction, it is itself a sort of observation" (Peirce, CP 2.227).

The mathematician, for example, constructs a diagram and by continuously modifying the diagram and observing a fix point or an invariant upon continuous modification he might then try and frame a hypothesis and formulate a theorem (Otte, 2003, 206ff). Peirce's epistemology of mathematics is analytical and experiential, rather than constructive. Kant gives the erroneous view "that ideas are presented separated and then thought together by the mind. This is his doctrine that a mental synthesis precedes every analysis. …" (Peirce, CP 1.384).

From this results Peirce's overwhelming logical interest in continuity or the continuum, because "the greater the number of distinct logical steps" into which mathematical inferences can be broken down, the closer the logician may come to identifying the "last elementary steps" of reasoning. Logic is not formal, it has, in contrast to pure mathematics, an object, it is "categorical in its assertions", rather than hypothetical, as mathematics. "Logic depends upon mathematics; still more on ethics; but its proper concern is with truths beyond the purview of either" (Peirce, CP 4.240).

Both, Bolzano, as well as Peirce recognized the fundamental significance of the symbol or representation and considered Kant's central question of "how is synthetic knowledge a priori possible?" as besides the point and both wanted to reconcile their theories of truth with pragmatic necessities, such that truth becomes more of a commitment or a necessity of rational behavior, than a matter of metaphysics.

2. The analytic/synthetic distinction

The fundamentally important and in the philosophical tradition much discussed distinction of the analytic and synthetic bears a malice, it seems treacherous, because it marks an intuitively plausible and easily perceptible difference. Two examples may illustrate this:

(A): Bachelors are unmarried.
(B): In Manaus in the Amazon it rains on 28 October 2013.

(A) is based on synonymy or sameness of meaning and is therefore analytical, while the verification of the truth of (B) appears to require empirical knowledge and is therefore considered synthetical. According to the conventional view, the predicates "analytic" or "synthetic" are not attributed to the sentences themselves, but are related to the truth conditions of the respective judgments. Neither Kant and Bolzano nor Peirce did endorse, however, a verification theory of truth (in the sense of Schlick or Ayer and the mainstream of analytical philosophy). Such a truth theory is characteristic of empiricist philosophy, which believes that empirical verification is absolutely necessary to sort out the meaningful sign combinations among those the mind had produced or could produce, because mind itself is a *tabula rasa* apart from purely combinatory faculties or principles (Katz, 1966, chap. 5).

As for the truth conditions, in the case of (A) you can look into a lexicon and in the other case (B) you better turn your face to the sky. So the matter seems to amount to the difference between communication and perception and this difference seems only relative or gradual (contrary to the views of Kant or Bolzano!), just as the difference between things and signs, the particular and the general is only relative. In Plato's dialogue *Cratylus*, Socrates forces Hermogenes to admit that any purposeful activity — even the efforts of a rhetorician or of a straightforward liar — are objectively constrained, if it wants to be successful. Mind and world are mediated or connected by the system of activities (including its means and goals). It follows from this, in particular, that words or signs, on the one side and objects and goals, on the other, are not as distinct and separated as one might suppose. To draw an absolute distinction between signs and objects, or between concepts and perceptions, the operative and receptive sides of the human mind, would amount to something like Xenon's paradox of the race between Achilles and the Tortoise (Peirce, CP. 5.157 and 5.181).

Kant understands knowledge as construction and objective or objectual activity and he refuses the description theory of reference, as it appears in Leibniz' 'principle of *Identity of Indiscernibles*' (Kant, B 320). Kant's intention in introducing the analytic/synthetic distinction was to say that we can never gain any knowledge by means of analytical reasoning from concepts and that therefore even *a priori* knowledge, like mathematics must be based on intuition. Now, in his *Critique* Kant gives the following description of the analytic/synthetic distinction:

> "In all judgments wherein the relation of the subject to the predicate is thought this relation is possible in two different ways. Either the predicate B belongs to the subject A, as something, which is contained (covertly) in this concept A; or B lies completely outside of the concept A, although it stands in connection with it. In the first instance, I term the judgment analytical, in the second synthetical" (B 11).

Kant presents the following example as an illustration:

> "When I say, *All bodies are extended*, this is an analytical judgment. … On the other hand, when I say, *All bodies are heavy*, the predicate is something totally different from that which I think in the mere concept of a body. By addition of such a predicate it becomes a synthetical judgment" (B 11).

It would be tempting to conclude that in an analytic judgment, the predicate remains an essential property of the object in question. Such a conclusion would represent the classical view as presented by Aristotle and also by Descartes. The entirety of Cartesian physics rests on the claim that extension is the primary attribute of body, and that nothing more is needed to explain or understand body. For Kant, this view is not quite correct, for two reasons. First, Kant differentiated sharply between concepts and objects, from whence the whole question about the analytic/synthetic distinction arises. Second, the judgment *bodies are extended* is analytic, because the extension of space belongs to pure intuition and is therefore one of the *a priori* conditions of this judgment. Space is subjective to Kant. "Space is a necessary idea *a priori,* which serves for the foundation of all external intuitions" (Kant, B 39), such that it becomes impossible to represent, or think a body without ascribing extension to it.

Peirce, as well as, Bolzano criticized Kant alleging, "that he confuses together a question of psychology with a question of logic" (Peirce, CP 4.85). This is somewhat doubtful, because Kant certainly intended to speak about the "epistemic" or transcendental, rather than the individual subject. A different more serious criticism, brought forward against Kant by both, concerns Kant's conception of logic as a theory of human thought or activity (for example: Peirce, CP 4.86).

Bolzano and Peirce differ profoundly, however, in their respective conceptions of logical inference. Peirce assumes a close connection between logic and mathematics and believes that Kant was right in distinguishing mathematics from philosophy by their respective methods and stating that that mathematical method "consists in studying constructions, or diagrams", while philosophy proceeds by constructing and analyzing concepts (Peirce, CP 3.556). Bolzano, in contrast, based logical reasoning on an analysis of the semantic relations between concepts and propositions. Bolzano understands the question of the difference between the analytic and synthetic as a questions concerning "the inner nature of the sentences" (Bolzano, WL § 133).

Bolzano was not concerned with epistemology and he blamed Kant for having confused mathematics as such with the manner we humans are gaining mathematical knowledge. His task was the developing of a "Doctrine of Science" (Wissenschaftslehre), that was to investigate into the true organization of knowledge and to present it in well written text books. It is said (for example, Misch 1901), that positivism came about through the efforts of the philosophers of the Enlightenment, like D'Alembert, Turgot and others searching for an adequate order of the new sciences. Their moral power sprang from the conviction that knowledge makes a difference and that the difference it makes is in this world. It had been the hope of 18th century Enlightenment that rational inquiry would provide those ideas that allowed the proper organization of all knowledge and the proper conduct of all human affairs. In this sense, Bolzano was a philosopher of the Enlightenment.

Bolzano recognizes Kant‘s insistence on the analytic/synthetic distinction as important and he drew a sharp a distinction between concept and object, like Kant. Bolzano also agreed with Kant's rejection of a "pre-established harmony" between our cognitions and the objective world in the sense of Leibniz. "It had exactly been Bolzano, who … had completely anti-platonically distinguished between the structure of being and the structure of cognition (Denkstruktur)" (Neemann, 1972, p. 81). And on this distinction the other one between analytic and synthetic propositions is crafted, because it has made both, Kant as well as Bolzano, aware of the errors of the traditional notion of a concept as something established by abstraction, wherefrom results the law of inverse relation between content and extension of concepts. Bolzano, refusing this law of inverse relationship, writes:

> "If I am so fortunate as to have avoided a mistake here which remained unnoticed by others, I will openly acknowledge what I have to thank for it, namely it is only the distinction Kant made between analytic and synthetic judgments, which could not be if all of the properties of an object had to be components of its representation" (WL, §120).

A proposition is obviously synthetic if its predicate contains a characteristic of the object, which is not already part of the presentation of the subject of that proposition. This could not happen if the concept would be just the set of all the characteristics of its objects. From such observations results Bolzano's definition of analytical propositions, which is as follows:

> "If there is a single representation (eine einzige Vorstellung) in a proposition which can be arbitrarily varied without disturbing its truth or falsity … then this character of the proposition is sufficiently remarkable to distinguish it from all others. I permit myself thence to call propositions of this kind, borrowing an expression from Kant, *analytic*, all others, however, *synthetic* propositions" (WL §148).

The variants of that "single representation" in question should be "objectual" variables of some sort, rather than being merely consistent, but otherwise arbitrary representations (WL §280). In consequence, one might say that a proposition is analytic if the sentential function which results from it by replacing one of its elements by a variable, whose meaning varies among the elements of a certain model world or universe of discourse, come all out true. The sentence, "Socrates is mortal" is synthetic, if God exists, for example, and is analytic otherwise. Now Bolzano certainly was in no doubt about God's existence and his "model-world" was constituted by the meanings of common language and understanding, while Kant's was that of objective activity and mental construction.

The definitions of Kant and Bolzano seemed very different from one another, at first sight. They do, however, try to capture the very same idea or problematic. The differences between them amount to contextual differences, rather than to different ideas of the analytic/synthetic distinction. A general idea or problem, being differently specified, according to different contextual interpretations, becomes an instrument for probing those very contexts. This situation sounds familiar to mathematicians.

Since Felix Klein's "Erlanger Program" of 1872, the idea of a "general triangle", for example, is commonly interpreted within a theoretical context. The idea of "general triangle" could have served as a probe to examine the views of philosophers since the days of the Locke/Berkeley controversy already. In 1710, Berkeley had asked the readers of Locke's *Essay concerning Human Understanding* to try and find out whether they could possibly have

> "an idea that shall correspond with the description here given of the general idea of a triangle, which is neither oblique, nor rectangle, equilateral, equicrural, nor scalenon, but all and none of these at once" (Berkeley 1975, p.70). Nowadays students are advised not to call a variable in algebra a "variable number", because such a variable number could be neither positive nor negative nor equal to zero nor even or uneven, etc.

And to this logical dilemma Berkeley proposed a "representational" solution, saying that "we shall acknowledge, that an idea, which considered in itself is particular, becomes general, by being made to represent or stand for all other particular ideas of the same sort" (Berkeley 1975, 70). The abstractionists, like Locke, maintain that abstract ideas are required for geometrical proofs. Berkeley argues that each proof makes use of some particular characteristics of a general idea only and never has to take into account an infinity of premises. He maintains that it is consistent with his theory of meaning to selectively attend to a single aspect of a complex idea.

Jesseph has characterized Berkeley's philosophy of geometry by the term "representative generalization" and he writes: "The most fundamental aspect of Berkeley's alternative is the claim that we can make one idea go proxy for many others by treating it as a representative of a kind" (Jesseph 1993, p. 33).

Berkeley insisting on the importance of the notion of representation or sign seems a kind of forerunner to both Bolzano and Peirce (something Peirce, for his part, had acknowledged)

3. Positivism, or exorcizing the Continuum

Positivist science and technology of modern times notably exorcised the Aristotelian continuum, based on which Aristotle had criticized Zeno's paradoxes of motion, and transforming the world into a vast collection of data. Does the world consist of separated particulars like a heap of sand, or is the principle of continuity real, rather than ideal and mental?

Aristotle is most often regarded as the great representative of a logic, that rests on the assumption of the possibility of clear divisions and rigorous classification. He seems to form his concepts by empirical abstraction, rather than proceeding from the general to the specific (Weyl, 2009, p. 150).

"But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language [...]" (Lovejoy, 1964/1936, p 58). Aristotle thereby became responsible for the introduction of the principle of continuity into natural history. "And the very terms and illustrations used by a hundred later writers down to Locke and Leibniz and beyond, show that they were but repeating Aristotle's expressions of this idea" (Lovejoy loc. cit.). There appears thus a dichotomy within classical rationalism and Zeno's paradoxes of motion express it.

Positivism, in contrast to the Aristotelian view, accepts the objective world only in as much as it fits certain descriptions, like those of mechanics. Science has, according to positivism, already found its definite form and character, such that progress could mean nothing but a filling in gaps of information, by furnishing new data. Knowledge development becomes exclusively data driven. The discrete world of data is, however, a world in which nothing is possible or has a right to be heard or taken into account, that is not a strictly defined datum or fact. The world of direct perception has no chance against the "prejudices" produced by the huge amounts of given data.

Maybe we only have become aware of this situation by the revelations of Edward Snowdon.

Positivism in mathematics has been firmly established by the so-called rigor movement of arithmetization. The program of rigorization by arithmetization, created by Bolzano and Cauchy attempted to solve the foundational problems in a reductionist manner, by defining all mathematical concepts in terms of set-theory. There is only one world for doing mathematics, the universe of sets.

Bolzano comments on Gauss' proof of the fundamental theorem of algebra of 1799, which was based on the intuition of the continuum:

> "There is certainly nothing to be said against the *correctness*, nor against the *obviousness* of this geometrical proposition. But it is also equally clear that it is an unacceptable offense against *good method* (ein nicht zu duldender

> Verstoss gegen die gute Methode) to try to derive truths of *pure* (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely *applied* (or special) part of it, namely *geometry*" (Bolzano in: Kolman, 1963, p. 176).

Arithmetization thus becomes a method to eliminate continuity and to furnish a model that turns continuity into "an arithmetical notion" (Coffa 1993, p. 28). To this end one must postulate some axiom of continuity in real analysis, as well as, in geometry (Dedekind; with respect to geometry see: Webb 1997, p. 6) such that nothing is gained really by arithmetization, except a greater explicitness. There is no rigorous proof now, where before had been sloppy intuition, rather one (implicit) axiom has been replaced by another explicit one. Explicitness of premises becomes important for various reasons. It is important as soon as one wants to generalize a theorem, or, in Bolzano's case, it complies to the wish to set forth the "objective grounds" of the theorem proved.

Bolzano's "*Wissenschaftslehre*" contains, in fact, a distinction between proofs that verify, being intended to create conviction or certainty, and others, which "derive the truth to be demonstrated from its objective grounds. Proofs of this kind could be called justifications (*Begründungen*) in difference to the others which merely aim at conviction (*Gewissheit*)" (WL §525). Only synthetic propositions are admitted as explanations and objective groundings. "Not a single basic theorem," says Bolzano, is an analytic truth" (WL §14).

So Bolzano is forced to replace the idea of the continuum by an explicit model of it. Now, one of the first observations in the definition of continuity of a mathematical function in the Bolzano-Cauchy manner concerns its local character (in contrast to the traditional concept of "uniform continuity"); the definition speaks about continuity at a certain point and then might generalize by quantifying over point-sets. Replacing the concept of uniform continuity — which implies that the limit of a convergent series of continuous functions is continuous itself — by a local concept, one proceeds from intensional to extensional mathematic. It became clear that a continuous mathematical function had to be conceived of as an equivalence class of concrete representations of it, rather than to be identified with some of its possible representations, — the axiom of extensionality furnishing the constitutive equivalence relation. The property of being continuous can be attributed to such a class only, rather than being a property of some representation of a function.

Moreover, a function becomes transformed from a concept into an object, it is nothing but a particular kind of set. A function from X to Y is a particular subset of the product set X*Y. And finally, the only requisite for the definitions and theorems about continuity is the availability of a notion of distance as a measure of proximity. This leads to the abstract notion of a metric space. A slightly deeper analysis of the relationship between a given metric and the collection of functions continuous with respect to that metric shows that it is not the metric which is significant, but only those subsets which are defined as "open". Two metrics on the same set X determine the same classes of continuous functions if a subset of X is open with respect to one of the metrics, if and only if, it is open with respect to the other. Thus the task of characterizing continuous functions is equivalent to choosing a topology, that is, to choosing a class of open sets. And if we choose the discrete topology, calling all individual points "open", all functions are continuous. The concept of continuity becomes formal and, in a sense, philosophically empty.

Continuity and the function concept have both become eliminated and have both been replaced by set theoretical notions. It is a curious thing to observe that the transformation of mathematics from *form* to *function*, as it occurred at the turn of the 18th century put mathematicians at a loss when asked to explain what a function essentially is (Otte 1994, pp. 285-289). The analytical viewpoint, on the one hand, liberated functionality and sense as relatively independent from structure or objective reference and eliminated them on the other hand in favor of set theory (Boutroux, 1920).

Peirce in contrast recognizes the continuum as real and as a realm of unactualized possibilities. The continuum is not a mathematical object, however, but is the a basic concept of meta-mathematics. A general, like the "famous" general triangle or like an apple is a free variable that can be further be specified as need might be. When we conceive of generalization as the introduction of variables, we can realize the difference between predicative generality and continuity by observing that in discrete mathematics and computer science variables are mere placeholders, while in continuous mathematics and the empirical sciences variables are objectual, "general", that is, incompletely determined objects. In a proposition like "an apple is a fruit" it would be unnatural to interpret "an apple" as a placeholder, because this presupposes that we have given individual names to all the apples in this world (Quine, 1974, chapter III).

There are ideas of an apple or a triangle in general, but they turn out to be ideas of particular triangles, put to a certain use. On such an account a general triangle is a free variable, like the terms in axiomatic descriptions, and not a collection of determinate triangles. It is an idea, that governs and produces its particular representations. Moreover which properties are essential to a "general triangle", depends on context, on the activity and its goals. If the task, for instance, is to prove the theorem that the medians of a triangle intersect in one point, the triangle on which the proof is to be based can be assumed to be equilateral, without loss of generality — because the theorem in case is a theorem of affine geometry and any triangle is equivalent to an equilateral triangle under affine transformations. This fact considerably facilitates conducting the proof because of such a triangle's high symmetry. Bishop Berkeley's discussion of the idea of "general triangle" had already made us aware of these things.

Generality is continuity and continuity offers possibilities of determination. The possible is not determined and fixed in every respect. Therefore Peirce refuses to accept the continuum as being constructed and built up from particulars, as in Cantorian set theory and arithmetized analysis. Peirce affirms that "no collection of points, thought it may be abnumerable to the billionth degree could fill a line so there would be room for no more points; and in that respect the line is truly general; no possible multitude of singulars is adequate to it" (Peirce, CP 5.530). Continuity represents in fact a central concept of the system of Peirce's evolutionary realism as well as his conception of logic and mathematics. And "it is the leading conception of science" (CP 1.62).

Now, if one believes in the inexhaustibility of objective reality then all our knowledge is fallible und never final "The principle of continuity is the idea of fallibilism objectified" (CP 1.171).

4. Evolutionary Realism

The notion of "possibility" is essential in evolution theory in particular and it seems no wonder therefore that we find the idea of continuity involved here. Since Kant, Peirce says,

> "it has been a very widespread idea that it is time and space which introduce continuity into nature. … Time and space are continuous because they embody conditions of possibility, and the possible is general, and continuity and generality are two names for the same absence of distinction of individuals" (Peirce, CP. 4.172).

The relations between continuity, variation and possibility influence all theories of evolution. Ernst Mayr, sometimes considered the "Darwin of the 20th century", for example, distinguishes between "typological thinking (essentialism)", founded, as he says, by Plato, and "population thinking", which he ascribes to Darwin. As an example of essentialism, he cites the famous "general triangle" from geometry. With respect to population thinking, he writes, "What we find among living organisms are not constant types, but variable populations … Within a population … every individual is uniquely different from every other individual". In addition, if the differences between individuals become sufficiently large, two species might suddenly break away where there had been just one before. Darwin's "basic insight was that the living world consists not of invariable essences (Platonic classes), but of highly variable populations. And it is the change of populations of organisms that is designated as evolution" (Mayr, 2001, chapter 5).

The opposition between the styles of thought, indicated by Mayr, could perhaps better be described as the distinction between a theory of prototypes (Rosch) and set-theory and definition by abstraction, because the opposition is not between essentialism and empirism, but between an Aristotelian and a positivistic, or exclusively data driven conception of biology. Peirce philosophy of the continuum and his evolutionary realism may be helpful in shedding some light on these issues. Peirce is "perhaps the most compelling example of a late 19th century thinker who sought to apply Charles Darwin's suggestion of evolution to other areas of science" (Alborn, 1989).

Peirce considers, like Mayr, the explanation of growth and diversification as essential to a theory of evolution. Any increase in variety points to spontaneity as an effective agency, because law never produces diversification (CP 1.174). So evolution theory must explain not only the phenomena, but also general facts, like natural laws or Platonic essences. Stated differently, evolution can be explained only if it is dealt with on many different levels, the genes, the cell, the individual and the population, etc.

Evolutionary theory implies that everything, laws as well as states of things, requires an explanation, for "evolution is the postulate of logic itself". Now, Peirce continues, "the only possible way of accounting for the laws of nature … is to suppose them results of evolution. This supposes them not to be absolute, not to be obeyed precisely. It makes an element of indeterminacy, spontaneity or absolute chance in nature" (Peirce, CP 6.13).

Evolution theory would therefore, in contrast to nominalism, have to explain how objectivity comes from arbitrariness, or, as Peirce formulated it, how "law ought to be explained as a result of spontaneity" (Peirce, MS 954). Peirce therefore assumes that "the laws are due to chance and repose on others far less rigid themselves due to chance … and so one in an infinite regress, the further we go back the more indefinite being the nature of the laws, and in this way we see the possibility of an infinite approximation toward complete explanation of nature. Chance is indeterminacy, is freedom. But the action of freedom issues in the strictest rule of law" (Peirce W4, 547ff, see also: Peirce, CP 1.175; CP 1.405).

In summary: "The endless variety in the world has not been created by law. It is not of the nature of uniformity to originate variation, nor of law to beget circumstance. When we gaze upon the multifariousness of nature we are looking straight into the face of a living spontaneity. A day's ramble in the country ought that bring home to us" (Peirce, CP. 6.553).

One might, however, be inclined to notice the other side too, nature's wholeness, its systemic character and the great harmony in it. All the species, mice and eagles, foxes and geese, wolves and deer, and the whole forest, everything together forms a perfect harmonious continuity. The great anthropologist Gregory Bateson once said, "The evolution of the horse from *Eohippus* was not a one-sided adjustment to life on grassy plains. Surely the grassy plains themselves were evolved *pari passu* with the evolution of the teeth and hooves of the horses and other ungulates. It is the *context* which evolves!" (Bateson, 1972, p. 155).

Spontaneity, randomness, development, imperfection and asymmetry, the evolution and the changes, all that appears only as soon as we focus on the individual or the particular population, on the individual animal, the individual plant, the individual fate. The realm of particular existents, or absolutely distinct entities, is governed by contingency and chance. In the end, however, one has to mediate between chance and continuity. Peirce hence distinguishes between various conceptions of evolution. "Evolution by fortuitous variation, evolution by mechanical necessity, and evolution by creative love. We may term them *tychastic* evolution, … *anancastic* evolution, and *agapastic* evolution, or *agapasm*" (Peirce, CP 6.302). These three types correspond to three fundamental categories of Peirce's logic and philosophy.

The purpose of thinking is to render things intelligible and scientists therefore try to discover lawfulness in nature and to achieve this purpose, one has to generalize and has therefore to employ some kind of continuity principle. The continuity principle is so important to Peirce that he decided to call his doctrine "Synechism" (CP 6.103). Synechism is the "tendency of philosophical thought which insists upon the idea of continuity as of prime importance in philosophy" (CP 6.169). And further:

> "I have proposed to make *Synechism* mean the tendency to regard everything as continuous. [---] I carry the doctrine so far as to maintain that continuity governs the whole domain of experience in every element of it" (EP 2:1, 1893).

Synechism means <u>mediation</u>, that is, what Peirce calls *Thirdness*, "and, in order to secure to thirdness, its really commanding function, I find it indispensable fully [to] recognize that it is a third, and that *Firstness*, or chance, and *Secondness*, or Brute reaction, are other elements, without the independence of which *Thirdness* would not have anything upon which to operate. Accordingly, I like to call my theory Synechism, because it rests on the study of continuity." (Peirce, CP 6.202).

So all development begins with chance, with a random idea or fluctuation of the mind, or the spontaneous decision to do this or that, and each such initial impinges on the response of another, an outside — for example, on a resistance or on the tracks of my yesterday's *Ego*, etc. The goal of knowledge, as well as of the real development is ultimately the mediation between impulse and reaction. It is the context that evolves!

The relations of analogy and similarity guide our thinking from one part to the next, while in our experience relations of contiguity or simple coincidence, without assignable reasons prevail.

One example: Each time people drank from that particular street pump near Golden Square in the city of London in that year 1854, they suddenly took sick and began dying from cholera. None of the authorities understood what cholera was or how it was transmitted. John Snow, however, a private physician caught the idea that the disease was caused by an unidentified agent that victims ingested, probably in contaminated drinking water. "Cholera was not something you inhaled. It was something you swallowed" (Johnson, 2005).

"Cholera" remained just a name for a bundle of symptoms until Snow made the connection to contaminated water and created a hypostatic abstraction, representing the cause of that contamination — an *ens rationis* — as Peirce used to call that. It is not the water as such, — the water could be cleaned — sterilized, one would say, based on today's knowledge — by boiling it — but an until then unknown bacterium *Vibrio cholerae*. Hypostatic abstractions are created by intuition, or by what Peirce calls, abductive inferences and are essential to the more complicated and profounder cases of mathematical and scientific reasoning. Think of examples like energy — of which heat and motion are different representations, and the law of energy conservation, for instance, — or of the electro-magnetic field, or of the already mentioned general triangle of geometry and other mathematical terms.

"What I call the theorematic reasoning of mathematics consists in introducing a foreign idea, using it, and finally deducing a conclusion from which it is eliminated. … The principal result of my closer studies of it has been the very great part, which an operation plays in it, which throughout modern times has been taken for nothing better than a proper butt of ridicule. It is the operation of abstraction, in the proper sense of the term, which, for example, converts the proposition "Opium puts people to sleep" into "Opium has a dormitive virtue." … I am able to prove that the most practically important results of mathematics could not in any way be attained without this operation of abstraction" (Peirce, NEM IV, 42-49).

Peirce, viewing the analytic/synthetic distinction from a genetic or evolutionary perspective considers it as only relative.

Now analytical thinking is based on "associations of similarity, synthetic reasoning upon associations of contiguity" (Peirce, CP 6.595). Or: "Phenomena that inward force puts together appear *similar*; phenomena that outward force puts together appear *contiguous*" (Peirce, CP 4.87). As we have learned, however, from our example of the London cholera epidemics and the discussion about the importance of hypostatic abstractions to science and mathematics, the difference between analytic and synthetic thinking is not as distinct as one might suppose. On another occasion, Peirce had expressed his views in the same direction:

> "The meanings of words ordinarily depend upon our tendencies to weld together qualities and our aptitudes to see resemblances, or, to use the received

> phrase, upon associations by similarity; while experience is bound together, …, by forces acting upon us, or, to use an even worse chosen technical term, by means of associations by contiguity" (Peirce, CP 3.419).

This means that analytical findings express a continuity in our thinking, while synthetic knowledge is based on actual coincidence. However, an idea might come up and a concept is created by hypostatic abstraction that helps to connect that which had appeared unintelligible, albeit related. One consequence of Peirce's Synechism is the belief that there is no absolute difference between our inner and outer world. Material and psychical phenomena are not entirely distinct (NEM 4, XVII; NEM 4, 355; CP 5.45; CP 8.261). "The main distinction between the inner and outer worlds is that inner objects promptly take any modifications we wish, while outer objects are hard facts. Yet tremendous as this distinction is, it is after all only relative" (Peirce, CP 5.45).

Accordingly the distinction between analytic and synthetic knowledge remains relative too.

5. The reality of discourse

Bolzano drew a sharp a distinction between concept and object, like Kant. Bolzano did, however, not believe in "empty thoughts". Bolzano was always concerned with reasoning, communication and teaching and hence his representationalism. Whenever we begin to reflect or to communicate there is already a world of *Vorstellungen* and propositions present. We do not encounter mere "Abracadabra". And this provides some reassuring comfort. This comfort comes at a certain price, however, because Bolzano's notion of "meaning in itself" etc. disguises the relationship between representation and object, between language and reality. Bolzano hypostasized the meaning of linguistic expressions. "representations as such" (Vorstellungen an sich) and "sentences as such" (Sätze an sich) are the subject-matter ("Stoff") of our subjective representations and our judgments (WL §280).

Bolzano's explanation of concepts, like "proposition as such", or "representation in itself" (Vorstellung an sich). always starts from a linguistic or imagined representation or statements, etc., then asking the reader to abstract from these incorporations. Bolzano ends up by saying that there neither exist two equal representations in themselves (WL §91), nor two equal propositions in themselves. (WL §150).

Propositions are the truth bearer of Bolzano's logic and it follows that truth or falsity are persistent properties of a proposition. A proposition is either true or false; and this permanently so. That certain propositions, like "this flower smells pleasant", or "a bottle of wine costs 10 thaler" appear as sometimes true and sometimes false, depending on circumstances, is due to our disregarding that the proposition in question does not remain the same. "This", for example, is an indexical sign with different referents depending on context. And in the second example we assume, says Bolzano, tacitly that there is a context of time and space when we hear somebody making such a judgment (WL §147) and the proposition therefore does not remain the same. Bolzano thus takes the pragmatic aspects of language into account.

Outside actual discourse pragmatics the proposition above — "a bottle of wine costs 10 thaler" should read: "At place X and time Y, a bottle of wine costs 10 thaler". If it would remain true forever and everywhere in the world that "a bottle of wine costs 10 thaler" then this proposition would be analytic, according to Bolzano's definition of analyticity (see part II.). The social and objective world, as it actually happens to be, is the arbiter of analyticity and propositions can be analytic by virtue of natural laws or even by virtue of mere accidental constellations. It may also happen that we do not know, whether a proposition is analytic or not (see also: Kneale, 1971, p. 366f).

One normally considers a sentence whose truth depends on the condition of the world as synthetic. For this reason, Kant tried to establish a new category of truths "synthetic a priori". To Bolzano, such a sentence is analytic, if the subject representation is just a placeholder for the subject representation of a whole class of true sentences. If one understands this variable subject as an objectual variable, however, as a universal object, than the sentence becomes synthetic (see part III.). "This stone shall fall to the ground, if I let it go" is analytic, whereas "A stone dropped will fall to the ground" is synthetic. In consequence, the distinction between the analytic and synthetic truths becomes relative, if one accepts that the difference between particular and general is only relative. And this is what Peirce concluded.

One should emphasize that Bolzano holds the very same view of analyticity and synthetiticity as Kant, if one is prepared to abstract from the fact that he relates these concepts to a different context than Kant.

Whenever we wish to say something about a certain thing we have to use a name or designation of that thing. This seems trivial enough, but the issues involved are not always clear. "The extreme pursuit of liberation of thought from all its subjective conditions has in Bolzano the effect that we in all our thoughts become to an even greater extent slaves of our verbal expressions. Because the subjective idea consists of parts, for example, he concludes that an objective representation must consist of parts too" (Palagyi 1902, p. 76). And Hugo Bergmann was saying that Bolzanos definition of analyticity signifies a complete "surrender to the arbitrariness of linguistic maneuvers" (Bergmann 1909, 76).

These criticisms seem somewhat hasty and superficial, however, for three reasons: First, analyticity depends on truth, not just on provability and formal consistency. Second, analyticity is as much a matter of reference as it is dependent on sense. Third our language and our social and communicative relations are as objective and immune to our arbitrary wishes and decisions as in the natural world.

6. Bolzano and Kant, once more

Bolzano praised Kant for his "deep insights" into the question of the analytic-synthetic distinction and its importance. It was Kant who, although not having discovered this distinction, "has provided it with the appropriate attention". After Bolzano Kant had studied thoroughly, he saw that the Kantian distinction between analytical and synthetical judgments rests essentially on the difference between concept components and characteristics of the object (WL § 120). A sentence is obviously synthetic, if its predicate contains characteristics of its object that do not already occur as components of the concept of that object, as it appears in the subject representation of the sentence.

In order to recognize that there are "characteristics of an object … that nevertheless are not presented by the concept of that object, it is only required to see adequately that distinction" (between analytic and synthetic truths) (WL §65). Kant had claimed "that all theorems of mathematics, physics etc are synthetic truths. He who understands this will also understand that there are innumerous characteristics of an object which can be deduced from the concept of that object, although we do not think of them as components of that concept" (WL §65).

"However", Bolzano complains, "while there are many followers of Kant's distinction, there are few who have since then properly distinguished between components (Bestandteilen) of the concept and characteristics (Merkmalen) of the object" (loc.cit, 289). Even Kant himself had not properly observed this difference (loc.cit, 292). "The possibility of assigning to a certain concept, like the concept of a triangle, for example, some further properties, like that of being equilateral, do not belong among the components of that concept, but are mere characteristics of it. It does not belong to the concept of triangle as a component, but is a consequence of that concept, that a triangle could be equilateral" (WL §65). Taken as the subject of predication the word "triangle" represents an object to which it characteristically belongs that it could possibly be equilateral.

Bolzano points out, for example, that according to Kant's definition the proposition "The father of Alexander, King of Macedonia, was King of Macedonia" (WL §148, p. 87) ought to be analytic, a consequence which seems absurd and which Kant certainly did not intend to be drawn. The whole argument seems completely alien to Kant and combines more with Bolzano's view of representation or proposition in itself and with his absolute rigid distinction between signs and things represented. To apply Kant's criterion of analyticity on a judgment of the form "q is P", we have to understand the subject of the sentence as something that is q and not simply as something that is arbitrarily defined as q. For Kant, the direct reference of a character is essential. Our intuitions and experiences refer ultimately to the things themselves, albeit in a way, relative and conditioned by our human constitution. Intuition remains an invaluable and powerful instrument of thought.

Bolzano's conception of analyticity is, as we have seen, broader than the common one of formal logic, which is based on the principle of non-contradiction, and his distinction between the analytic and synthetic does not at all coincide with the distinction between conceptual and factual truths. He warns us to note that it may not always be easy, to decide if a sentence is analytic or synthetic. "For example, the sentence, *a learned man is a man* — in the sense in which one interprets it, to find it useful, is not analytical" (WL § 148, Ann. 1), because it just says that even a learned person is "fallible".

Conversely, Bolzano emphasizes, in contrast to Hume and Kant, that the phrase "any effect has a cause" is an analytical proposition, "for as one understands by *effect* exactly that which is caused by another, and by the phrase, *have a cause,* being effected by some other, that sentence has only the meaning of what is effected by another is effected by another" (ibid.). Here we see again the extreme language dependency of Bolzano's logic.

Like Kant, Bolzano considers the sentence "The sum of angles in a triangle is 180°" as synthetic. But the proposition: "This triangle has an angle sum of two right angles" is analytic. In fact all particular propositions, having the form: "This A has b", are analytic, if the general proposition "All A have b", is true, and are synthetic otherwise. Every proposition which represents just a particular example of a more general truth where the subject "belongs to a certain kind of thing" (WL §33) is analytic.

But, the mathematician, starting from the case of the triangle and then generalizing the theorem about the angle sum of a triangle, gets: "The angle-sum of a *n-gon* = *2(n–2)* right angles". And this proposition is analytic according to Bolzano. But by substituting *n* by *3* one gets the proposition: "The angle-sum of a *3-gon* = *2(3–2)* right angles". Mathematicians would consider this last proposition as equivalent to saying: "The angle sum in a triangle equals 2 right angles".

Thus we have two propositions, one analytic the other synthetic and both representing the very same mathematical facts. Even the general theorem could be reformulated to make it a synthetic proposition: "The sum of the interior angles of a convex polygon is equal to the number of its sides minus two multiplied by a straight angle".

The analytic/synthetic difference depends on the extensions, as well as, on the intensions. Like in linguistics, Bolzano conceives of the extensions and intensions of terms, that is, of reference and sense as being independent from each other (WL §120).

Contemporaneous authors had already argued that if one defines a triangle by saying it is a geometrical figure "which has an angle-sum of 180 degrees", then the theorem about the angle-sum of a triangle would come out analytic.

"I think differently here", responds Bolzano, "as I do not consider a proposition a mere conjunction of words, but intend it as the sense of the statement, I do not admit that the proposition remains the same if one assigns to the word *triangle* at times this one and at other times a different concept. Such habits would be analogous to our pronouncing the proposition: "Euclid was a famous mathematician" intending by the name Euclid at one moment of time Ptolemäus teacher of geometry in Alexandria and at a different moment think of Euclid of Megara the student of Socrates. … In order to distinguish propositions from one another it suffices that they consist of different representations (Vorstellungen) even though they might refer to the very same objects" (WL §148).

Bolzano considers analytical propositions as particularly useful to the pragmatics of mathematical discourse. Although they cannot serve as foundations (axioms) they are useful to enlarge our universe of discourse beyond the empirically given. Bolzano writes:

> "Mathematicians profit most from this generalization (to 'imaginary' representations) especially in the theory of equations. Equations are, taking the notion at face value nothing else than statements about the equivalence of two representations (Gleichgültigkeit zweier Vorstellungen). Stating that $4+5 = 11-2$, just means to say that the representation $4+5$ comprises the very same object as the representation $11-2$. Would one stop, however, at this conception, equations like $2-2 = 0$ or $1/i = -i$ and similar ones would not be admitted, although their meaning is easy to explain according to what has been said already" (WL §108).

A mathematician might ask: why is the equation $2-2 = 0$ put into the same category as $1/i = -i$ and is to be distinguished categorically from $4+5 = 11-2$? Zero is not a number or at least is not an "objectual representation" according to Bolzano (WL §49, §55) and the square root of -1 is even an inconsistent representation to him. It is nonetheless worthwhile, Bolzano believes, to occupy oneself with such "imaginary" ideas and mathematicians "attract great advantages" from it.

"Not every representation that contains an imaginary number as a component, has therefore to be imaginary itself.… To give just one example, the notion…: ‘The mathematician who first applied the concept of the square root of – 1’, surely contains the imaginary notion ‘square root of – 1’, as a part, and yet it is undeniably a denotative representation” (WL § 71).

And “if we pronounce the sentence: ‘The representation of the square root of – 1 is composed’ then the part which represents the subject of the sentence, that is, that part, which is given by the words, ‘the representation of the square root of – 1’ indicates an objective representation, its subject matter namely is the representation, ‘the square root of – 1’, this ‘square root of – 1’ itself is, however, no denotative representation”.

In fact, within an equation referring to object-less representations, these representations must be replaced by variables which enable one to substitute the imaginary representations by denotative ones. “The equation $^1/_2$ *worker* = $^2/_4$ *worker* consists as it appears here of objectless representations and we may grasp its correctness only after having interpreted the representation *worker* as variable, in which case it becomes obvious that any substitution, producing objective representations, gives ones which represent the very same” (Groessenlehre, § 49).

The analytical propositions thus have the important function of rendering the practice of mathematics independent from too many questions about reference. We must not actually be able to interpret each statement concretely and could use mathematical statements freely if only they had meaning (make sense). We see here a certain substitute for the meaning holism of axiomatic theories, which states that only the theory as a whole must be put to test and compared with reality, not every individual concept or particular sentence.

7. Is mathematics extensional or intensional? Is it both?

Set-theoretical mathematics seems to be extensional, whereas logic is intensional. An intensional theory or theory of meanings does allow for objects, but an object represented in two different ways appears as two different objects within such an intensional theory. One might argue that mathematics is intensional like logics, because no mathematical object is given independently from its theoretical representation, but mathematics, nevertheless searches for objective truths, rather than for formal consistency between propositions about “representations in themselves”.

Frege had, against Husserl, argued in favor of the extensionality of mathematics:

> “For the mathematician, it is no more correct and no more incorrect to define a conic section as the circumference of the intersection of a plane and the surface of a right circular cone than as a plane curve whose equation with respect to rectangular co-ordinates is of degree 2. Which of these two definitions he

> chooses, or whether he chooses another again, is guided solely by grounds of convenience, although these expressions neither have the same sense nor evoke the same ideas".

With respect to the growth, as well as concerning the foundations of knowledge, it seems very relevant or even essential, however, which definition is chosen, which perspective is taken or how a problem is represented. How a mathematician defines something, a conic section, for instance, to take Frege's example, is certainly important. Two concepts A and B are not the same, even if contingently or necessarily all A's are B's and vice versa, because different concepts help to establish different kinds of relationships and thus influence cognitive development in quite different ways. Two concepts could be extensionally equivalent and yet might function differently within a certain cognitive or communicative context.

Frege himself had introduced the distinction between sense and reference to deal with the puzzle about how '$A = A$' and true '$A = B$' statements can differ semantically. Since Frege uses A and B merely to identify a particular individual descriptively, these signs can be replaced with others that have the same referent, whereas an attributive use of a representation cannot, for reasons of cognitive or emotional dynamics, for example be so replaced. The problem with this is that Fregean senses are dependent of reference, thus that "it is not immediately clear why the distinction between sense and reference should survive" (Katz, 2004, p. 12).

One might therefore suspect that mathematics — at least considered from a dynamical or genetic perspective — is characterized by a complementarity of the intensional and extensional aspects of its representations. Sense and reference are distinguished by their complementary roles in the development of knowledge. We use our language terms attributively as well as referentially, because an entity is not just the sum of its attributes and we do in fact encounter things sometimes directly without knowing much of how to describe them. Let us take a closer look at arithmetics. In his *Beyträge* (1810, p. 146f) Bolzano writes:

> "The … majority of propositions of arithmetic are, according to Kant's correct observation, synthetic propositions. But who cannot feel how contrived it is, that Kant, in order to carry through his theory of intuitions generally, has to assert that even these propositions are based on intuition, … Kant gave the proposition, 7 + 5 = 12, instead of which, to make it easier, we shall take the shorter, 7 + 2 = 9. The proof of this proposition is not difficult as soon as we assume the general proposition. a + (b + c) = (a + b) + c. i.e. that with an arithmetic sum one only looks at the number of terms not their order (certainly a wider concept than sequence in time). … Having accepted it, the proof of the above proposition can be carried out in the following way: the statements 1 + 1 = 2, 7 + 1 = 8, 8 + 1 = 9 are mere definitions and conventions. Therefore, 7 + 2 = 7 + (1 + 1). (per def.) = (7 + 1) + 1, (per propos. praeced.) = 8 + 1, (per def) = 9, (per def)".

And in his *Wissenschaftslehre*, he repeats this critique almost verbatim. By means of the associative law of arithmetic, Bolzano writes, “as well as by the definition (Erklärung) that 7 + 1 = 8, 8 + 1 = 9, etc., the proposition 7 + 5 = 12 would (emphasis added) come out as a purely analytical truth” (WL §305).

This means that Bolzano alleges Kant should have qualified the proposition 7 + 5 = 12 as analytic (WL §305), presupposing an axiomatic foundation of arithmetic. Bolzano himself considers arithmetic as synthetic in accordance with his definition above.

In an axiomatical presentation one conceives of the intensions of the basic terms as comprising all the deductive consequences from the axioms (something which Bolzano did not accept, however). Axiomatics thus is a kind of algebra and algebra is an analytic science. On this account modern axiomatic theories in the sense of Hilbert or Peano are analytic theories in that the theory as a whole determines, by means of a set of postulates the intensions of its theoretical terms and the intensions determine the extensions, that is, the intended applications or objects about which the theory speaks. Anything that obeys Peano’s axioms of arithmetic must be called a number. But nevertheless the axioms do not determine the particular numbers. One cannot deduce from the axioms that there exists number with a certain property, for example.

Each algebra requires, however, an “arithmetic”, a universe of discourse. In 1810, Bolzano had proposed a new definition of mathematics as “the science which deals with the general laws (forms) to which things must conform in their existence” (Bolzano 1810, § 8). In his mature work he thought this definition as not sufficiently general and he came back to defining mathematics as the science of quantity (WL §7; see also his “doctrine of quantity” = Groessenlehre). A quantity in Bolzanos sense is just defined in terms of equality and difference (GL, p. 26) rather than being given in terms of some more or less concrete characteristics. Quantities are just sets, where the identity of sets is based on the axiom of extensionality.

So Bolzano wanted the number concept to be based on the notion of cardinality of sets. And as soon as one formulates the concept of arithmetical sum in terms of the cardinality of sets (intuitively assuming the existence of the latter), the axioms are established as objective truths, as laws, and the arithmetical theorems in question thus become synthetic (Cassirer 1907). Arithmetical propositions like 7 + 5 = 12 are synthetic according to Bolzanos definition of syntheticity (as given above).

In summary: As soon as the whole numbers are constructed, completely from the concept of ordinal numbers, introducing the concept of sum axiomatically and recursively on the basis of the successor operation of ordinal numbers, arithmetical propositions become analytical. So both ways of founding the number concept become complementary to each other.

The rigor movement of arithmetization criticized, for example, that the axiomatic characterization of numbers leads to a situation where "every number-symbol becomes infinitely ambiguous" (Russell). These philosophers, like Russell, seemed, however, not to have perceived clearly that the axiomatic procedure "was a very general method of mathematics and that it was therefore in need of a set-theoretical foundation, if set-theoretical thinking was to be — as claimed — an omni-comprehensive basis for all mathematics. On the other hand strictly formalistic mathematics, as it was developed by Hilbert's school, did not pay sufficient attention to all that burden of set-theoretic tools which were strictly connected with axiomatics and which can be summarized in the word 'model'" (Casari, 1974, p. 52).

Martin has related the difference between Kant and Bolzano to the two sides of the development of modern mathematics mentioned:

> "One can characterize the difference between Kant and Bolzano meaning that for Kant axiomatization, and that for the Bolzano arithmetization has been the ultimate goal. Felix Klein is right, when he says: 'Bolzano is one of the fathers of the 'arithmetization' of our science'. By the keywords arithmetization and axiomatization the viewpoints are given for a specific assessment of the researchers involved in these investigations. These viewpoints also make understandable, Hilbert's appreciation of Kant, on the one hand, and Couturat's, on the other" (Martin 1956, p. 103).

8. Peirce

Peirce is not a foundationalist with respect to mathematics, like Kant or Bolzano and it seems that the analytic/synthetic distinction does not have the same importance to him, as it had for Kant or Bolzano.

Mathematical proofs are neither explanations nor justifications in the sense of Bolzanos *Begründungen*. Such a view, Peirce says, is due to an erroneous conception of meaning. Meaning, according to Peirce, is to be seen as the endless process of construction of interpreting representations. Besides mathematics does not contain, in Peirce' view, categorical affirmations at all, but consist entirely of hypothetic-deductive reasoning. And concerning Bolzano's views, Peirce might have said with some justification that Bolzano is confusing mathematics with logic or philosophy (something Bolzano would not deny but would consider a virtue rather than an error). Philosophy strives for meaningfulness and explanation, mathematicians or scientists try to generalize and search for clues to carry on their business. And with respect to the differences in argumentation and proof Peirce writes:

> "While all the philosophers follow Aristotle in holding no demonstration to be thoroughly satisfactory except what they call … a demonstration *why* …

> The mathematicians on the contrary entertain a contempt for that style of reasoning and glory in what philosophers stigmatize as mere indirect demonstrations or demonstrations *that*" (Peirce CP 4.233).

Bolzano's "*Wissenschaftslehre*" contains, in contrast, a distinction between proofs that verify, being intended to create conviction or certainty, and others, which "derive the truth to be demonstrated from its objective grounds. Proofs of this kind could be called justifications (*Begründungen*) in difference to the others which merely aim at conviction (*Gewissheit*)" (WL §525).

Finally, a mathematical proof cannot, according to Peirce, proceed exclusively on the basis of reasoning from concepts.

> "A different reasoning is demanded. Here it will not do to confine oneself to general terms. It is necessary to set down, or to imagine, some individual and definite schema or diagram. ... This schema is constructed so as to conform to a hypothesis set forth in general terms in the thesis of the theorem. ... After the schema has been constructed according to the precept .... the assertion of the theorem is not evidently true, not even for the individual schema, nor will any hard thinking of the philosopher's corrollorial kind ever render it evident. Thinking in general terms is not enough. It is necessary that something should be DONE. In geometry subsidiary lines are drawn. In algebra permissible transformations are made. Thereupon the faculty of observation is called into play" (Peirce CP 4.233).

The mathematician constructs and manipulates or modifies a diagrammatic representation of the premises in order to find out that foreign idea — to use Peirce's expression — which must be added to the set of explicit premises already available. Peirce calls such kind of reasoning theorematic reasoning. Theorematic reasoning implies generalization, that is, the introduction of new conceptions or ideal objects, which are a result of a process called "hypostatic abstraction" (with respect to the fundamentally important notion of hypostatic abstraction see also: CP 4.234, 4.235, 4.463, 4.549, 5.447, 5.534 and NEM IV, 49). In theorematic reasoning; differently from mere corollarial reasoning, which only unfolds what is already given in the premises; the mathematician must handle an abductive strategy capable of integrating the missing information.

Mathematics "studies nothing but pure hypotheses, and is the only science which never inquires what the actual facts are; while philosophy, although it uses no microscopes or other apparatus of special observation, is really an experimental science, resting on that experience which is common to us all; so that its principal reasoning are not mathematically necessary at all, but are only necessary in the sense that all the world knows beyond all doubt those truths of experience upon which philosophy is founded" (Peirce, CP 3.560).

9. Continuity again

Whereas Bolzano tries to eliminate the continuum from pure mathematics, Peirce considers it to the contrary as the very key to a logical understanding of mathematical reasoning. The supreme point of the Bolzano-Cantor approach to the foundation of mathematics consists in the hypothesis that the real numbers represent the best model of the (linear) continuum. But it should be clear — irrespective of the importance of set theory and the continuum hypothesis — that a given particular approach to a universal concept cannot hope to capture its richness. A universal, like a sign, a concept or a theory is nothing but the totality of its possible interpretations. This totality is, however, not a well defined set, but includes all possible future instantiations or applications (Peirce, CP 5.526). Peirce claims in fact that the continuum is more of a means than an object. Continuity is essentially involved in our processes of concept formation too and this requires the perception of similarities, which are neither objective nor arbitrary.

> "I desire to point out," says Peirce, "that it is by taking advantage of the idea of continuity, of the passage from one form to another by insensible degrees, that the naturalist builds his conceptions. … And it will be found everywhere that the idea of continuity is a powerful aid to the formation of true and fruitful conceptions. By means of it, the greatest differences are broken down and resolved into differences of degree, and the incessant application of it is of the greatest value in broadening our conceptions" (Peirce, CP 2.646).

Peirce praised Kant for having asked the question of "how are synthetical judgments a priori possible?", how can reason provide synthetic knowledge, because "by the mere asking of it, current philosophy of that time was shattered and destroyed, and a new epoch in its history was begun". And even the analytical philosophers recognized that a "high rank" must be attributed to Kant "because of his question concerning the problem of a synthetic a priori" (Reichenbach 1951, 40). Kant had realized, in contrast to his predecessors that neither knowledge nor experience are a direct and unmediated result of the impact of external reality.

Now, synthetic knowledge depends on the firmness of its premises while the analytical shows the power of deductive reasoning. The mathematician takes more pride in the latter aspect, whereas the empirical scientist is concerned with the synthetic. Peirce accordingly says that Kant before asking that question about the synthetic a priori, he ought to have asked the more general one: "*How are any synthetical judgments at all possible?* How is it that a man can observe one fact and straightaway pronounce judgment concerning another different fact not involved in the first? Such reasoning … has, at least in the usual sense of the phrase, no definite probability; how, then, can it add to our knowledge?" (Peirce, W3, p. 304).

And he answers this latter question by stating that "cognitions whose conditions are the same will have the same general characters" (p. 305). This is an analogue to the continuity principle in the sense of Leibniz, which governs analytical judgments: equal or similar antecedents yield equal or similar consequences. It states that there exist laws of nature, although there is no perfect "Uniformity of Nature". But "the tendency to obey laws has always and will always be growing" (Peirce CP 1.405). Laws do not apply themselves. I maintain", says Peirce "the existence of law as something real and general. But I hold there is no reason to think that there are general formulae to which the phenomena of nature always conform, or to which they precisely conform" (Peirce, CP 6.588).

Therefore Peirce does not deny, "that geometry contains propositions which may be understood to be synthetical propositions a priori. .... But the difficulty is that considered as applicable to the real world they are false. Possibly the three angles of every triangle make exactly 180 degree; but nothing more unlikely can be conceived" (Peirce, NEM IV, 82).

10. In Place of a Conclusion

As Gregory Bateson once said, "It is the *context* which evolves!" It does not make much sense therefore to critizise Kant because his "inprecise definition of the analytic/synthetic definition" or to ask why Bolzano had cast this distinction into this strange definition, etc. etc. As was said, the distinction between analytic and synthetic truths became a great issue in course of the *Scientific Revolution* and because of this it renders itself as a probe to investigate the wider philosophical and historical contexts. The differences between the three philosophers amount to fundamental philosophical differences, due to their different historical situation, rather than reflecting different ideas of the analytic/synthetic distinction itself.

References

ALBORN, T. (1989). Peirce's Evolutionary Logic, *TCSPS,* vol. 25, pp. 1-27

BATESON, G. (1972). *Steps to an Ecology of Mind*, Chandler Publ. Company.

BERGMANN, H. (1909). *Das Philosophische Werk Bernard Bolzanos*, Halle a.S.: Niemeyer.

BERKELEY, G. (1975). *Philosophical Works*, London, Dent & Sons.

BOLZANO, B. (1981). *Wissenschaftslehre* (WL), 4 vols., Sulzbach: Wolfgang Schultz. Reprint Scientia Verlag Aalen.

BOLZANO, B. (1975). *Groessenlehre,* Nachgelassene Schriften, Band 7, Stuttgart, Verlag Frommann.

BOLZANO, B. (1810). *Beytraege zu einer begruendeteren Darstellung der Mathematik*, Prag, Reprint Darmstadt: Wiss. Buchgesellschaft.

BOLZANO, B. (1849). *Was ist Philosophie?*, Braumüller Wien.
BOUTROUX, P. (1920). *L'Ideal Scientifique des Mathematiciens*, Paris: Alcan.
CASARI, E. (1974). Axiomatical and Set-Theoretical Thinking, *Synthese* 27, 49-61.
CASSIRER, E. (1907), Kant und die Moderne Mathematik, *Kant-Studien*, Bd 12, 1-40.
COFFA, A. (1993). *The Semantic Tradition from Kant to Carnap: To the Vienna Station*, Cambridge UP.
HACKING, I. (1980). Proof and Eternal Truth: Descartes and Leibniz in: *Descartes – Philosophy, Mathematics and Physics*, Stephen Gaukroger (ed), The Harvester Press, Sussex, 169-180.
HEIJENOORT, J.v. (1967). Logic as calculus and logic as language, *Synthese* 17, 324-330.
HINTIKKA, J. (1997). *Lingua Universalis vs. Calculus Ratiocinator*, Selected Papers, vol. 2, Kluwer, Dordrecht.
JESSEPH, D.J. (1993). Berkeley's Philosophy of Mathematics, Chicago: The Univ. Of Chicago Press.
JOHNSON, S. (2005). *The Ghost Map, The Story of London's Most Terrifying Epidemic — and How It Changed Science, Cities, and the Modern World.* Riverhead Books.
KANT, I., *Kritik der Reinen Vernunft*, Frankfurt 1956 (citation after the 1.(A) and 2.(B) edition).
KATZ, J. (1966). *The Philosophy of Language*, N.Y., Harper & Row.
KATZ, J. (2004). *Sense, Reference, and Philosophy*, Oxford UP.
KNEALE, W. & M. (1971). *The Development of Logic*, Oxford UP.
KOLMAN, A. (1963). *Bernard Bolzano*, Akademie-Verlag Berlin.
LOVEJOY, A. (1964/1936). *The Great Chain of Being*, Harvard University Press.
MARTIN, G. (1956). *Arithmetik und Kombinatorik bei Kant*, Berlin, de Gruyter.
MAYR, E.E. (2001). *What Evolution is*, Basic Books N.Y.
MISCH G., (1901). Zur Entstehung des Positivismus, *Archiv für Philosophie*, vol. XIV, pp. 1-39 and 156-209.
NEEMANN, U. (1972). *Bernard Bolzanos Lehre von Anschauung und Begriff*, Schöningh Paderborn.
OTTE, M. (1994). *Das Formale, das Soziale und das Subjektive*, Frankfurt, Suhrkamp.
OTTE, M. (1997). Analysis and Synthesis in Mathematics from the Perspective of Charles S. Peirce's Philosophy, in: Otte, M./M. Panza (eds.), *Analysis and Synthesis in Mathematics*, Boston Studies in the Phil. Of Science, vol. 196, Kluwer, 327-364.
OTTE, M. (2003). Does Mathematics have Objects? In what Sense?, *Synthese* 134, 181-216.
OTTE, M. (2006). Proof-Analysis and Continuity, *Foundations of Science*, 11: 121-155.
PALAGYI, M. (1902). *Kant und Bolzano*, Verlag Niemeyer, Halle
PEIRCE, CH. S.:
CCL = The Cambridge Conferences Lectures of 1898, in: Ch. S. Peirce, Reasoning and the Logic of Things, ed. by K.L. Ketner, with an Introduction by K.L. Ketner and H. Putnam, Harvard UP, Cambridge/London 1992.
CP = Collected Papers of Charles Sanders Peirce, Volumes I-VI, ed. by Charles Hartshorne and Paul Weiß, Cambridge, Mass. (Harvard UP) 1931-1935, Volumes VII-VIII, ed. by Arthur W. Burks, Cambridge, Mass. (Harvard UP) 1958 (quoted by no. of volume and paragraph).

NEM = Carolyn Eisele (ed.), The New Elements of Mathematics by Charles S. Peirce, Vol. I-IV, The Hague-Paris/Atlantic Highlands, N.J. (Mouton/ Humanities Press).

W = Writings of Charles S. Peirce. A Chronological Edition, Vol. 1-5, Bloomington (Indiana University Press) 1982 ff.

POPPER, K. (1972). *Objective Knowledge*, Oxford UP.

QUINE, W.V.O. (1974). *The Roots of Reference*, Open Court, LaSalle.

REICHENBACH, H. (1961). *The Rise of Scientific Philosophy*, U. of California Press.

WEBB, J. (1997.) Hilbert' Formalism and Arithmetization of Mathematics, *Synthese* 110, 1-14.

WEYL, H. (2009). *Philosophy of Mathematics and Natural Science*, Princeton UP.

Michael OTTE
Universität Bielefeld
Universitätsstraße 25
33615 Bielefeld
Deutschland
michaelontra@ad.com